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The Generalized Criterion of Dieudonné for Valuated p-Groups

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Abstract. We prove that if G is an abelian p-group with a nice subgroup A so that G/A is a Σ-group, then G is a Σ-group if and only if A is a Σ-subgroup in G provided that A is equipped with a valuation induced by the restricted height function on G. In particular, if in addition A is pure in G, G is a Σ-group precisely when A is a Σ-group.

This extends the classical Dieudonné criterion (Portugal. Math., 1952) as well as it supplies our recent results in (Arch. Math. Brno, 2005), (Bull. Math. Soc. Sc. Math. Roumanie, 2006) and (Acta Math. Sci., 2007).

1. Introduction

Let all groups under discussion be p-primary abelian groups, that are commutative groups each element of which has a finite order equal to a power of p, written additively as is customary when regarding such groups.

In 1952, Jean Dieudonné [8] proves his remarkable generalization of the classical Kulikov's criterion [10] for direct sums of torsion cyclic groups (e.g., cf. [9] too). His generalized version possesses a rather convincing for applications form, which shows that the structure of a group to be a direct sum of cycles depends on this how its subgroup is situated inside the whole group such that the quotient of the group modulo its subgroup is a direct sum of cycles.

The Dieudonné's affirmation was strengthened by us in a subsequent series of papers ([3]-[7]) for various exotic classes of primary groups. More especially, in [4] and [5] respectively, we have proved the valuated versions of the *Generalized Dieudonné Criterion* for σ-summable groups as well as for summable groups and totally projective groups both with countable lengths, respectively.

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Our purpose here is to give an enlarged formulation, in terms of valuated subgroups, of the corresponding result in [7] for Σ-groups. Also, the approach in constructing the generating subgroups is quite different than that of [7].

All our other unexplained specially notions and notation will be the same as those in [9].

2. The main result

Before formulating the chief theorem, we need a few conventions. First of all, we recollect a necessary and sufficient condition for a group to be a Σ-group.

Criterion ([1]). *The group H is a Σ-group $\iff$ $H[p] = \cup_{n<\omega} H_n, H_n \subseteq H_{n+1} \leq H[p] : H_n \cap p^n H \subseteq p^\omega H = \cap_{i<\omega} p^i H$, $\forall\ n \geq 1$.*

Assume that $K < H$. Define $K(\alpha) = K \cap p^\alpha H$ for any ordinal number α. We shall say that K is a proper valuated subgroup of H by using the valuation inherited from the height function on G. Thereby, K is a Σ-group with respect to this valuation (more precisely K is a Σ-subgroup in H), provided that $K[p] = \cup_{n<\omega} K_n, K_n \subseteq K_{n+1} \leq K[p] : K_n \cap p^n H \subseteq p^\omega H$, i.e. $K_n \cap K(n) \subseteq K(\omega)$. In particular, if K is pure in H, K being a Σ-subgroup in H reduces to K is a Σ-group.

And so, we have done much of the groundwork necessary to proceed by proving the following assertion (compare with the corresponding result from [7]).

Theorem. *Suppose G is a group with a nice valuated subgroup A endowed with the valuation produced by the restricted height valuation of G. If G/A is a Σ-group, then G is a Σ-group if and only if A is a Σ-subgroup in G. In particular, under these circumstances, G is a Σ-group only when A is a Σ-group, provided A is pure in G.*

Proof. Foremost, we treat the necessity. According to the foregoing criterion, we write $G[p] = \cup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1} \leq G[p]$ and $G_n \cap p^n G \subseteq p^\omega G$ for each $n \geq 1$. Therefore $A[p] = \cup_{n<\omega} A_n$, where we put $A_n = G_n \cap A$. Thus $A_n \subseteq A_{n+1} \leq A[p]$ and $A_n \cap A(n) = A_n \cap p^n G = G_n \cap p^n G \cap A \subseteq p^\omega G \cap A = A(\omega)$. By virtue of the comments alluded to above, we are finished.

As for the sufficiency, we write down via the preceding criterion that $A[p] = \cup_{n<\omega} A_n$, where, for every natural number n, $A_n \subseteq A_{n+1} \leq A[p]$ and $A_n \cap p^n G \subseteq p^\omega G$. Hypothesis also implies that $(G/A)[p] = \cup_{n<\omega}(G_n/A)$, where, for all positive integers n, $(G_n/A) \cap p^n(G/A) = (G_n/A) \cap [(p^n G + A)/A] \subseteq p^\omega(G/A) = (p^\omega G + A)/A$; equivalently, by using the modular law from [9], we have $G_n \cap p^n G \subseteq p^\omega G + A$. Furthermore, since $(G[p] + A)/A \subseteq (G/A)[p]$, it is a routine technical exercise to obtain that $G[p] = \cup_{n<\omega} G_n[p]$.

Next, we select a family of groups $(C_n)_{n<\omega}$ so that $C_n \subseteq C_{n+1} \leq G[p]$, so that $C_n \cap A = 0$ and so that $(C_n \oplus A)/A = (G_n/A) \cap [(G[p] + A)/A]$. Utilizing the classical modular law from [8], the last equality is equivalent to $C_n \oplus A = G_n[p] + A$ where $C_n \leq G_n[p]$.

We claim that $G[p] = \cup_{n<\omega}(C_n \oplus A_n)$. In order to verify that, letting $g \in G[p]$. Hence $g + A \in (G_k/A) \cap [(G[p] + A)/A]$ for some $k \in \mathbb{N}$. It is obvious then that $g + A \subseteq C_m \oplus A$, whence $g \in C_m \oplus A$. Finally, $g \in C_t \oplus A_t$ for some $t \geq 1$, so the claim sustained.

Now, we choose an ascending chain $(P_n)_{n<\omega}$ of subgroups of $G[p]$ such that, for every $n \geq 1$, $P_n \subseteq C_n$ with $\cup_{n<\omega} P_n = \cup_{n<\omega} C_n$ and such that $(P_n \oplus A_n) \cap p^n G \subseteq p^\omega G + A_n$. The choice is possible because of the inclusions $(C_n \oplus A_n) \cap p^n G \subseteq G_n \cap p^n G \subseteq p^\omega G + A$.

It is straightforward that $G[p] = \cup_{n<\omega}(P_n \oplus A_n)$ where $(P_n \oplus A_n)_{n<\omega}$ forms an increasing tower of subgroups. What suffices to argue is that $(P_n \oplus A_n) \cap p^n G \subseteq p^\omega G$. In fact, $(P_n \oplus A_n) \cap p^n G \subseteq p^\omega G + A_n$, hence, with the modular law from [9] at hand, we derive $(P_n \oplus A_n) \cap p^n G \subseteq (p^\omega G + A_n) \cap p^n G = p^\omega G + (p^n G \cap A_n) = p^\omega G$. As a final step, we apply our criterion to complete the proof. *QED*

As a direct consequence, we yield the following affirmation which was proved in ([2], p. 267, Theorem) by the usage of another technique.

Corollary ([2]). *Let G be a group with a balanced (= nice and isotype) subgroup A such that G/A is a Σ-group. Then G is a Σ-group if and only if A is a Σ-group.*

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