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The Generalized Criterion of Dieudonné for Primary Valuated Groups

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Abstract. Let G be an abelian reduced p-group with limit length and let A be its valuated nice subgroup endowed with a valuation induced by the height valuation on G. If both A and G/A are either summable or totally projective groups of countable lengths, then G is either summable or totally projective.

This extends the classical Kulikov's criterion for direct sums of cyclic groups (Mat. Sbornik, 1945) and its important generalization due to Dieudonné (Portugal. Math., 1952) as well as it supports our recent results in (Acta Math. Univ. Comen., 2005), (Bull. Math. Soc. Sc. Math. Roumanie, 2006), (Portugal. Math., 2008) and (Algebra Colloq., 2008).

1. Introduction

Throughout this paper, suppose G is an abelian p-group, where p is an arbitrary but fixed prime, with a subgroup A. One of the most significant and applicable criteria in the abelian group theory is the so-termed Kulikov's criterion for direct sums of cycles ([Ku] and [Fu, v. I, p. 110, Theorem 18.1]). Later on, Dieudonné strengthens in [Di] this necessary and sufficient condition to a remarkable assertion, named as Dieudonné criterion, in which the structure of the whole group to be a direct sum of cyclic groups depends on a special subgroup and the factor-group modulo this subgroup. Specifically, assume that G is an abelian p-group with a subgroup A so that G/A is a direct sum of cycles. Then G is a direct sum of cyclic groups precisely when A is contained in a basic subgroup of G.

Recently, we have enlarged in [D] this statement to the class of so-called σ-summable groups and also in [Da] for the valuated variant of σ_λ-summable groups (λ is an ordinal cofinal with ω). In this aspect, we improved in [Danc] and [Danch] the attainment of Dieudonné for other independent classes of torsion abelian groups.

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The aim of this brief note is to generalize some of the affirmations from [Dan], [Danc] and [Danch], especially for summable and totally projective groups, in the form of valuated subgroups having the standard height valuation.

For some basic concepts about valuated groups, used here, we refer for instance to [Da].

Before beginning, we need some preliminary technicalities.

Criterion ([Ho]; [F, v. II, p. 123, Theorem 84.1]). *Let H be a reduced abelian p-group of countable length. Then H is summable if and only if $H[p] = \cup_{n<\omega} H_n$, $H_n \subseteq H_{n+1} \leq H$ and, for each non-negative integer n, H_n is with a finite number of height values in H.*

We recall that if $C \leq H$, then $C(\alpha) = C \cap p^\alpha H$ for some ordinal number α.

2. Main Result

We are now prepared with the following (see the corresponding affirmation from [Danch] as well).

Theorem. *Suppose that G is a reduced abelian p-group (with limit length) and A is its nice valuated subgroup endowed with a valuation produced by the restricted height valuation of G. If A and G/A are summable groups of countable lengths, then G is summable.*

Proof. We observe that there is a countable ordinal δ with the property that $0 = p^\delta(G/A) \supseteq (p^\delta G + A)/A$, hence $p^\delta G \subseteq A$. But $p^\nu A = 0$ for some countable ordinal ν, whence $p^{\delta+\nu} G = 0$. Thus $length(G) \leq \delta+\nu$ and consequently G possesses countable length.

In conjunction with the listed above Honda's criterion, we write $(G/A)[p] = \cup_{n<\omega}(G_n/A)$, where $G_n \subseteq G_{n+1} \leq G$ and, for every $n \geq 0$, $(G_n/A) \setminus \{A\} \subseteq [p^{\alpha_1}(G/A) \setminus p^{\alpha_1+1}(G/A)] \cup \cdots \cup [p^{\alpha_n}(G/A) \setminus p^{\alpha_n+1}(G/A)]$ for some ordinals $\alpha_1, \cdots, \alpha_n$. Moreover, we write down $A[p] = \cup_{n<\omega} A_n$, $A_n \subseteq A_{n+1} \leq A$ and, for any $n \geq 0$, $A_n \setminus \{0\} \subseteq [A(\beta_1) \setminus A(\beta_1+1)] \cup \cdots \cup [A(\beta_n) \setminus A(\beta_n+1)]$ for some ordinals $\beta_1, \cdots, \beta_n$. Since $(G[p]+A)/A \subseteq (G/A)[p]$, we easily obtain that $G[p] = \cup_{n<\omega} G_n[p]$. We now choose a family of groups $(C_n)_{n<\omega}$ such that $C_n \subseteq C_{n+1} \leq G[p]$, such that $C_n \cap A = 0$ and such that $(C_n \oplus A)/A = (G_n/A) \cap [(G[p]+A)/A]$. By the utilization of the modular law, the last equality is equivalent to $C_n \oplus A = G_n[p] + A$ where $C_n \leq G_n[p]$.

We claim that $G[p] = \cup_{n<\omega}(C_n \oplus A_n)$. In order to check this, letting $g \in G[p]$, hence $g + A \in G_m/A$ for some $m \geq 1$. It is therefore obvious that $g + A \subseteq C_m \oplus A$, whence $g \in C_m \oplus A$. Finally, $g \in C_i \oplus A_i$ for some $i \geq 1$ which substantiates our claim.

Furthermore, because of the niceness of A in G, we have that $[(A \oplus C_n)/A] \setminus \{A\} \subseteq [((p^{\alpha_1}G + A)/A) \setminus ((p^{\alpha_1+1}G + A)/A)] \cup \cdots \cup [((p^{\alpha_n}G + A)/A) \setminus ((p^{\alpha_n+1}G + A)/A)]$. Consequently, $(A \oplus C_n) \setminus A \subseteq [(p^{\alpha_1}G+A) \setminus (p^{\alpha_1+1}G+A)] \cup \cdots \cup [(p^{\alpha_n}G+A) \setminus (p^{\alpha_n+1}G+A)]$. Since for each ordinal δ it is fulfilled that $(p^\delta G+A) \setminus (p^{\delta+1}G+A) \subseteq (p^\delta G \setminus p^{\delta+1}G) + A$, we easily obtain that $(A \oplus C_n) \setminus A \subseteq [(p^{\alpha_1}G \setminus p^{\alpha_1+1}G) + A] \cup \cdots \cup [(p^{\alpha_n}G \setminus p^{\alpha_n+1}G) + A] = [(p^{\alpha_1}G \setminus p^{\alpha_1+1}G) \cup \cdots \cup (p^{\alpha_n}G \setminus p^{\alpha_n+1}G)] + A$. Thus $C_n \setminus A = C_n \setminus \{0\} \subseteq [(p^{\alpha_1}G \setminus p^{\alpha_1+1}G) \cup \cdots \cup (p^{\alpha_n}G \setminus p^{\alpha_n+1}G)] + A$.

Now, we select an ascending tower of groups $(P_n)_{n<\omega}$ so that $P_n \subseteq C_n$ with $\cup_{n<\omega} P_n = \cup_{n<\omega} C_n$, so that $P_n \setminus \{0\} \subseteq [(p^{\alpha_1}G \setminus p^{\alpha_1+1}G) \cup \cdots \cup (p^{\alpha_n}G \setminus$

$p^{\alpha_n+1}G)] + A_n$ and so that $(P_n \oplus A_n + p^{\gamma+1}G) \cap A[p] \subseteq A_n$ for each ordinal $\gamma \notin \{\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n\}$.

It is plainly verified that $G[p] = \cup_{n<\omega}(P_n \oplus A_n)$ and that $(P_n \oplus A_n)_{n<\omega}$ also forms an increasing sequence of subgroups.

What remains to prove is that every member $P_n \oplus A_n$ of the union is height-finite in G. In doing that, given $x \in P_n \oplus A_n$, hence $x = c_n + a_n$, where $0 \neq c_n \in C_n \cap [(p^{\alpha_1}G \setminus p^{\alpha_1+1}G) \cup \cdots \cup (p^{\alpha_n}G \setminus p^{\alpha_n+1}G)]$ and $0 \neq a_n \in A_n$ whence $a_n \in [(A \cap p^{\beta_1}G) \setminus (A \cap p^{\beta_1+1}G)] \cup \cdots \cup [(A \cap p^{\beta_n}G) \setminus (A \cap p^{\beta_n+1}G)]$.

Notice that if either $c_n = 0$ or $a_n = 0$, we are done.

Moreover, if the heights of c_n and a_n as calculated in G are different, we see that the height of x lies in the finite set of ordinals $\{\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n\}$. Otherwise, if $height_G(c_n) = height_G(a_n)$, we process like this. Assume in a way of contradiction that $(P_n \oplus A_n) \cap (p^\gamma G \setminus p^{\gamma+1}G) \neq \emptyset$ for some γ which does not belong to $\{\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n\}$, i.e. that x has height in G equal to γ. Since $x + A \in [(G_n/A) \setminus \{A\}] \cap [(p^\gamma G + A)/A] = [(G_n/A) \setminus \{A\}] \cap [p^\gamma(G/A)]$, it must be that $x + A \in p^{\gamma+1}(G/A) = (p^{\gamma+1}G + A)/A$, hence $x \in p^{\gamma+1}G + A$, we write that $x + g_\gamma = a$ for some $g_\gamma \in p^{\gamma+1}G[p]$ and $0 \neq a \in A[p]$. By what we have asked before, $a \in A_n$ and it possesses height γ. Therefore $a \in A_n(\gamma) \setminus A_n(\gamma+1)$ or, in other words, $(A_n \setminus \{0\}) \cap (A(\gamma) \setminus A(\gamma+1)) \neq 0$. But this is impossible.

We thus finally conclude that $(P_n \oplus A_n) \setminus \{0\} \subseteq (p^{\alpha_1}G \setminus p^{\alpha_1+1}G) \cup \cdots \cup (p^{\alpha_n}G \setminus p^{\alpha_n+1}G) \cup (p^{\beta_1}G \setminus p^{\beta_1+1}G) \cup \cdots \cup (p^{\beta_n}G \setminus p^{\beta_n+1}G)$, as desired. So, the proof is complete. *QED*

As immediate consequences, we yield the following.

Corollary ([Dan]). *Let G be a reduced abelian p-group with countable (limit) length. If $p^\alpha G$ and $G/p^\alpha G$ are both summable, then G is summable.*

Proof. It is clear that $p^\alpha G$ is a nice valuated subgroup of G equipped with a valuation induced by the height function in G. Henceforth, the Theorem works. *QED*

Corollary ([Danch]). *Let G be an abelian reduced p-group with countable (limit) length and A a balanced subgroup of G. If A and G/A are summable groups, then so is A.*

Proof. Since A is balanced in G, it is by definition nice and isotype; so all heights may be computed in G. Hereafter, we apply the Theorem. *QED*

By using the criterion for total projectivity from [HU, Theorem 2.1] and via the same reasoning appeared in the proof of the first theorem, we can record the following parallel statement (see also the corresponding assertion from [Danc] and [Danch], respectively).

Theorem ([Danch]). *Suppose G is a reduced abelian p-group (of limit length) with a nice valuated subgroup A endowed with a restricted valuation induced by the height function on G. If A and G/A are totally projective groups of countable length, then G is totally projective of countable length.*

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