Mathematica Slovaca

Joseph M. Szucs

The order of a zero of a Wronskian and the theory of linear dependence

Mathematica Slovaca, Vol. 57 (2007), No. 6, [515]--522

Persistent URL: `http://dml.cz/dmlcz/136975`

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0043-2
Math. Slovaca **57** (2007), No. 6, 515–522

THE ORDER OF A ZERO OF A WRONSKIAN AND THE THEORY OF LINEAR DEPENDENCE

J. M. SZUCS

(*Communicated by Michal Fečkan*)

ABSTRACT. Every root of the top Wronskian of a Wronskian matrix whose rank at the root is equal to the number of columns, is of integer order even if the highest derivatives exist only at the root. If the rank of a Wronskian matrix is constant and smaller than the number of rows, then the number of independent linear relations between the functions in the first row is equal to the number of functions minus the rank. These results were proved under additional assumptions by Bôcher, Curtiss, and Moszner. Their proofs are simplified.

1. Introduction

Every derivative function is equal to a Wronskian of dimension at least two, since $f' = W(1, f)$. The converse is not true because while derivative functions have the Darboux property, some Wronskians of dimension at least two do not, as was shown in this journal by Banas and El-Sayed [1]. However, Wronskians share several properties with derivative functions. For example, we showed in this journal [13] that the sets on which Wronskians of dimension at least two can vanish identically without vanishing identically everywhere, coincide with the sets defined similarly for derivative functions rather than Wronskians. In this paper we prove an analog, for Wronskians, of the following property of roots of real functions: If the kth ($k > 0$) derivative of a real function f exists and is finite and not zero at a point x_0, then there is a (unique) nonnegative integer ρ such that $\lim\limits_{x \to x_0} [f(x) / (x - x_0)^{\rho}]$ exists, is finite and nonzero, that is, the order of the root x_0 of f is ρ. (Of course, the limit here is equal to $f^{(s)}(x_0)/s!$,

2000 Mathematics Subject Classification: Primary 26A06; Secondary 15A03.
Keywords: order of a root of a Wronskian, Wronskian matrix, linear independence of functions.

where $f^{(s)}$ is the lowest-order derivative, starting from zeroth order, that does not vanish at x_0. If $f(x_0) \neq 0$, then $\rho = 0$.)

A kth-order Wronskian matrix is one whose successive rows are n functions and their successive derivatives stopping at order k. As an analog of the above property of functions for Wronskians, we show that every zero of the top n-dimensional Wronskian (matrix) of a kth-order, $k \geq n$, Wronskian matrix of n functions is of integer order if the rank of the kth-order Wronskian matrix is n at the zero even if the kth derivatives exist only at the zero. If the functions involved are k times continuously differentiable in an entire neighborhood of the zero of the Wronskian, then this was proved by Bôcher [3, IX, p. 58]. The proof given in this paper relies on an identity of Christoffel [5, pp. 297 299] and on a result of elementary calculus [9, (1), p. 290].

An important consequence is that if the rank of a kth-order Wronskian matrix of n functions is equal to a constant $m\,(\leq k)$ on an interval, then the number of independent linear relations between the functions is $n - m$. Our proof unifies, simplifies, and generalizes those of Curtiss [7, Theorem X, p. 296] and Moszner [10, Théorème (T), p. 177]. Under the additional assumption of continuity of the kth derivatives of the functions involved, Curtiss proved this result for $k \geq n-1$ and Moszner for $k \leq n-1$. Another unified proof was given by the author in [12].

The kth order Wronskian matrix of $f_1, \dots, f_n$ is denoted by $M_k = M_k(f_1, \dots$ $\dots, f_n)$. This matrix has $k+1$ rows and the ith row is the row of $(i-1)$st derivatives: $f_1^{(i-1)}, \dots, f_n^{(i-1)}$. The Wronskian $W = W(f_1, \dots, f_n)$ of $f_1, \dots, f_n$ is $\det M_{n-1}(f_1, \dots, f_n)$. All functions considered are real-valued functions of a real variable, defined on a nondegenerate interval of the real line $\mathbb{R}$. However, the results extend to complex-valued functions (of a real variable).

2. Zeros of Wronskians

It follows easily from the local form of Taylor's theorem [9, (1), p. 290] that if $f^{(k)}(x_0) \neq 0$ for a function f and a positive integer k at a point $x_0 \in \mathbb{R}$, then there is a nonnegative integer ρ such that $\lim\limits_{x \to x_0} [f(x)/(x-x_0)^{\rho}] \neq 0, \pm\infty$ (and the limit exists), even if $f^{(k)}(x)$ exists only at $x = x_0$. Bôcher [3, IX, p. 58] proved that a ρ satisfying such a limit relation exists for $f = W(f_1, \dots, f_n)$ at any x_0 at which $M_k(f_1, \dots, f_n)$, $k \geq n$, has the full rank n, provided that $f_1^{(k)}, \dots, f_n^{(k)}$ exist and are continuous in a neighborhood of x_0. Theorem 3 below moves this result closer to the one from calculus by freeing it from the assumption of existence and continuity of $f_1^{(k)}, \dots, f_n^{(k)}$ on an entire neighborhood of x_0.

The proof of Theorem 3 relies on two lemmas. Lemma 1 is a generalization, to higher derivatives, of the following observation: If f is differentiable at $x_0 \in \mathbb{R}$ and $h(x) = (f(x) - f(x_0)) / (x - x_0)$ $(x \neq x_0)$, $h(x_0) = f'(x_0)$, then h is continuous at x_0. Lemma 1 could be stated in any elementary calculus text made honest. It is surprising that its proof is not entirely routine. The part of Lemma 2 about W is due to Christoffel [5, p. 297–299] and about M_k to Chaundy [4].

Lemma 1. *Let a function f be k, $k \geq 1$, times differentiable at a point x_0. The function h equal to $(f(x) - f(x_0)) / (x - x_0)$ at $x \neq x_0$ and to $f'(x_0)$ at x_0 is $k-1$ times* continuously *differentiable at x_0 and $(l+1) h^{(l)}(x_0) = f^{(l+1)}(x_0)$, $l = 0, \ldots, k-1$. This result extends to one-sided derivatives.*

Proof. We only prove the two-sided case, since the one-sided extension automatically follows from it. Without loss of generality we assume that $x_0 = f(x_0) = 0$. Then the equality $f(x) = xh(x)$ in a neighborhood U of 0 where f is $k-1$ times differentiable, implies that for $x \in U$, $x \neq 0$, $l = 0, 1, \ldots, k-1$ (with an arbitrary $h^{(-1)}$),

$$f^{(l)}(x) = lh^{(l-1)}(x) + xh^{(l)}(x). \tag{1}$$

If in particular, $l > 0$, then

$$\begin{aligned} h^{(l)}(x) &= \left(f^{(l)}(x) - lh^{(l-1)}(x)\right)/x \\ &= \left(f^{(l)}(x) - f^{(l)}(0)\right)/x - T_l(x), \end{aligned} \tag{2}$$

where

$$\begin{aligned} T_l(x)\, x^{l+1} &= x^l \left(lh^{(l-1)}(x) - f^{(l)}(0)\right) \\ &= lx^{l-1}\left(f^{(l-1)}(x) - (l-1)\, h^{(l-2)}(x)\right) - f^{(l)}(0)\, x^l, \end{aligned} \tag{3}$$

$$\begin{aligned} \left(T_l(x)\, x^{l+1}\right)' &= lx^{l-1} f^{(l)}(x) + l(l-1)\, x^{l-2} f^{(l-1)}(x) - l(l-1)\, x^{l-1} h^{(l-1)}(x) \\ &\quad - l(l-1)^2 x^{l-2} h^{(l-2)}(x) - lx^{l-1} f^{(l)}(0) \\ &= lx^{l-1}(f^{(l)}(x) - f^{(l)}(0)). \end{aligned} \tag{4}$$

(The second equalities in (3) and (4) follow from (1) with l replaced by $l-1$.)

We prove by induction on l that for $l = 0, \ldots, k-1$,

$$\lim_{x \to 0} h^{(l)}(x) = f^{(l+1)}(0) / (l+1). \tag{5}$$

For $l = 0$, (5) is just the observation before Lemma 1. Let $1 \leq l \leq k-1$. From (5) with l replaced by $l-1$ (the induction hypothesis) and from (3) we obtain that $\lim\limits_{x \to 0} [T_l(x)\, x^{l+1}] = 0$. Then by l'Hôpital's rule and by (4),

$\lim_{x\to 0} T_l(x) - \lim_{x\to 0}[T_l(x)x^{l+1}/x^{l+1}] \quad \lim_{x\ 0}\left[l\left(f^{(l)}(x) - f^{(l)}(0)\right) \quad ((l+1)x)\right]$ $lf^{(l+1)}(0)/(l+1)$. Therefore, (5) holds by (2).

Since the derivative of a continuous function cannot have removable discontinuitie , the conclu ion of the lemma follows by induction on l from (5) for $l - 0,\ldots\ k-1$.

Lemma 2. (Christoffel [5] and Chaundy [4]) *We have* $W(\varphi g_1,\ldots,\varphi g$ $\varphi^n W(g_1,\ldots,g_n)$ *and if* $g_1 - 1$ *everywhere, then* $W(\varphi g_1,\ldots,\varphi g$ $\cdot\ \varphi^n W(g_2',\ldots,g_n')$. *Let* $k \geq n$ *The n-rowed determinants of* $M_k(\varphi g_1,\ldots,\varphi g_r$ *are linear combinations (with variable coefficients) of those of* $M_k(g_1,\ldots,g_n)$. *If, in particular,* $g_1 - 1$ *everywhere, then they are linear combinations of the $(n-1)$-rowed determinant of* $M_{k\ 1}(g_2',\ldots,g_n')$.

Sketch of Proof. The first statements about W and M_k follow from the fact that every determinant is a linear function of each of its row vectors and from Leibniz' rule for the higher derivative of a product. The second state ments follow from the first ones by expanding $W(g_1,\ldots,g_n)$ and every n-rowed determinant of $M_k(g_1,\ldots,g_n)$ by their first column and observing that only the top entry in that column is different from zero if $g_1 \quad 1$ everywhere. □

Theorem 3. *Let $f_1,\ldots,f_n$ be $n-1$ times differentiable functions on a nondegenerate interval I. If at a point $x_0 \in I$ the matrix $M_k(f_1,\ldots f_n;x_0)$ exists for some $k \geq n$ and its rank is n then there is an integer ρ, $\rho\ \_\ 0$, such that the following limit exists and*

$$\lim_{x\ I,x\ x_0} \frac{W(f_1,\ldots,f_n)}{(x\quad x_0)^\rho} \neq 0, \pm\infty. \tag{6}$$

Proof. We denote Theorem 3 for given n, k, $n \leq k$, by $\mathbf{T}_{n,k}$. It is clear that $\mathbf{T}_{1,k}$ is equivalent to the calculus fact quoted at the beginning of this section. In the next paragraph we deduce $\mathbf{T}_{,k}$, $2 \leq n < k$, from $\mathbf{T}_{n-1,k\ 1}$ when $|f_1(x_0) + \cdots + |f_n(x_0)| > 0$. In the last paragraph we show that if $f_1(x_0) \quad \cdots \quad f_n\ x_0$ 0, then $\mathbf{T}_{n,n}$, $n \geq 2$, holds and $\mathbf{T}_{n\ k}$, $2 \leq n < k$, follows from $\mathbf{T}_{n\ k-1}$. Based on these results, the proof is completed by verifying the conjunction $\bigwedge_n \mathbf{T}_{r\ +s}$ using induction on $\ , s\quad 0,1,\ldots$

Let the hypothe es of $\mathbf{T}_{n,k}$, $k\ \_\ n \geq 2$, be satisfied and let $f_1(x_0)| + \ldots + |f_n(x_0)| > 0$. We a ume that $f_1(x_0) \neq 0$, since this i only a matter of notation. Then f_1 does not vani h on an entire nondegenerate interval J containing x_0. Let $g_i = f_i/f_1$ on J. It follows from Lemma 2 (take $\varphi - f_1$) that the rank of $M_{k\ 1}(g_2',\ldots,g_n';x_0)$ is $n \quad 1$. Therefore, $\lim_{x\ I,x\to x_0}(W(g_2',\ldots,g_n') \quad (x \quad x_0)$ $\neq 0,\pm\infty$ for some integer ρ, $\rho \geq 0$, if $\mathbf{T}_{n\ 1,k-1}$ hold . Then (6) holds (with the same ρ) because $W(f_1,\ldots,f_n) \quad f_1^n W(g_2',\ldots,g_n')$ by Lemma 2.

Let the hypotheses of $\mathbf{T}_{n,k}$, $k \geq n \geq 2$, be satisfied and let $f_1(x_0) = \cdots = f_n(x_0) = 0$. The n-rowed determinants of the matrix $M_k(f_1, \ldots, f_n; x_0)$ containing its first row are all zero. Consequently, the rank of $M_{k-1}(f_1', \ldots, f_n'; x_0)$ is n. We write each f_i by Lemma 1 as $f_i = h_i(x - x_0)$, where h_i is $k-1$ times differentiable on I. It follows from Lemma 1 that the n-rowed determinants of $M_{k-1}(h_1, \ldots, h_n; x_0)$ and the n-rowed determinants of $M_{k-1}(f_1', \ldots, f_n'; x_0)$ differ from each other only by nonzero multiplicative constants. Therefore, the rank of $M_{k-1}(h_1, \ldots, h_n; x_0)$ is n. On the other hand, from Lemma 2 we obtain that

$$W(f_1, \ldots, f_n) = (x - x_0)^n W(h_1, \ldots, h_n), \qquad x \in I. \tag{7}$$

If $k = n$, then this shows that (6) is true with $\rho = n$, since $\lim_{x \in I,\, x \to x_0} W(h_1, \ldots, h_n) = W(h_1, \ldots, h_n; x_0)$ by the continuity of $h_i^{(k-1)}$, $i = 1, \ldots, n$, at x_0 (see Lemma 1), and since $W(h_1, \ldots, h_n; x_0) \neq 0$ because the rank of $M_{n-1}(h_1, \ldots, h_n; x_0)$ is n. If $k > n$, then $\lim_{x \in I,\, x \to x_0} [W(h_1, \ldots, h_n)/(x - x_0)^\rho] \neq 0, \pm\infty$ for some nonnegative integer ρ if $\mathbf{T}_{n,k-1}$ holds. Then by (7), relation (6) holds with ρ replaced by $n + \rho$. □

3. Consequences

Corollary 4. *If the zeros of $W(f_1, \ldots, f_n)$ cluster at a point $x_0 \in I$, then the rank of $M_k(f_1, \ldots, f_n; x_0)$ is less than n for every k ($\geq n$) for which $M_k(f_1, \ldots, f_n; x_0)$ exists.*

Corollary 5. *If the zeroes of $W(f_1, \ldots, f_n)$ cluster at a point x_0 of an interval I, then $W(f_1, \ldots, f_n, g_1, \ldots, g_m; x_0) = 0$ provided that $f_1, \ldots, f_n, g_1, \ldots, g_m$ are $n + m - 1$ times differentiable at x_0 and $m \geq 1$.*

Corollary 6. (Curtiss [7]) *If $f_1, \ldots, f_n$ are k ($\geq n$) times differentiable and $W(f_1, \ldots, f_n) \equiv 0$ on a nondegenerate interval I, then* rank $M_k(f_1, \ldots, f_n) < n$ *throughout I.*

Corollary 7. (Curtiss [6]) *If $f_1, \ldots, f_n, f_{n+1}$ are n times differentiable functions on a nondegenerate interval I, then $W(f_1, \ldots, f_n; x) \equiv 0$, $x \in I$, implies $W(f_1, \ldots, f_n, f_{n+1}; x) \equiv 0$, $x \in I$.*

Proof of Corollaries 4--7. Corollary 4 is the contrapositive of Theorem 3 when the latter is taken with the weaker conclusion that the zeroes of $W(f_1, \ldots, f_n)$ do not cluster at x_0. Corollary 5 follows from Corollary 4 by expanding $W(f_1, \ldots, f_n, g_1, \ldots, g_m; x_0)$ by its first n columns. Corollary 6 is a special case of Corollary 4 and Corollary 7 is a special case of Corollary 5. □

Corollary 4 when $f_1^{(k)}, \dots, f_n^{(k)}$ exist and are continuous throughout I, was stated and proved by Curtiss [7, Th. IV, pp. 285 289]. Thereby he reproved a weaker version of Bôcher's result mentioned in our Introduction. Chaundy [4, (C), p. 6] assumed only the existence (without the continuity) of $f_1^{(k)}, \dots, f_n^{(k)}$ throughout I to prove the conclusion of Corollary 4. He also simplified Curtiss' proof. (Actually, Chaundy [4, (C), p. 6] states Corollary 4 in its general form as above, but he uses the existence of $f_1^{(k)}, \dots, f_n^{(k)}$ throughout I in his proof. He only works out his induction step from 1 to 2 and from 2 to 3 and seems to need a general lemma like Lemma 1 above to make his proof complete.) Both Curtiss [7, Th. VII, p. 293] and Chaundy [4, (A), p. 5] deduce Corollary 6 from their version of Corollary 4. For Chaundy, Corollary 6 is a trivial special case but for Curtiss, Corollary 6 represents extra work because he assumes the continuity of $f_1^{(k)}, \dots, f_n^{(k)}$ in his version of Corollary 4. Of course, Corollary 7 follows from Corollary 6 by determinant expansion. (The converse is a little more complicated.) Corollary 7 was first proved by Bôcher [2, Theorem VIII, p. 148] assuming continuous nth derivatives and by Curtiss [6] in the general case. Our proof of Theorem 3 is shorter than Chaundy's proof of his version of Corollary 4 and uses some of the ideas of that proof.

The author presented a proof of Corollary 7 in a preceding issue of this journal ([13]). That proof relies on properties of Wronskian matrices different from the one expressed in Theorem 3.

Corollary 8 below is due to Curtiss [7, Theorem X, p. 296] (for $k \geq n-1$) and Moszner [10, Théorème (T), p. 177] (for $k \leq n-1$) when $f_1^{(k)}, \dots, f_n^{(k)}$ are continuous throughout I. (Actually, in his announcement [6, Th. V, p. 484], Curtiss does not assume this continuity but in [7, Theorem X, p. 296] he does.) The unified proof of Corollary 8 below without assuming the continuity of $f_1^{(k)}, \dots, f_n^{(k)}$ is simpler than Curtiss' or Moszner's proof. In the proof of Corollary 8 we use a classical theorem of Peano (see Theorem 9), a proof of which is included here for completeness.

The author presented a proof of Corollary 8 in [12]. That proof is different from the one below and is based on properties of Wronskian matrices different from the one expressed in Theorem 3. The author also showed (see [12, Corollary 4]) that Corollary 7 can be deduced from Corollary 8.

COROLLARY 8. *Let $f_1, \dots, f_n$ be k times differentiable functions on a nondegenerate interval I and let $m = \operatorname{rank} M_k(f_1, \dots, f_n; x)$ be independent of $x \in I$ and not larger than k. The number of independent linear relations for $f_1, \dots, f_n$ on I is equal to $n - m$.*

Proof. If $m = 0$ or $m = n$, then Corollary 8 is obvious. Let $0 < m < n$ and let $x_0 \in I$. Since $\operatorname{rank} M_k = m$, there are m linearly independent columns in $M_k(x_0)$, say the first m columns. Then it follows from Theorem 3 that there is an open interval J containing x_0, such that $W(f_1, \ldots, f_m; x) \neq 0$ if $x \in J \cap I$ and $x \neq x_0$. However, $W(f_1, \ldots, f_m, f_i; x) = 0, x \in I, m < i \leq n$, because rank $M_k = m$. Therefore, it follows from Theorem 9 below that each f_i, $m < i \leq n$, is a linear combination of $f_1, \ldots, f_m$ on the left and right components of $(J \cap I) \setminus \{x_0\}$ separately, and then separately on their closures by continuity. Since rank $M_k = m$, we obtain that the number of independent linear relations between $f_1, \ldots, f_n$ on the closure of each of these components is $n - m$.

Since $x_0 \in I$ was arbitrary, a Borel covering argument shows the following: I is the union of a finite or infinite chain $\mathscr{C}$ of adjacent nondegenerate subintervals on each of which the number of independent linear relations between $f_1, \ldots, f_n$ is $n-m$. For every $J \in \mathscr{C}$, let V_J be the vector space of constant vectors $\langle c_1, \ldots, c_n \rangle$ such that $c_1 f_1(x) + \cdots + c_n f_n(x) = 0$, $x \in J$. If $\langle c_1, \ldots, c_n \rangle \in V_J$ and a is an endpoint of J belonging to J, then k-fold one-sided differentiation shows that $c_1 f_1^{(s)}(a) + \cdots + c_n f_n^{(s)}(a) = 0, s = 0, \ldots, k$. Therefore, $V_J \subseteq \mathscr{M}_k^{\perp}(f_1, \ldots, f_n; a)$, where $\mathscr{M}_k^{\perp}(f_1, \ldots, f_n; a)$ denotes the orthogonal complement, in $\mathbb{R}^n$, of the vector subspace $\mathscr{M}_k(f_1, \ldots, f_n; a)$ spanned by the rows of $M_k(f_1, \ldots, f_n; a)$. Then $V_J = \mathscr{M}_k^{\perp}(f_1, \ldots, f_n; a)$, since rank $M_k = m$ and since by the paragraph above, $\dim V_J = n - m$. Similarly, $V_K = \mathscr{M}_k^{\perp}(f_1, \ldots, f_n; a)$ for the subinterval $K \in \mathscr{C}$ on the other side of a, if such a K exists. Consequently, V_J and V_K are the same for adjacent subintervals $J \in \mathscr{C}$ and $K \in \mathscr{C}$, and thus V_J is independent of $J \in \mathscr{C}$. This means that the number of independent linear relations for $f_1, \ldots, f_n$ on the entire interval I is $n - m$. □

Theorem 9. (Peano [11]) *If $W(f_1, \ldots, f_{n-1}; x) \neq 0$, $x \in I$, and $W\big(f_1, \ldots, f_n; x\big) = 0$, $x \in I$, on a nondegenerate interval I, then f_n is a unique linear combination of $f_1, \ldots, f_{n-1}$.*

Proof after Frobenius. ([8, p. 238]) Since $W(f_1, \ldots, f_{n-1}; x) \neq 0$, $x \in I$, by Cramer's rule there are unique differentiable $c_1(x), \ldots, c_{n-1}(x)$ such that $f_n^{(s)}(x) = c_1(x) f_1^{(s)}(x) + \cdots + c_{n-1}(x) f_{n-1}^{(s)}(x)$, $x \in I$, $s = 0, \ldots, n-2$. Since it follows from the hypotheses that the last row of $W(f_1, \ldots, f_n; x)$ is a linear combination of its first $n-1$ rows, this equality holds for $s = n-1$, as well. Differentiation of the equalities for $s = 0, \ldots, n-2$ yields that $f_n^{(s+1)}(x) - c_1(x) f_1^{(s+1)}(x) + \cdots + c_{n-1}(x) f_{n-1}^{(s+1)}(x) + c_1'(x) f_1^{(s)}(x) + \cdots + c_{n-1}'(x) f_{n-1}^{(s)}(x)$, $x \in I$, $s = 0, \ldots, n-2$. Comparison of the two expressions thus obtained for $f_n^{(s+1)}(x)$, $s = 0, \ldots, n-2$, gives that $c_1'(x) f_1^{(s)}(x) + \cdots + c_{n-1}'(x) f_{n-1}^{(s)}(x) = 0$, $x \in I$, $s = 0, \ldots, n-2$. The determinant of this system of $n-1$ homogeneous linear equations for the $n-1$ unknowns $c_1'(x), \ldots, c_{n-1}'(x)$ is $W(f_1, \ldots, f_{n-1}; x)$ $\neq 0$. Therefore, $c_1'(x) = \cdots = c_{n-1}'(x) = 0$, $x \in I$: the c_i are constant and thus f_n is a linear combination of $f_1, \ldots, f_{n-1}$. Since $W(f_1, \ldots, f_{n-1}; x) \neq 0$, $x \in I$, this linear combination is unique. □

REFERENCES

[1] BANAS, J.—EL-SAYED, W. G.: *Darboux property of the Wronski determinant*, Math. Slovaca **45** (1995), 57–61.

[2] BÔCHER, M.: *Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence*, Trans. Amer. Math. Soc. **2** (1901), 139–149.

[3] BÔCHER, M.: *On Wronskians of functions of a real variable*, Bull. Amer. Math. Soc. **8** (1901), 53–63.

[4] CHAUNDY, T. W.: *The vanishing of the Wronskian*, J. London Math. Soc. **8** (1933), 4–9.

[5] CHRISTOFFEL, E. B.: *Ueber die lineare Abhängigkeit von Functionen einer einzigen Veränderlichen*, J. Reine Angew. Math. **55** (1858), 281–299.

[6] CURTISS, D. R.: *On certain properties of Wronskians and related matrices*, Bull. Amer. Math. Soc. **12** (1906), 482–485.

[7] CURTISS, D. R.: *The vanishing of the Wronskian and the problem of linear dependence*, Math. Ann. **65** (1908), 282–298.

[8] FROBENIUS, G.: *Über den Begriff der Irreductibilität in der Theorie der linearen Differentialgleichungen*, J. Reine Angew. Math. **76** (1873), 236–270.

[9] HARDY, G. H.: *A Course of Pure Mathematics* (10th ed.), Cambridge University Press, Cambridge, 1952. [Reissued in: Cambridge Mathematical Lib., 1992, reprinted 1994.]

[10] MOSZNER, Z.: *Sur le wronskien et la dépendance linéaire des fonctions*, Bull. Sci. Math. (2) **85** (1961), 161–190.

[11] PEANO, G.: *Sul determinante Wronskiano*, Rend. Reale Accad. Lincei (5) **6** (1897), 413–415.

[12] SZUCS, J. M.: *When dim = constant implies subspace = const*, Acta Sci. Math. (Szeged) **66** (2000), 541–552.

[13] SZUCS, J. M.: *The size of the set of zeros of a Wronskian*, Math. Slovaca **51** (2001), 69–74.

Received 16. 1. 2006

Texas A&M University at Galveston
P.O. Box 1675
Galveston, TX 77553-1675
USA

E-mail: szucsj@tamug.edu