Vladimír Baláž; Oto Strauch; Tibor Šalát

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Remarks on several types of convergence of bounded sequences

V. Baláž, O. Strauch, and *T. Šalát*

Abstract. In this paper we analyze relations among several types of convergences of bounded sequences, in particulars among statistical convergence, $\mathcal{I}_u$-convergence, φ-convergence, almost convergence, strong p-Cesàro convergence and uniformly strong p-Cesàro convergence.

1. Introduction

Generalized approach to convergence was presented in [B, p.99] by means of the notion of a filter $\mathcal{F}$ of subsets of positive integer numbers $\mathbb{N}$. The same approach we can obtain by means of a dual notion of filter, what is an ideal $\mathcal{I}$ (i.e. *ideal* is an additive and hereditary class of sets). A sequence of real numbers $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is said to be $\mathcal{I}$-*convergent* to L provided that for every $\varepsilon > 0$ the set A_ε belongs to $\mathcal{I}$, where $A_\varepsilon = \{n \in \mathbb{N}; |x_n - L| \geq \varepsilon\}$. We write $\mathcal{I} - \lim x_n = L$ (see [KŠW]). The notion of $\mathcal{I}$-convergence is in certain sense equivalent to the notion of μ-statistical convergence (see [C1]).

The aim of this paper is investigate relations among different types of convergences of real sequences. There are convergences defined by means of densities on the set $\mathbb{N}$ (i.e. *density* is a finitely additive measure) on one side. Let ν be a density then $\mathcal{I}_\nu = \{A \subset \mathbb{N}; \nu(A) = 0\}$ is an associated ideal to the density ν which generates the $\mathcal{I}_\nu$-convergence. In this paper we will use the asymptotic density and the uniform density, sometime called Banach density. On the other side, there are convergences that cannot be defined by means of any ideal $\mathcal{I}$. We take into consideration almost convergence, strong p-Cesàro convergence, uniformly strong p-Cesàro convergence and φ-convergence (for all definitions see Section 2).

The paper consists of four sections with the new results in Sections 3, where for each studied type of convergence we assign a set of all in this way convergent

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sequences, these sets create linear subspaces of the linear space of all bounded sequences. We will study their mutual position and describe their structure from the point of view of topological properties, for instance separability and porosity. In the Section 4 we give an additional information for uniform density and for not so much known φ-convergence.

2. Definitions and basic properties

We recall some known notions. Denote by m the linear normed space of all bounded sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$ of real numbers with the supremum norm $||\mathbf{x}|| = \sup_{n\in\mathbb{N}} |x_n|$. Let $A \subset \mathbb{N}$. If $m, n \in \mathbb{N}$ by $A(m, n)$ we denote the cardinality of the set $A \cap [m, n]$.

(i) If there exists the limit $\lim_{n\to\infty} \frac{A(1,n)}{n} = d(A)$, then $d(A)$ is said to be *the asymptotic density* of A.

(ii) The following limits exist

$\lim_{n\to\infty} \frac{\min_{m=0,1,\dots} A(m+1,m+n)}{n} = \underline{u}(A)$,

$\lim_{n\to\infty} \frac{\max_{m=0,1,\dots} A(m+1,m+n)}{n} = \overline{u}(A)$,

and they are called the *lower* and *upper uniform density* of the set A, respectively. If $\underline{u}(A) = \overline{u}(A) = u(A)$ then $u(A)$ is called the *uniform density* of A, (see [BF, BF1]). It is clear that if there exists $u(A)$, then also there exists $d(A)$ and $u(A) = d(A)$. The converse is not true (forinstance Example 2 in the Section 3).

(iii) Put $\mathcal{I} = \mathcal{I}_d = \{A \subset \mathbb{N}; d(A) = 0\}$, then $\mathcal{I}_d$-convergence coincides with the *statistical convergence*, which was introduced by H. Fast (1951)[F] (see also [C, Fr, P, S, Š]). If $\mathbf{x} = (x_n)_{n=1}^{\infty}$ converges statistically to L then we write $\lim -\text{stat }x_n = L$ and $\lim -\text{stat }x_n = \mathcal{I}_d - \lim x_n$. By m_0 we denote the set of all bounded statistical convergent sequences (see [Š]).

(iv) In the case if $\mathcal{I} = \mathcal{I}_u = \{A \subset \mathbb{N}; u(A) = 0\}$ we obtain $\mathcal{I}_u$-*convergence.* If $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is $\mathcal{I}_u$-convergent to L we write $\mathcal{I}_u - \lim x_n = L$. By m_1 we denote the set of all bounded $\mathcal{I}_u$-convergent sequences.

Further we recall the notions of strong p-Cesàro convergence, uniformly strong p-Cesàro convergence that is generalization of notion of strong almost convergence (see [M]) and almost convergence.

(v) We say that a bounded sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is *almost convergent* or *fast convergent* to a number L if $\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} x_{n+i} = L$, uniformly in n what is equivalent to condition that every Banach limit [1] of $\mathbf{x}$ is equal to L (see [MO], [KN p.216], [P, p. 59-62]). By F we denote the set of all almost convergent sequences.

(vi) A sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is said to be *strong p-Cesàro convergent* $(0 < p < \infty)$ to a number L if $\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} |x_i - L|^p = 0$ (see [C]). A sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is said to be *uniformly strong p-Cesàro convergent* $(0 < p < \infty)$ to a number L if $\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} |x_{n+i} - L|^p = 0$ uniformly in n. This notion was introduced in [BŠ] and it is a generalization of a notion of strong almost convergence in [M]. As usual $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is Cesáro sumable if there exists $(C, 1) - \lim x_n = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = L$. By w_p resp. uw_p denote the set of all strong p-Cesàro convergent sequences,

[1] A Banach limit is a bounded linear functional on the space m of all bounded sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$ such that the sequence $x_n = 1$ has the Banach limit 1, and $\mathbf{x}$ and shifted $\mathbf{x}' = (x_{n+1})_{n=1}^{\infty}$ have the same Banach limit (if exists).

uniformly strong p-Cesàro convergent sequences, respectively. It is immediate that $uw_p \subset w_p$ $(0 < p < \infty)$ and Example 2 shows that the inclusion is strict. As usual by c resp. c_1 denote the set of all convergent sequences, Cesàro sumable, respectively.

Notion φ-convergence had been introduced by I.J. Schoenberg (1959)[S].

(vii) A sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is said to be φ-convergent to a number L if $\lim_{n\to\infty} \frac{1}{n} \sum_{d|n} \varphi(d) x_d = L$, where $\varphi(n)$ is Euler function i.e. $\varphi(n)$ is the number of elements from $\{1, 2, \ldots, n\}$ coprime to n and $\sum_{d|n}$ is sum over the positive divisors d of n. By c_φ we denote the set of all φ-convergent sequences.

The notion of porosity in a metric space is introduced in conformity with the definition of porosity in line (see [T, p. 183-190] and [Z]) as follows.

(viii) Let (X, d) be a metric space and $Y \subset X$, $x \in X$, $\delta > 0$, then symbol $\gamma(x, \delta, Y)$ denotes the supremum of all $t > 0$ for which there exists $y \in X$ such that $B(y, t) \subset B(x, \delta) \setminus Y$. Here $B(x, \delta)$ denotes a ball centered at $x \in X$ with the radius $\delta > 0$. If there exist no such $t > 0$, then $\gamma(x, \delta, Y) = 0$. The numbers

$$\underline{p}(x, Y) = \liminf_{\delta \to 0^+} \frac{\gamma(x, \delta, Y)}{\delta}, \quad \overline{p}(x, Y) = \limsup_{\delta \to 0^+} \frac{\gamma(x, \delta, Y)}{\delta},$$

are called the *lower* and *upper porosity* of the set Y at x. We say that Y is *porous* or *very porous* at x if $\overline{p}(x, Y) > 0$ or $\underline{p}(x, Y) > 0$, respectively. If for each $x \in X$ we have $\overline{p}(x, Y) > 0$ or $\underline{p}(x, Y) > 0$, then Y is said to be *porous* or *very porous* in X, respectively. Obviously every porous set in X is nowhere dense in X. If $\overline{p}(x, Y) \geq c > 0$ or $\underline{p}(x, Y) \geq c > 0$ then Y is called *c-porous* or *very c-porous* at x, respectively. If Y is c-porous or very c-porous at x for each $x \in X$, then Y is called c-porous or very c- porous in X, respectively. In the case, that the number $p(x, Y) = \lim_{\delta \to 0^+} \frac{\gamma(x, \delta, Y)}{\delta}$ exists, it is called the porosity of Y at x.

3. Results

In this section we analyse relations among types of convergence of bounded sequences defined above and describe the structure of the following spaces

c - the set of all convergent sequences,

c_φ - the set of all φ-convergent sequences,

m_1 - the set of all $\mathcal{I}_u$-convergent sequences,

uw_p - the set of all uniformly strong p-Cesàro convergent sequences $(0 < p < \infty)$,

w_p - the set of all strong p-Cesàro convergent sequences $(0 < p < \infty)$,

m_0 - the set of all statistical convergent sequences,

F - the set of all almost convergent sequences,

c_1 - the set of all Cesàro sumable sequences,

m - the set of all bounded sequences,

equipped by the sup-norm from point of view of topological properties as subspaces of all bounded sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$ of real numbers with the same norm.

I.J. Maddox (1974)[M1] has been shown that $m_0 = w_p$, $0 < p < \infty$ and $m_0 \subset c_1$ (see also [C, S]). In [BŠ] can be found that $m_1 \subset F$ and $m_1 = uw_p \subset m_0$ $(0 < p < \infty)$. The inclusion $c_\varphi \subset m_0$ was proved in [S].

In this paper we obtain the following relations:

$$\begin{gathered} c \subset m_1 = uw_p \subset m_0 = w_p \subset c_1 \subset m, (0 < p < \infty), \\ c \subset m_1 \subset F \subset c_1 \subset m, \\ c \subset c_\varphi \subset m_0 \subset m_1 \subset m. \end{gathered} \tag{1}$$

Following examples show, that all inclusions in (1) are strict.

Example 1 Let P be the set of all primes. Define $x_n = 1$ for $n \in P$ and $x_n = 0$ otherwise. For the reason that $u(P) = 0$ (see [BF1]), we have that $\mathbf{x} = (x_n)_{n=1}^\infty$ is $\mathcal{I}_u$-convergent to 0 but as we can see it is not convergent. Therefore $c \neq m_1$.

Example 2 It is easy to see that for the set $A = \cup_{k=1}^\infty([10^k + 1, 10^k + k] \cap \mathbb{N})$ we have $d(A) = 0$, $\underline{u}(A) = 0$, $\overline{u}(A) = 1$. Put $x_n = 1$ for $n \in A$ and $x_n = 0$ for $n \notin A$. Then $\mathcal{I}_d - \lim x_n = 0$ but $\mathbf{x} = (x_n)_{n=1}^\infty$ is not $\mathcal{I}_u$-convergent. Therefore $m_1 \neq m_0$.

Example 3 Let $\mathbf{x} = (x_n)_{n=1}^\infty$ be the sequence defined by $x_n = 1$ if n is even and $x_n = 0$ if n is odd. The sequence $\mathbf{x}$ is Cesàro sumable to $1/2$ but it is not statistically convergent. Therefore $m_0 \neq c_1$.

Example 3 simultaneously shows that $m_1 \neq F$ and also Example 2 shows that $F \neq c_1$. To show that $c \neq c_\varphi$ we use the following P. Erdös' solution (see [E]) of the problem 6090, AMM 1976, p. 385 proposed by T. Šalát and O. Strauch.

Example 4 Let $P = \{p_1 < p_2 < \cdots < p_k < \dots\}$ be the set of all primes. Put $A = \{p_1, p_1p_2, \dots, p_1p_2\cdots p_k, \dots\}$ and define $x_n = 1$ for $n \in A$ and $x_n = 0$ otherwise. Then the sequence $\mathbf{x} = (x_n)_{n=1}^\infty$ is φ-convergent to zero but it is not convergent.

Example 1 simultaneously shows that $c_\varphi \neq m_0$. To show that $\mathbf{x} = (x_n)_{n=1}^\infty$ is not φ-convergent we use [S, p.366, Th.2].

In [MO] is shown that almost convergence and statistical convergence are not compatible neither in the case of bounded sequences. Moreover Example 2 shows that $m_0 \backslash F \neq \emptyset$ and Example 3 simultaneously shows that $F \backslash m_0 \neq \emptyset$. On the basis of (1) the following question arise, what is mutual position between m_1 and c_φ or c_φ and F, respectively. First of all we show that $\mathcal{I}_u$-convergence and φ-convergence are not compatible. Example 1 shows that $m_1 \setminus c_\varphi \neq \emptyset$ and the following Example 5 shows that $c_\varphi \setminus m_1 \neq \emptyset$.

Example 5 Directly by E. Kováč [Ko, Th.6.5]: Clearly, there exists an increasing sequence $\mathbf{a} = (a_k)_{k=1}^\infty$ of positive integers such that greatest common divisor $(a_i, a_j) = 1$ for every $i \neq j$ and $\varphi(a_k)/a_k \to 0$, such a sequence $(a_k)_{k=1}^\infty$ can be construct by multiplication of sufficiently long interval of consecutive primes. To this sequence $(a_k)_{k=1}^\infty$ it can be construct an increasing sequence $(b_k)_{k=1}^\infty$ of positive integers such that $b_{k+1} > b2_k$ and $a_{k+1}|b_k + 1, a_{k+2}|b_k + 2, \dots, a_{k+k}|b_k + k$. This follows from Chinese remainder theorem. Put $A = \cup_{k=1}^\infty([b_k + 1, b_k + k] \cap \mathbb{N}$ and let $\mathbf{x} = (x_n)_{n=1}^\infty$ be the characteristic function of A thus $x_n = 1$ for $n \in A$ and $x_n = 0$ otherwise. Then $\mathbf{x} = (x_n)_{n=1}^\infty$ is φ-convergent to 0 but it is not $\mathcal{I}_u$-convergent since $\underline{u}(A) = 0$, $\overline{u}(A) = 1$.

Both φ-convergence and almost convergence are not compatible, as we can see Example 5 and Example 3 simultaneously show that $c_\varphi \setminus F \neq \emptyset$ and $F \setminus c_\varphi \neq \emptyset$, respectively.

It is well known that if E_0 is a closed linear subspace of a linear normed space E and $E_0 \neq E$, then E_0 is a nowhere dense set in E (see [G], [K, p.37, Ex.4]). This fact evokes the question about the porosity of E_0. The solution is given by the following theorem (see [KMŠS, Th.2.5]):

Theorem 1 *Suppose that E is a linear normed space and E_0 is its closed linear subspace, $E_0 \neq E$. Then E_0 is a very porous set in E, in more detail*

a) *If $x \in E \setminus E_0$, then $p(x, E_0) = 1$,*
b) *If $x \in E_0$, then $p(x, E_0) = 1/2$.*

In [Š] is proved that m_0 is a closed linear subspace of the space m, $m_0 \neq m$. That is why m_0 is a very porous set in the space m. According to (1) we have the following Lemma.

Lemma 1 *Each of sets m_1, uw_p and w_p $(0 < p < \infty)$ is a very porous set in m.*

In [L] is proved that F is a closed linear subspace of the space m, $F \neq m$. On that account F is a very porous set in the space m. Also from this fact we get that each of sets m_1 and uw_p $(0 < p < \infty)$ is a very porous set in m.

For the proof of the next Theorem 2 we first prove the following lemma.

Lemma 2 *The set m_1 is closed in m.*

Proof Let $\mathbf{x}^{(k)} = (x_j^{(k)})_{j=1}^\infty$ $(k = 1, 2, \dots)$ belong to m_1, $\mathbf{x}^{(k)} \to \mathbf{x}$, $\mathbf{x} = (x_j)_{j=1}^\infty$ in m i.e. $||\mathbf{x}^{(k)} - \mathbf{x}|| \to 0$ by $k \to \infty$. Since $m_1 \subset F$ and F is a closed set in m we get that $\mathbf{x} \in F$. Hence $\mathbf{x}^{(k)} \in m_1 \subset F$ we have $\mathbf{x}^{(k)}$ is almost convergent to some number L_k for all $k = 1, 2, \dots$. We shall prove that the sequence $(L_k)_{k=1}^\infty$ is convergent to some number L and the sequence $\mathbf{x} = (x_j)_{j=1}^\infty$ is $\mathcal{I}_u$-convergent to L. A simple estimation gives

$$\begin{aligned} |L_k - L_r| \leq & \left| \frac{x_{n+1}^{(k)} + x_{n+2}^{(k)} + \dots + x_{n+p}^{(k)}}{p} - L_k \right| + \\ & + \left| \frac{x_{n+1}^{(k)} + x_{n+2}^{(k)} + \dots + x_{n+p}^{(k)}}{p} - \frac{x_{n+1}^{(r)} + x_{n+2}^{(r)} + \dots + x_{n+p}^{(r)}}{p} \right| + \\ & + \left| \frac{x_{n+1}^{(r)} + x_{n+2}^{(r)} + \dots + x_{n+p}^{(r)}}{p} - L_r \right| . \end{aligned} \tag{2}$$

Let $\varepsilon > 0$. Since $\mathbf{x}^{(k)} \to \mathbf{x}$ in m, there exists an n_0 such that for $k, r > n_0$ we have $||\mathbf{x}^{(k)} - \mathbf{x}^{(r)}|| \leq \frac{\varepsilon}{3}$. Let us choose fixed k, r such that $k, r > n_0$. Since $\mathbf{x}^{(k)} = (x_j^{(k)})_{j=1}^\infty$ is almost convergent to L_k and $\mathbf{x}^{(r)} = (x_j^{(r)})_{j=1}^\infty$ is almost convergent to L_r, there exists an p_0 such that the first and third summand in (2) is less than $\frac{\varepsilon}{3}$

for $p > p_0$, $n = 1, 2, \dots$, respectively. The second summand is also less than $\frac{\varepsilon}{3}$, what can be shown as follows:

$$\left| \frac{x_{n+1}^{(k)} + x_{n+2}^{(k)} + \cdots + x_{n+p}^{(k)}}{p} - \frac{x_{n+1}^{(r)} + x_{n+2}^{(r)} + \cdots + x_{n+p}^{(r)}}{p} \right| \leq$$

$$\leq \frac{1}{p} \sum_{j=1}^{p} |x_{n+j}^{(k)} - x_{n+j}^{(r)}| \leq \frac{1}{p} \sum_{j=1}^{p} ||\mathbf{x}^{(k)} - \mathbf{x}^{(r)}|| = ||\mathbf{x}^{(k)} - \mathbf{x}^{(r)}|| < \frac{\varepsilon}{3}.$$

Consequently for $k, r > n_0$ we obtain $|L_k - L_r| < \varepsilon$. Since $(L_k)_{k=1}^{\infty}$ is a Cauchy sequence, then there exists an L such that $L = \lim_{k\to\infty} L_k$. Further, let $\eta > 0$. Put $A_\eta = \{k \in \mathbb{N}; |x_k - L| \geq \eta\}$. Since $\mathbf{x}^{(k)} \to \mathbf{x}$ in m, there exists an $r \in \mathbb{N}$ such that $||\mathbf{x}^{(r)} - \mathbf{x}|| < \frac{\eta}{3}$ and $||L_r - L|| < \frac{\eta}{3}$ simultaneously. Put $B_\eta = \left\{k \in \mathbb{N}; |x_k^{(r)} - L_r| \geq \frac{\eta}{3}\right\}$. For an arbitrary $k \in \mathbb{N}$ we have

$$|x_k - L| \leq |x_k - x_k^{(r)}| + |x_k^{(r)} - L_r| + |L_r - L| < \frac{2\eta}{3} + |x_k^{(r)} - L_r|. \qquad (3)$$

Hence if $k \in A_\eta$, according to (3) we have $k \in B_\eta$, thus $A_\eta \subset B_\eta$. Using the fact that $u(B_\eta) = 0$ we get $u(A_\eta) = 0$ and therefore $\mathcal{I}_u - \lim x_k = L$ thus $\mathbf{x} = (x_k)_{k=1}^{\infty} \in m_1$. □

Theorem 2 *The set m_1 is a perfect, very porous and not separable set in m_0.*

Proof The facts that m_0 is a closed set in m and Lemma 2 imply, that m_1 is a closed set in m_0. Since m_1 is a linear space and $m_1 \neq m_0$ (see Example 2) by Theorem 1 we obtain that m_1 is a very porous set in m_0. Further, if $\mathbf{x} = (x_j)_{j=1}^{\infty} \in m_1$ then for every $\eta > 0$ a sequence $x_1 + \eta, x_2, x_3, \dots, x_j, \dots$ also belongs to m_1. So we get that m_1 is dense in itself. To prove that m_1 is not separable it is sufficient to construct uncountable many sequences belong to m_1 having the distance 1 from each other. Let $B \subset \mathbb{N}$ be an infinite set such that $u(B) = 0$ (see Example 1). Put M the set of those sequences $\mathbf{x} = (x_j)_{j=1}^{\infty}$, $\mathbf{x} \in m_1$ for which x_j is equal 0 or 1 if $j \in B$ and $x_j = 0$ otherwise. Evidently card $M = c$ - power of continuum and for all $\mathbf{x}, \mathbf{y} \in M$ such that $\mathbf{x} \neq \mathbf{y}$ we have $||\mathbf{x} - \mathbf{y}|| = 1$. □

Remark Since $m_1 = uw_p$, $(0 < p < \infty)$ we have that uw_p is a perfect, very porous and not separable set in m_0.

It is easily to verify that c_1 is a closed linear subspace of the space m and $m_0 \neq c_1$ (see Example 3). So we get the following proposition.

Theorem 3 *The set m_0 is a perfect, very porous and not separable set in c_1.*

Proof The proof is analogous as the proof of the previous theorem. Non separability of m_0 immediately is implied by the non separability of m_1 . □

Remark Again since $m_0 = w_p$, $(0 < p < \infty)$ we have that w_p is a perfect, very porous and not separable set in c_1.

As we mentioned (see [L]), F is a closed and non separable set in m, moreover $F \subset c_1$, $F \neq c_1$. So we have a similar proposition as Theorem 3.

Theorem 4 *The set F is a perfect, very porous and not separable set in c_1.*

The following theorem investigates topological properties of subspace c_φ in m_0.

Theorem 5 *The set c_φ is a perfect, very porous and not separable set in m_0.*

Proof First of all we shall prove that the set c_φ is a closed set in m. Let $\mathbf{x}^{(k)} = (x_j^{(k)})_{j=1}^\infty$ $(k = 1, 2, \dots)$ belong to c_φ, $\mathbf{x}^{(k)} \to \mathbf{x}$, $\mathbf{x} = (x_j)_{j=1}^\infty$ in m. Since $\mathbf{x}^{(k)} \in c_\varphi$, we have $\mathbf{x}^{(k)}$ is φ-convergent to some number L_k , for all $k = 1, 2, \dots$. To prove that $\mathbf{x} \in c_\varphi$ it is sufficient to show that the sequence $(L_k)_{k=1}^\infty$ is convergent to some number L and the sequence $\mathbf{x} = (x_j)_{j=1}^\infty$ is φ-convergent to L. For $n, k, j \in \mathbb{N}$ we put

$$S_n(\mathbf{x}^{(k)}, L_k) = \left| \frac{\sum_{d|n} \varphi(d) x_d^{(k)}}{n} - L_k \right|,$$

$$S_n(\mathbf{x}^{(k)}, \mathbf{x}^{(j)}) = \left| \frac{\sum_{d|n} \varphi(d) x_d^{(k)}}{n} - \frac{\sum_{d|n} \varphi(d) x_d^{(j)}}{n} \right|.$$

For $n = 1, 2, \dots$ a simple estimation gives

$$S_n(\mathbf{x}^{(k)}, \mathbf{x}^{(j)}) \le \frac{\sum_{d|n} \varphi(d) |x_d^{(k)} - x_d^{(j)}|}{n} \le ||\mathbf{x}^{(k)} - \mathbf{x}^{(j)}|| \frac{\sum_{d|n} \varphi(d)}{n} = ||\mathbf{x}^{(k)} - \mathbf{x}^{(j)}|| \tag{4}$$

Since $\mathbf{x}^{(k)} = (x_j^{(k)})_{j=1}^\infty$ is Cauchy sequence then for $\varepsilon > 0$ there exists a k_0 such that for arbitrary $k, j > k_0$ we have $S_n(\mathbf{x}^{(k)}, \mathbf{x}^{(j)}) \le \frac{\varepsilon}{3}$ for $n = 1, 2, \dots$. Let us choose fixed k, j such that $k, j > k_0$. Since $\mathbf{x}^{(k)} = (x_j^{(k)})_{j=1}^\infty$ is φ-convergent to L_k and $\mathbf{x}^{(j)} = (x_i^{(j)})_{i=1}^\infty$ is φ-convergent to L_j, there exists an n_0 such that for each $n > n_0$ we have $S_n(\mathbf{x}^{(k)}, L_k) < \frac{\varepsilon}{3}$, $S_n(\mathbf{x}^{(j)}, L_j) < \frac{\varepsilon}{3}$ and the simple estimation yields

$$|L_k - L_j| \le S_n(\mathbf{x}^{(k)}, L_k) + S_n(\mathbf{x}^{(k)}, \mathbf{x}^{(j)}) + S_n(\mathbf{x}^{(j)}, L_j) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The fact that $(L_k)_{k=1}^\infty$ is a Cauchy sequence implies the existence of a number L such that $L = \lim_{k\to\infty} L_k$. Further, let $\eta > 0$. Since $\mathbf{x}^{(k)} \to \mathbf{x}$ in m, there exists an $r \in \mathbb{N}$ such that $||\mathbf{x}^{(r)} - \mathbf{x}|| \le \frac{\eta}{3}$ and $|L_r - L| < \frac{\eta}{3}$ simultaneously. Let r be a fixed natural number. Because $\mathbf{x}^{(r)} = (x_j^{(r)})_{j=1}^\infty$ is φ-convergent to L_r, so there exists an n_0 such that for each $n > n_0$ we have $S_n(\mathbf{x}^{(r)}, L_r) < \frac{\eta}{3}$. From this and applying (4) we have

$$S_n(\mathbf{x}, L) \le S_n(\mathbf{x}, \mathbf{x}^{(r)}) + S_n(\mathbf{x}^{(r)}, L_r) + |L_r - L| < \eta$$

for $n > n_0$ and thus the sequence $\mathbf{x} = (x_j)_{j=1}^\infty$ is φ-convergent to L. The fact that c_φ and m_0 are closed sets in m implies that c_φ is a closed set in m_0. Since c_φ is a linear space and $c_\varphi \ne m_0$ (see Example 1) by Theorem 1 we obtain that c_φ is a very porous set in m_0.

Further, if $\mathbf{x} = (x_j)_{j=1}^\infty \in c_\varphi$ then for every $t > 0$ a sequence $x_1 + t, x_2, x_3, \dots$ $\dots, x_j, \dots$ also belongs to c_φ. So we get that c_φ is dense in itself.

Again to prove that c_φ is not separable it is sufficient to construct uncountable many sequences belong to c_φ having the distance 1 from each other. Let $A = \{p_1, p_1p_2, \dots, p_1p_2 \dots p_k, \dots\}$ be a set defined in Example 4. Then the sequence $\mathbf{x} = (x_n)_{j=n}^\infty$ defined as $x_n = 1$ for $n \in A$ and $x_n = 0$ otherwise is φ-convergent to 0. Consider K the set of those sequences $\mathbf{y} = (y_k)_{k=1}^\infty$, for which $y_k = 0$ or 1 if $k \in A$ and $y_k = 0$ otherwise. Evidently card $K = c$. For each $\mathbf{y} \in K$ we have

$\sum_{d|n}\varphi(d)y_d \le \sum_{d|n}\varphi(d)x_d$ for $n = 1,2,\dots$. Thus $\mathbf{y} = (y_k)_{k=1}^{\infty}$ is φ-convergent to 0 and so $\mathbf{y} \in c_{\varphi}$. Moreover if $\mathbf{z},\mathbf{y} \in K$ such that $\mathbf{x} \neq \mathbf{y}$ we have $||\mathbf{z}-\mathbf{y}|| = 1$. □

4. Concluding remarks

This section contains a brief addition of results and problems concerning the uniform density and φ-convergence.

• The notion of uniform density $u(A)$ defined in Section 2(ii), can be found in different pats of number theory. It is also known as Banach density, since $u(A) = \lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m} c_A(n+i)$, uniformly in n, see Section 2(v). Here $c_A(x)$ is the characteristic function of A.

In [KN, p.40, Def. 5.1] was studied the concept of well-distributed sequence as follows: A sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ in $[0,1)$, is said to be *well-distributed sequence* if for every interval $I \subset [0,1)$ we have $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} c_I(x_{n+i}) = |I|$ uniformly in n. Here $|I|$ is the length of I. In [KN, p. 42, Ex. 5.2] is shown that the sequence $\mathbf{x} = (\{n\theta\})_{n=1}^{\infty}$ with θ irrational is well distributed. Here $\{y\}$ is fractional part of y. Thus for every interval $I \subset [0,1)$, $|I| > 0$, the set $A = \{n \in \mathbb{N}; x_n \in I\}$ has uniform density $u(A) = |I|$.

Another examples of sets of positive integers having uniform density are Hartman sequences. For general definition see [KN, p. 295, Def. 5.6] and [W], but we give here only the following equivalent property (see [KN, p.296; Ex. 5.11]): An increasing integer sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is *Hartman sequence* if and only if $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i t x_n} = 0$ holds for all $t \in [0,1]$. For instance (see [Wie]) $\mathbf{x} = ([n\log n])_{n=1}^{\infty}$, $\mathbf{x} = ([n^{3/2}])_{n=1}^{\infty}$, $\mathbf{x} = ([n2/\log n])_{n=2}^{\infty}$, and *lacunary sequence* $\mathbf{x} = (x_n)_{n=1}^{\infty}$ (i.e. $x_{n+1}/x_n \ge c > 1$) are Hartman sequences, where $[y]$ means the integer part of y. Of course the uniform density of such sequences is 0. *Beatty sequence* $\mathbf{x} = ([n\beta+\gamma])_{n=1}^{\infty}$, where $\beta > 1$ is irrational and γ is appropriate is also Hartman.

• As we mentioned in Section 2(vii) φ-convergence was introduced by I.J. Schoenberg (1959)[S]. Denote the φ-transformation of $\mathbf{x} = (x_n)_{n=1}^{\infty}$ as $\mathbf{y} = (y_n)_{n=1}^{\infty}$ where $y_n = \frac{1}{n}\sum_{d|n}\varphi(d)x_d$ and $x_n \to L$ denotes the classical limit. Schoenberg's main results are:

(i) If $y_n \to L$ then $x_{n_k} \to L$ for every subsequence $(x_{n_k})_{k=1}^{\infty}$, for which $\varphi(n_k)/n_k \ge \delta > 0$.

(ii) If $y_n \to L$ then x_n is statistical convergent to L.

(iii) $x_n = \frac{1}{\varphi(n)}\sum_{d|n}\mu(\frac{n}{d})dy_d$, that is a result of Möbius' inversion formula. [2]

These results lead to the following open problems:

- Test some number-theoretic statistical convergent sequences whether they are also φ-convergent, e.g. the sequence $\frac{\omega(n)}{\log\log n}$ which is statistical convergent to 1, see also [SP, p. 2–35, 2.3.23].
- Input any convergent sequence $y_n \to L$ into the inversion formula (iii), then the result is a sequence x_n which is statistical convergent and φ-convergent to L, simultaneously. Find $y_n \to L$ such that x_n is no $\mathcal{I}_u$-converges to L (different from Example 5).

[2] The Möbius function is defined by $\mu(n) = (-1)^{\omega(n)}$ for square-free n and $\mu(n) = 0$ others. Here $\omega(n)$ is the number of different primes dividing n.

5. Obituary notice

Professor Tibor Šalát died on 2005 May 14th.

References

[BS] BALÁŽ, V. – ŠALÁT, T., *Uniform density u and corresponding I_u-convergence,* Math. Communication **11** (2006), 1–7.

[BF] BROWN, T.C. – FREEDMAN, A.R., *Arithmetic progressions in lacunary sets,* Mountain J. Math. **17** (1987), 587–596.

[BF1] BROWN, T.C. – FREEDMAN, A.R., *The uniform density of sets of integers and Fermat's Last Theorem,* C. R. Math. Ref. Acad. Sci. Canada **XII** (1990), 1–6.

[B] BOURBAKI, N., *Éléments De Mathématique Topologie Générale Livre III,* Russian translation: Obščaja topologija Osnovnye struktury, Nauka, Moskva, (1968).

[C] CONNOR, J.S., *The statistical and strong p-Cesàro convergence of sequences,* Analysis **8** (1988), 47–63.

[C1] CONNOR, J.S., *Two valued measures and summability,* Analysis **10** (1990), 1–6.

[E] ERDÖS, P., *Solution of advanced problems: φ-convergence,* Amer. Math. Monthly **85** (1978), 122-123.

[F] FAST, H. *Sur la convergence statistique,* Coll. Math. **2** (1951), 241–244.

[Fr] FRIDY, J.A., *On statistical convergence,* Analysis **5** (1985), 301-313.

[G] GOFFMAN, C., *Functions of finite Baire type,* Amer. Math. Monthly **67** (1960), 164-165.

[K] KOLODZIEJ, W., *Selected Parts of Mathematical Analysis,* (Polish), PWN, Warszawa, (1970).

[KŠW] KOSTYRKO, P. – ŠALÁT, T. – WILCZINSKI, W., *$\mathcal{I}$-convergence,* Real. Anal. Exchange **26** (2000–2001), 669-686.

[KMŠS] KOSTYRKO, P. – MÁČAJ, M. – ŠALÁT, T. – SLEZIAK, M., *$\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points,* Math. Slovaca **55** (2005), no. 4,, 443-464.

[Ko] KOVÁČ, E., *On φ-convergence and φ-density,* Math. Slovaca **55** (2005), no. 3, 139-150.

[KN] KUIPERS, L. – NIEDERREITER, H., *Uniform Distribution of Sequences,* John Wiley & Sons, New York, (1974), reprint edition, Dover Publications, Inc., Mineola, New York, (2006).

[L] LORENTZ, G.G., *A contribution to theory of divergent sequences,* Acta Math. **80** (1948), 167-190.

[M] MADDOX, I.J., *A new type of convergence,* Math. Proc. Cambridge Phil. Soc. **83** (1978), 61-64.

[M1] MADDOX, I.J., *Steinhaus type theorems for sumability matrices,* Proc. Amer. Math Soc. **45** (1974), 209-213.

[MO] MILLER, H.I. – ORHAN, C., *On almost convergent and statistically convergent subsequences,* Acta Math. Hung. **43** (2001), 135-151.

[P] PETERSEN, G., *Regular Matrix Transformations,* Mc-Graw Hill Publ. Comp., London-New York-Toronto-Sydney, (1966).

[S] SCHOENBERG, I.J., *The integrability of certain functions and related sumability methods,* Amer. Math. Monthly **66** (1959), 361–375.

[SP] STRAUCH, O. - PORUBSKÝ, Š., *Distribution of Sequences: A Sampler,* Schriftenreihe der Slowakischen Akademie der Wissenschaften, Band 1, Peter Lang, Frankfurt am Main, (2005).

[Š] ŠALÁT, T., *On statistically convergent sequences of real numbers,* Math. Slovaca **30** (1980), 139-150.

[T] THOMSON, B.S., *Real Functions,* Springer-Verlag, Berlin-Göttingen-Heidelberg-New York-Tokyo, (1985).

[Z] ZAJÍČEK, L., *Porosity and σ-porosity,* Real. Anal. Exchange **13** (1987–88), 314-350.

[Wie] WIERDL, M., *Almost everywhere convergence and recurrence along subsequences in ergodic theory,* Ph.D. Thesis, The Ohio State University

[W] WINKLER, R., *Hartman sets, functions and sequences - a survey,* Advanced Studies in Pure Mathematics **43** (2006), 1–27.

Author(s) Address(es):

Vladimír Baláž Department of Mathematics of the Chemical Technology Faculty, Ralinskeho 9, SK-812 37 Bratislava, Slovak republic

E-mail address: `vladimir.balazstuba.sk`

Oto Strauch Mathematical Institute of the Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava, Slovak republic

E-mail address: `strauchmat.savba.sk`