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INFINITE PRODUCTS OF FILTERS

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ABSTRACT. Stability of some classes of filters under the (infinite, Tikhonov) product operation is investigated. Applications to productivity of some types of set valued maps are given.

1. Introduction

Novak [N] and Terasaka [T] have shown, by providing suitable examples, that the product of two countably compact spaces does not have to be countably compact. A search for additional conditions under which some forms of productivity hold is therefore natural and, not surprisingly, there is quite a great deal of literature concerning this subject. We pick the survey paper [V2] of Vaughan as our starting point, because it reports adequately on earlier research and, at the same time, seems to indicate that for a long time the central notion under scrutiny was that of a totally countably compact space, one possible definition of which follows.

Definition 1.1. A topological space X is *totally countably compact* if every countably based filter on X admits a finer compact countably based filter.

We recall that a filter $\mathcal{F}$ is compact if every filter finer than $\mathcal{F}$ has a cluster point.

Totally countably compact spaces are countably productive. They are strictly contained in the class of those spaces whose product with any countably compact space is countably compact [V2, Example 3.13]. The problem of an internal characterization of the latter class of spaces seems to be open to this day. Recently, it has been re-examined in a more general framework. Although an internal

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characterization remains elusive for *spaces*, a satisfactory answer was provided in [JLM] "at a *filter* level". Here is the key notion used there.

Definition 1.2. Let $\mathbb{D}$, $\mathbb{J}$ be classes of filters on a topological space X. A filter $\mathcal{X}$ on X is said to be *compactly $\mathbb{D}$ to $\mathbb{J}$ meshable*, if, for each filter $\mathcal{D} \in \mathbb{D}$ meshing with $\mathcal{X}$, there exists a compact filter $\mathcal{J} \in \mathbb{J}$ meshing with $\mathcal{D}$.

It is not difficult to see, by taking $\mathbb{D} = \mathbb{J}$ to be the class of countably based filters and the filter $\mathcal{X}$ to be the principal filter of the set X, that the second definition is one of possible variants of the first definition translated to the more general setting of filters. In the countable case, the characterization "at the filter level" alluded to above reads as follows.

Theorem 1.3. *The product filter $\mathcal{X} \times \mathcal{Y}$ is countably compact for every countably compact countably based filter $\mathcal{Y}$ if and only if $\mathcal{X}$ is compactly $\mathbb{D}/\mathbb{D}$-meshable, with $\mathbb{D}$ being the class of countably based filters.*

Thus convinced that the notion of compact $\mathbb{D}$ to $\mathbb{J}$-meshability of filters is not too far-fetched and taking into account the classical results on infinite products of totally countably compact spaces, we embarked on research[1] of infinite product of filters. As a result, we present rather general theorems on stability of classes of filters under product whose potential remains to be fully explored. We only investigate the productivity properties of some classes of (set-valued) maps which seem to provide a natural arena of applications for our results on products.

2. Compactness

Our general reference for undefined terms is [K]. In particular, the notion of compactness does not presuppose Hausdorffness.

We use script to denote families of sets, filters in particular, and we use blackboard bold to denote classes of families. Let X be a topological space, $A \subset X$ and let $\mathcal{B}$ denote a fixed but otherwise arbitrary nonempty family of nonempty subsets of X. Let $\mathbb{P}$ and $\mathbb{Q}$ denote fixed, but otherwise arbitrary classes of families of *open* subsets of X.

(1) $\mathcal{B}$ is *$\mathbb{P}/\mathbb{Q}$-compact at A* if for every $\mathcal{P} \in \mathbb{P}$ covering A there exists $B \in \mathcal{B}$ and a refinement $\mathcal{Q} \in \mathbb{Q}$ of $\mathcal{P}$ such that $B \subset \bigcup \mathcal{Q}$.

(2) $\mathcal{B}$ is *$\mathbb{P}/\mathbb{Q}$-midcompact at A* if the same condition holds but the latt r containment takes the form $B \subset \bigcup\{\overline{Q} : Q \in \mathcal{Q}\}$.

[1] A part of first author's PhD-dissertation prepared at the University of Mississippi.

These notions are further extended by replacing the set A by a family of sets $\mathcal{A}$. $\mathcal{B}$ is said to be $\mathbb{P}/\mathbb{Q}$-compact (resp. midcompact) at $\mathcal{A}$, if it is so at A for each $A \in \mathcal{A}$. In particular,

(3) $\mathcal{B}$ is $\mathbb{P}/\mathbb{Q}$-*selfcompact* if it is $\mathbb{P}/\mathbb{Q}$-compact at $\mathcal{B}$.

Classical cases do not display the class $\mathbb{Q}$, but use the following names instead:

(4) $\mathbb{Q}$ = all subfamilies (of "covers" $\mathcal{P} \in \mathbb{P}$) of finite cardinality: $\mathbb{P}$-*compactness*,
(5) $\mathbb{Q}$ = countable subfamilies: $\mathbb{P}$-*Lindelöfness*,
(6) $\mathbb{Q}$ = locally finite families (of refinements): $\mathbb{P}$-*paracompactness*,
(7) $\mathbb{Q}$ = point finite families (of refinements): $\mathbb{P}$-*metacompactness*.

In case (4) the usual conventions apply when $\mathbb{P}$ is specified. For instance, if $\mathbb{P}$ is the class of all covers, $\mathbb{P}$ is dropped and $\mathcal{B}$ is said to be *compact at* A. If $\mathbb{P}$ is $\mathbb{G}_{\mathfrak{m}}$ i.e., the class of open families of cardinality (strictly) less than a cardinal number $\mathfrak{m}$, we speak about $\mathfrak{m}$-*compactness*. Because we use the strict inequality when dealing with cardinals, *finite compactness* refers to $\mathfrak{m} = \aleph_0$ and *countable compactness* to $\mathfrak{m} = \aleph_1$. We note that the definition of $\mathbb{P}/\mathbb{Q}$-compactness as given here, although quite general, is a special case of "cover-compactoidness" considered in [D, Sec. 7].

Recall that if $\mathcal{A}$, $\mathcal{B}$ are families of sets, we write $\mathcal{A}\#\mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ *mesh* if $A \cap B \neq \emptyset$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$.

Each time $\mathbb{P}$-compactness occurs (note that we assume $\mathbb{Q} = \mathbb{G}_{\aleph_0}$), its *dual form* in terms of the class $\mathbb{P}_\star$ of filters meshing with $\mathcal{B}$ can also be stated. The duality is determined by the fact that (some of) the families of complements of the members of $\mathbb{P}$ form the bases of filters. The class of filters they generate is denoted $\mathbb{P}_\star$. This leads to the adoption, for a class of filters $\mathbb{D}$, of the following definition of "filter-compactness". The family $\mathcal{B}$ is $\mathbb{D}$-*compact at* $\mathcal{A}$ if

$$\mathcal{D} \in \mathbb{D},\ \mathcal{D}\#\mathcal{B} \implies \operatorname{adh}\mathcal{D}\#\mathcal{A}.$$

If, in the above condition, the filter $\mathcal{D}$ *meshing with* $\mathcal{B}$ is replaced by the filter $\mathcal{D}$ *finer than* $\mathcal{B}$, then we speak about *near* $\mathbb{D}$-compactness of $\mathcal{B}$ at $\mathcal{A}$. Dealing with filters or filter bases in a fixed space X, it is customary to drop A if $A = X$. Thus a filter $\mathcal{B}$ is $\mathbb{D}$-*compact* if each filter meshing with $\mathcal{B}$ has a cluster point. One must, however, be cautious when dealing with sets. The convention cannot be applied in this case without inducing an error. For instance, a set $B \subset X$ identified with its principal filter is compact at X whenever it is *relatively compact* as a subset of X and so "at X" cannot be dropped.

Concerning the duality between compactness in terms of covers versus filters, a reader in need of further detail may consult [L] and [DaL]. We only mention that in terms of filters, $\mathfrak{m}$-compactness is determined by the class $\mathbb{F}_{\mathfrak{m}}$ of all filters that admit a base of cardinality less than $\mathfrak{m}$. Similarly, $\mathfrak{m}$-midcompactness is determined by the class $\mathcal{O}(\mathbb{F}_{\mathfrak{m}})$ of all filters that admit a base of cardinality less than $\mathfrak{m}$ which is composed of open sets.

Remark. In some papers (including [D] and [L]), the term “compactoid” instead of “compact” was used.

3. A product theorem for classes of filters

We denote by $\mathbb{F}$ the class of all filters (with unspecified underlying set) and by $\mathbb{F}(X)$ the class of all filters on a set X. If $\mathcal{H}$ is a filter on X and $\mathcal{G}$ is a filter on $X \times Y$, we define $\mathcal{G}\mathcal{H}$ to be the (possibly degenerate) filter on Y generated by $\{GH : H \in \mathcal{H}, G \in \mathcal{G}\}$, where $GH = \{y : \exists x \in H, (x, y) \in G\}$. Thus each $G \in \mathcal{G}$ is treated as a relation $G\colon X \rightrightarrows Y$ and we consider the filter generated by the images via all relations in $\mathcal{G}$ of all sets in $\mathcal{H}$.

Now let $\mathfrak{m}$ again be a cardinal number. A class $\mathbb{D}$ of filters is said to be $\mathfrak{m}$-*productive* if the product of (strictly) less than $\mathfrak{m}$ $\mathbb{D}$-filters is in $\mathbb{D}$. Since we denote by $\mathbb{F}_{\mathfrak{m}}$ the class of filters with a base of cardinality less than $\mathfrak{m}$, $\mathbb{F}_{\aleph_0}$ is the class of principal filters and $\mathbb{F}_{\aleph_1}$ the class of filters with countable base. If $\mathcal{D}\mathcal{J} \in \mathbb{D}(Y)$ for $\mathcal{D} \in \mathbb{D}(X \times Y)$ and $\mathcal{J} \in \mathbb{J}(X)$, we write $\mathbb{D}\mathbb{J} \subset \mathbb{D}$ for short.

Let $\{X_\iota : \iota \in I\}$ be a family of non-empty sets and let $X = \prod_{\iota \in I} X_\iota$ be its product. We denote by p_ι the ι-th projection $X \to X_\iota$ and say that a class $\mathbb{D}$ of filters is *stable under projections* if $p_\iota(\mathcal{D}) \in \mathbb{D}(X_\iota)$ for any $\mathcal{D} \in \mathbb{D}(X)$ and any product X.

A class $\mathbb{D}$ *is* $\mathbb{J}$-*steady* if for each filter $\mathcal{D} \in \mathbb{D}$ and each filter $\mathcal{J} \in \mathbb{J}$, their supremum $\mathcal{D} \vee \mathcal{J}$ belongs to $\mathbb{D}$. $\mathbb{D}$ *contains spaces* if, for each space X, the filter $\{X\} \in \mathbb{D}(X)$.

We now proceed to state our main theorem on productivity of classes of filters. As the theorem is rather general and abstract, it can be hard to swallow. For this reason, following the advice of the referee, we decided to move its proof and the development of related criteria to the last section of the paper. The hope is that knowing a few applications of the theorem, the potential reader will be motivated enough to go into technicalities.

A class of filters whose productivity is under scrutiny is the intersection of two classes. One class, say $\mathbb{J}$, is “set-theoretical” while the other class, say $\mathbb{K}$, is “topological”. We are asserting the productivity of the class $\mathbb{K} \cap \mathbb{J}$. Yet, we often say that the filter $\mathcal{F} \in \mathbb{K} \cap \mathbb{J}$ is the $\mathbb{K}$-*type* $\mathbb{J}$-*filter* because, in practice, “$\mathbb{K}$-type” is used as an adjective. This terminology may seem “heavy” at first glance, however, we experimented with other options and did not find anything comparably flexible or convenient.

In order to deal successfully with infinite products, the Definition 1.2 needs to be slightly modified. Let $\mathbb{D}$, $\mathbb{J}$ and $\mathbb{K}$ be classes of filters. We say that a filter $\mathcal{F}$ is $\mathbb{K}$-*type nearly* $\mathbb{D}/\mathbb{J}$-*meshable* if for any filter $\mathcal{D} \in \mathbb{D}$ finer than $\mathcal{F}$ there exists a $\mathbb{K}$-type filter $\mathcal{J} \in \mathbb{J}$ which is meshing with $\mathcal{D}$.

THEOREM 3.1 (Main Theorem). *Suppose that* $\mathbb{D}$ *is* $\mathbb{J}$-*steady and stable under projections,* $\mathbb{J}$ *contains spaces and is* $\mathfrak{m}$-*productive, and* $\mathbb{K}$ *is* $\mathfrak{m}$-*productive. Then the class of* $\mathbb{K}$-*type nearly* $\mathbb{D}/\mathbb{J}$-*meshable filters is* $\mathfrak{m}$-*productive.*

In a few concrete examples which we propose to examine, the class $\mathbb{K}$ has actually a slightly more complicated structure. It involves one more parameter, the filters in $\mathbb{K}$ being dependent on a specified subset of the space in which they are defined.

Given X and $A \subset X$, we consider

The class $\mathbb{K}_A$ of filters that are compact at A;

The class $\mathbb{M}_A$ of filters that are midcompact at A;

The class $\mathbb{A}_A$ of filters that are *adherent at* A, i.e., admit a cluster point in A.

There is also a sort of "resonance of productivity": each time an $\mathfrak{m}$-productive class of filters, say $\mathbb{X}$, is found by applying the theorem, the theorem can be applied again with $\mathbb{K} = \mathbb{X}$.

As mentioned, in concrete situations it is convenient to use "$\mathbb{K}$-type" as an adjective. We will speak, respectively, of filters that are compactly at A, midcompactly at A, adherently at A, nearly $\mathbb{D}/\mathbb{J}$-meshable.

Here is a theorem asserting that the three mentioned $\mathbb{K}$-types are indeed productive in an appropriate sense.

THEOREM 3.2. *Let* $(X_\iota)_{\iota \in I}$ *be a family of topological spaces. Suppose that, for each* $\iota \in I$, *the filter* $\mathcal{X}_\iota$, *defined on* X_ι, *is compact (resp. midcompact, resp. adherent) at* $A_\iota \subset X_\iota$. *Then the product filter* $\mathcal{X}$ *on* $X = \prod_{\iota \in I} X_\iota$ *is compact (resp. midcompact, resp. adherent) at* $A = \prod_{\iota \in I} A_\iota$.

Proof. The "compact case" can be proven following the lines of Bourbaki's proof of the Tikhonov theorem. We now give a proof of the "midcompact case" which, although similar in spirit, is technically more demanding.

Keeping the conventions of Section 2 in place, a family of sets $\mathcal{B}$ is said to be (cover) *midcompact at* A if, for each open cover $\mathcal{P}$ of A there exist $B \in \mathcal{B}$ and a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that B is contained in the closure of $\bigcup \mathcal{P}_0$. As mentioned, the dual statement asserts that for each openly based filter $\mathcal{G}$ which is meshing with $\mathcal{B}$, $\mathcal{G}$ has a cluster point in A. It is the latter form that we use in the following argument.

Denote by $\mathcal{O}$ the topology of the product space X. Suppose $\mathcal{G}$ is a filter that is meshing with $\mathcal{X}$ and also admits a base of open sets. Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{G} \vee \mathcal{X}$. Denote by $\mathcal{O}(\mathcal{U})$ the base (also called filter in $\mathcal{O}$) formed as $\{O \in \mathcal{O} : \exists U \in \mathcal{U} \text{ with } U \subset O\}$. We claim that $\mathcal{O}(\mathcal{U})$ is an ultrafilter in $\mathcal{O}$.

Let G_1, G_2 be open sets such that their union belongs to $\mathcal{O}(\mathcal{U})$. Then there is $U \in \mathcal{U}$ such that $U \subset G_1 \cup G_2$. Let $U_1 = G_1 \cap U$ and $U_2 = G_2 \cap U$. As $\mathcal{U}$ is an ultrafilter, U_1 or U_2 is in $\mathcal{U}$. Suppose U_2 is. Then G_2 is in $\mathcal{O}(\mathcal{U})$ which shows that $\mathcal{O}(\mathcal{U})$ is an ultrafilter in $\mathcal{O}$.

Having shown our claim and taking into account that the projections are open, $\mathcal{O}(\mathcal{U})_\iota = \mathrm{pr}_\iota(\mathcal{O}(\mathcal{U}))$ is a base of an ultiafilter (in $\mathcal{O}$) which is meshing with $\mathcal{X}_\iota$ and therefore converges to $x_\iota \in A_\iota$. Hence $\mathcal{O}(\mathcal{U})_\iota \geq \mathcal{N}(x_\iota)$ and one infers that $\mathcal{O}(\mathcal{U})$ has a limit point $x = (x_\iota)$. This limit point is the needed cluster point of $\mathcal{G}$.

The third case, i.e., the one of adherent filters is rather obvious. □

Remark. The "compact case" is the filter form, essentially due to Pettis [P], of the Tikhonov theorem. The "midcompact case" is the filter form of a theorem of Chevalley and Frink [CF]. It has been announced by Dolecki and Greco ([DG, Theorem 3.1]). However, as far as we know, a proof was never published. The definition of being "absolutely closed" given in [DG] slightly differs from the definition of midcompactness here, but the two notions are equivalent.

The next result is the Main Theorem for the concrete $\mathbb{K}$-types that we now consider. Its assumptions about the classes $\mathbb{D}$ and $\mathbb{J}$ are kept.

THEOREM 3.3. *Let $A_\iota \dot{\subset} X_\iota$ and let, for each $\iota \in I$, $\mathcal{X}_\iota$ be a compactly (resp. midcompactly, resp. adherently) at A_ι nearly $\mathbb{D}/\mathbb{J}$-meshable filter on X_ι. Then the product filter $\mathcal{X}$ is compactly (resp. midcompactly, resp. adherently) at $\prod_{\iota \in I} A$ nearly $\mathbb{D}/\mathbb{J}$-meshable $\mathbb{J}$-filter on X.*

We get, as a special case, a classical result on products of "compact-like" spaces discussed in the introduction, compare [V1, Theorem 1] and [S, Theorem 5.4]. We note that its "midcompact case" seems to be new even in the framework of spaces.

COROLLARY 3.4. *Let $\mathfrak{m}$ be a cardinal number. The class of compactly (resp. midcompactly) $\mathbb{F}_\mathfrak{m}/\mathbb{F}_\mathfrak{m}$-meshable spaces is $\mathfrak{m}$-productive.*

4. Maps

Recall that if Y, X are topological spaces, then a continuous onto function $f \colon Y \to X$ is said to be *perfect* if it is closed and has compact fibers. It is well-known and can be readily checked that f (continuous or not) is *closed* if and only if the set-valued map $\Gamma = f^{-1} : X \rightrightarrows Y$ is *upper semicontinuous.* Without any further assumptions on f, the values of Γ are essentially arbitrary. If f is continuous, the values of Γ will be closed provided Y is T_1. If f is perfect, the values of Γ are compact. Thus, the inverse of a perfect map serves as a model for a usco map defined as follows. A set valued map $F\colon X \rightrightarrows Y$ is said to be a *usco map* if it is usc and takes non-empty compact values. The acronym "usco" is a play on "Upper Semi-Continuous and Compact".

We move to a more general environment. For the next proposition, we assume that the class $\mathbb{P}$ *contains all open sets* (i.e., if O is open, then $\{O\} \in \mathbb{P}$) and the corresponding class $\mathbb{Q}$ of open refinements has the property that a *finite union of its members is still in the class.* Let $\Gamma\colon X \rightrightarrows Y$ be a map taking non-empty values. We will say that Γ is $\mathbb{P}/\mathbb{Q}$*-usco* if Γ is usc and, for each $x \in X$ the value $\Gamma(x)$ is a $\mathbb{P}/\mathbb{Q}$-selfcompact subset of Y. $\mathcal{N}(x)$ denotes the filter of neighborhoods of a point $x \in X$. The next proposition can be treated as a general form of [L, Proposition 1.2].

PROPOSITION 4.1. *The following conditions are equivalent.*

(i) Γ *is* $\mathbb{P}/\mathbb{Q}$*-usco.*

(ii) $\Gamma(\mathcal{N}(x))$ *is* $\mathbb{P}/\mathbb{Q}$*-compact at* $\Gamma(x)$.

(iii) *If* $\mathcal{B}$ *is a family of sets that is compact at* $A \subset X$*, then* $\Gamma(\mathcal{B})$ *is* $\mathbb{P}/\mathbb{Q}$*-compact at* $\Gamma(A)$.

Proof.

(i) implies (iii). Let $\mathcal{P} \in \mathbb{P}$ be a cover of $\Gamma(A)$. Pick, for each $x \in A$, a $\mathbb{Q}$-cover $\mathcal{Q}_x$ of $\Gamma(x)$. This is possible, because the values $\Gamma(x)$ are $\mathbb{P}/\mathbb{Q}$-compact. Define $Q_x = \bigcup \mathcal{Q}_x$ and find, by upper semicontinuity of Γ, an open neighborhood N_x of x such that $\Gamma(N_x) \subset Q_x$. Then $\{N_x : x \in A\}$ is an open cover of A. As $\mathcal{B}$ is compact at A, we can find $x_1, x_2, \ldots, x_n$ and $B \in \mathcal{B}$ such that $B \subset \bigcup_{i=1}^{n} N_{x_i}$. Observe that $\bigcup_{n=1}^{n} \mathcal{Q}_{x_i}$ is still a member of $\mathbb{Q}$ and moreover covers $\Gamma(B)$.

(iii) implies (ii) is trivial (because $\mathcal{N}(x)$ is compact at $\{x\}$).

(ii) implies (i). We show that $\Gamma(x)$ is $(\mathbb{P}/\mathbb{Q})$-selfcompact. Let $\mathcal{P} \in \mathbb{P}$ be a cover of $\Gamma(x)$. By (ii), we can find $N \in \mathcal{N}(x)$ and $\mathcal{Q} \in \mathbb{Q}$ such that $\Gamma(N)$ is contained in $\Gamma(\bigcup \mathcal{Q})$. But $\Gamma(x) \subset \Gamma(N)$.

Now, let O be an open set containing $\Gamma(x)$. By (ii), there exists a refinement $\mathcal{Q}$ of the cover $\{O\}$ and $N \in \mathcal{N}(x)$ such that $\Gamma(N)$ is contained in $\bigcup \mathcal{Q}$. But the latter union is contained in O. Hence Γ is usc. $\square$

A known theorem of Bourbaki and Frolík asserts that perfect maps are stable under products. Here is a more general statement (compare e.g. [DL, Proposition 3.6]).

THEOREM 4.2. *For each $\iota \in I$, let $\Gamma_\iota \colon X_\iota \rightrightarrows Y_\iota$ be a usco-map. Then the product map*

$$\Gamma \colon \prod_{\iota\in I} X_\iota \rightrightarrows Y = \prod_{\iota\in I} Y_\iota$$

defined by $\Gamma((x_\iota)_{\iota\in I}) = \prod_{\iota\in I} \Gamma_\iota(x_\iota)$ is also a usco-map.

Proof. Let $\mathcal{N}_\iota = \mathcal{N}(x_\iota)$ be the neighborhood filter of x_ι. Then, by 4.1, the image filter $\Gamma_\iota(\mathcal{N}_\iota)$ is compact at $\Gamma_\iota(x_\iota)$. Hence the product filter $\prod_{\iota\in I} \Gamma_\iota(\mathcal{N}\ )$ is compact at $\prod_{\iota\in I} \Gamma(x_\iota)$. This means also that $\left(\prod_{\iota\in I} \Gamma_\iota\right)\left(\prod_{\iota\in I} \mathcal{N}_\iota\right) = \Gamma(\mathcal{N}(x))$ is compact at $\Gamma(x)$, where $x = (x_\iota)_{\iota\in I}$. Apply 4.1 to conclude that Γ is usc. $\square$

In fact, we did not use the full force of our product theorem. The productivity of the class of filters that are compact at a set was all we needed. However, the product theorem indicates one more way, in which a generalization of usco maps could be attempted. This is here that we take advantage of the "resonance of productivity" mentioned in the previous section.

Let us say that Γ is $\mathfrak{m}$-*productively usco* if, for each $\mathcal{F} \in \mathbb{F}_\mathfrak{m}$ and $\mathcal{F} \geq \mathcal{N}(x)$ there exists a $\mathbb{F}_\mathfrak{m}$-filter $\mathcal{H}$ such that $\mathcal{H}\#\mathcal{F}$ and $\mathcal{H}$ is compact at $\Gamma(x)$. In other words, applying the just defined notion to coordinate maps, we obtain the maps Γ_ι such that, for each x_ι, the filter $\Gamma[\mathcal{N}(x_\iota)]$ is compactly at $\Gamma(x_\iota)$ nearly $\mathbb{F}_\mathfrak{m}/\mathbb{F}_\mathfrak{m}$-meshable. By the "canonical" proof of 4.2 and our product theorem, we conclude that the just defined class of maps is indeed $\mathfrak{m}$-*productive.*

Unfortunately, a characterization of $\mathfrak{m}$-productively usco maps in terms of their values remains an open question. For the purpose of the present discussion, let us accept that a subset E of Y is *totally* $\mathfrak{m}$-*compact*, if for any filter $\mathcal{F}$ in the class $\mathbb{F}_\mathfrak{m}$ on E, E equipped with the induced topology, there exists a filter $\mathcal{E} \in \mathbb{F}_\mathfrak{m}$ on E such that $\mathcal{E}$ is compact at E and is meshing with $\mathcal{F}$. Note that this means that a totally $\mathfrak{m}$-compact subset of Y, considered as a *space* E, is *compactly* $\mathbb{F}_\mathfrak{m}/\mathbb{F}_\mathfrak{m}$*-meshable.* Evidently, one would like $\mathfrak{m}$-productively usco maps to be just usc maps with totally $\mathfrak{m}$-compact values. However, the relationship between these two classes is unclear.

Remark. As we have seen, the third case, i.e., the one in which $\mathbb{K}$ is the class of filters that are *adherent at a set* is covered by our general considerations. Recall also that a continuous onto function $f \colon Y \to X$ is said to be *biquotient* if for each filter base $\mathcal{B}$ having a cluster point $x \in X$, the filter $f^{-1}(\mathcal{B})$ has a cluster point in $f^{-1}(x)$. Instead of the theorem of Bourbaki and Frolík, we have Michael's theorem asserting that biquotient maps are productive. We only signal the full analogy that is at work here. The details concerning this case will hopefully be presented in the forthcoming thesis of the first author.

5. Product theorems

In this section we present a proof of the main theorem and related material on infinite products of filters. Below I is an index set, $(X_\iota)_{\iota\in I}$ are topological spaces, $\mathcal{X}_\iota$ is a filter on X_ι, $X = \prod_{\iota\in I} X_\iota$ and $\mathcal{X} = \prod_{\iota\in I} \mathcal{X}_\iota$.

THEOREM 5.1. *Let* $\operatorname{card}(I) < \mathfrak{m}$ *and* $\mathbb{D}$, $\mathbb{J}$ *and* $\mathbb{K}$ *be classes of filters such that*

(i) $\mathbb{D}$ *is* $\mathbb{J}$*-steady and stable under projections.*

(ii) $\mathbb{J}$ *contains spaces and is* $\mathfrak{m}$*-productive.*

(iii) $\mathbb{K}$ *is* $\mathfrak{m}$*-productive.*

If the filters $\mathcal{X}_\iota$ *are nearly* $\mathbb{D}/\mathbb{J}\cap\mathbb{K}$*-meshable, then so is* $\mathcal{X}$.

Remark. Instead of being $\mathbb{J}$-steady, we could require that for a filter $\mathcal{J}$ meshing with $\mathcal{D}$ there exists a finer $\mathbb{D}$-filter.

Proof. Let ν (resp. μ) be the least ordinal whose cardinality is that of I (resp. $\mathfrak{m}$). We identify I with ν, write $\{\mathcal{X}_\alpha : \alpha < \nu\}$ and $\mathcal{X} = \prod\{\mathcal{X}_\alpha : \alpha < \nu\}$.

We identify the principal filter of a set with the set itself. In particular, under this identification $X_\alpha = \{X_\alpha\}$, $X = \prod\{X_\alpha : \alpha < \nu\}$ and we will use the notation

$$X^\alpha = \prod\{X_\gamma : \alpha \le \gamma < \nu\}.$$

Let $\mathcal{D} \in \mathbb{D}$ be finer than $\mathcal{X}$. We need to show the existence of a $\mathbb{J}$-filter $\mathcal{J}$ such that $\mathcal{J}\#\mathcal{D}$ and $\mathcal{J} \in \mathbb{K}$. We proceed by induction on the ordinals less than μ.

Base step. The filter $p_0(\mathcal{D}) \ge \mathcal{X}_0$ and, as $\mathbb{D}$ is stable under projections, belongs to $\mathbb{D}$. As $\mathcal{X}_0 \in \mathbb{J}$ and $\mathbb{D}$ is $\mathbb{J}$-steady, $p_0(\mathcal{D}) \vee \mathcal{X}_0$ is in $\mathbb{D}$. As $\mathcal{X}_0$ is nearly $\mathbb{D}/\mathbb{J}\cap\mathbb{K}$-meshable, there exists $\mathcal{J}_0 \in \mathbb{J}(X_0)$ which is a $\mathbb{K}$-type filter and meshes with $p_0(\mathcal{D}) \vee \mathcal{X}_0$. We set $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 = \mathcal{D} \vee (\mathcal{J}_0 \times X^1)$.

Inductive step. Suppose that for all α, $\alpha < \beta$ with $\beta < \mu$, we have already defined

$$\mathcal{D}_\alpha := \mathcal{D} \vee \left\{\prod\{\mathcal{J}_\gamma : \gamma < \alpha\} \times X^\alpha\right\}$$

such that $\mathcal{J}_\gamma$'s are $\mathbb{K}$-type filters. We note that $(\mathcal{D}_\alpha)$ is a transfinite sequence of linearly ordered by refinement $\mathbb{D}$-filters which are finer than $\mathcal{D}$.

Case A: β is a limit ordinal.

We define $\mathcal{D}_\beta = \mathcal{D} \vee \left(\prod\{\mathcal{J}_\alpha : \alpha < \beta\} \times X^\beta\right)$. The filter $\mathcal{D}_\beta$ is well defined (i.e., is non-degenerate). This follows from the fact that $\mathcal{D}_\beta = \sup\{\mathcal{D}_\alpha : \alpha < \beta\}$. Indeed, it is clear that it is finer than any $\mathcal{D}_\alpha$ for all $\alpha < \beta$. Conversely, let H be a (basic) set of $\mathcal{D}_\beta$. It is of the form $D \in \mathcal{D}$ intersected with

$$J_{\alpha_1\alpha_2\ldots\alpha_n}$$

where the latter set denotes the X-cylinder determined by the sets

$$J_{\alpha_1} \in \mathcal{J}_{\alpha_1},\ J_{\alpha_2} \in \mathcal{J}_{\alpha_2},\ \dots,\ J_{\alpha_n} \in \mathcal{J}_{\alpha_n}.$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \beta$.

As β is a limit ordinal, there exists α larger than α_n and smaller than β. Then $D \cap J_{\alpha_1\alpha_2\dots\alpha_n} \in \mathcal{D}_a \subset \sup\{\mathcal{D}_\alpha :\ \alpha < \beta\}$.

Case B: $\beta = \gamma + 1$.

That is, β is a successor ordinal and $\mathcal{D}_\gamma = \mathcal{D} \vee (\prod\{\mathcal{J}_\alpha :\ \alpha < \gamma\} \times X^\gamma)$. It is clear, by the very form of $\mathcal{D}_\gamma$, that $\mathcal{D}_\gamma \in \mathbb{D} \vee \mathbb{J} \subset \mathbb{D}$. Moreover, $p_\gamma(\mathcal{D}_\gamma) \geq p_\gamma(\mathcal{D}) \geq p_\gamma(\mathcal{X}) = \mathcal{X}_\gamma$. $\mathcal{X}_\gamma$ being nearly $\mathbb{D}/\mathbb{J} \cap \mathbb{K}$-meshable, there exists $\mathcal{J}_\gamma \in \mathbb{J}$ meshing with $p_\gamma(\mathcal{D}_\gamma)$ which is $\mathbb{K}$-type. We define

$$\mathcal{D}_\beta = \mathcal{D} \vee \Big(\prod\{\mathcal{J}_\alpha :\ \alpha < \gamma\} \times \mathcal{J}_\gamma \times X^\beta\Big)$$

and see that the induction continues.

To finish the proof, consider $\mathcal{J} = \prod\{\mathcal{J}_a :\ \alpha < \nu\}$. It is clear that $\mathcal{J} \in \mathbb{J} \cap \mathbb{K}$. Moreover, as $D_\nu = \mathcal{D} \vee \prod\{\mathcal{J}_a :\ \alpha < \nu\} = \mathcal{D} \vee \mathcal{J}$ exists, we see that $\mathcal{J} \# \mathcal{D}$. The proof is complete. □

We say that a filter $\mathcal{X}$ is $\mathbb{D}/\mathbb{J}$-*refinable* ($\mathcal{X} \in (\mathbb{D}/\mathbb{J})_{\geq}$), if, for each $\mathcal{D} \in \mathbb{D}$ meshing with $\mathcal{X}$, there exists a filter $\mathcal{J} \in \mathbb{J}$ such that $\mathcal{J}$ is finer than both $\mathcal{D}$ and $\mathcal{X}$. With the assumptions of the theorem, we have:

Corollary 5.2. *Suppose moreover that the product filter $\mathcal{X}$ is $\mathbb{D}/\mathbb{D}$-refinable. Then $\mathcal{X}$ is $\mathbb{D}/\mathbb{J}$-meshable.*

Indeed, by the theorem, $\mathcal{X}$ is nearly $\mathbb{D}/\mathbb{J}$-meshable. Combining this with the fact that $\mathcal{X}$ is now $\mathbb{D}/\mathbb{D}$-refinable, we can drop "nearly" in the conclusion.

We also have a corollary for the case of weak inequality, although we loose information about the class of the filter $\mathcal{J}$.

Corollary 5.3. *Let $\mathbb{K}$ be productive and* $\operatorname{card}(I) \leq \mathfrak{m}$. *If the filters $\mathcal{X}_\iota$ are nearly $\mathbb{D}/\mathbb{J} \cap \mathbb{K}$-meshable, then $\mathcal{X}$ is nearly $\mathbb{D}/\mathbb{K}$-meshable.*

Remarks.

(1) The use of transfinite induction is quite classical, compare [V1] and the survey articles of Vaughan and Stephenson in the Handbook [H].

(2) In [JM] a class of filters $\mathbb{J}$ is $\mathbb{D}$-*composable* if $\mathcal{D}\mathcal{J} \in \mathbb{J}(Y)$ whenever $\mathcal{D} \in \mathbb{D}(X \times Y)$ and $\mathcal{J} \in \mathbb{J}(X)$ (i.e., if $\mathbb{D}\mathbb{J} \subset \mathbb{J}$). If $\mathbb{D} = \mathbb{J}$, $\mathbb{J}$ *is composable.* Note that if a class $\mathbb{D}$ is $\mathbb{F}_{\aleph_0}$-composable, then it is stable under projections.

One would like to give a corollary in the special case of $\mathbb{D}$ equal $\mathbb{J}$. In that case, in view of the preceding remark 2, we may write

Corollary 5.4. *Let $\mathbb{D} \supset \mathbb{F}_{\aleph_0}$ be a composable $\mathfrak{m}$-productive class of filters and suppose* $\mathrm{card}(I) < \mathfrak{m}$. *If the filters $\mathcal{X}_\iota$ are nearly $\mathbb{D}/\mathbb{D} \cap \mathbb{K}$-meshable, then so is $\mathcal{X}$.*

However, assuming that each filter $\mathcal{X}_\iota$ on X_ι is in the class $\mathbb{D}$, a direct argument gives more.

Theorem 5.5. *Let $\mathbb{D}$, $\mathbb{K}$ be $\mathfrak{m}$-productive classes of filters, $\mathbb{D}$ being also composable. If* $\mathrm{card}(I) < \mathfrak{m}$ *and $\mathcal{X}_\iota$ are nearly $\mathbb{D}/\mathbb{D} \cap \mathbb{K}$-meshable $\mathbb{D}$-filters, then so is $\mathcal{X}$. If $\mathbb{D}$ contains $\mathbb{F}_{\aleph_0}$, the "nearly" in the conclusion can be dropped.*

Proof. The filter

$$\mathcal{X}^\alpha = \prod\{\mathcal{X}_\gamma : \alpha \le \gamma < \nu\}$$

will play the role of X^α in the proof of Theorem 5.1.

Let $\mathcal{D} \in \mathbb{D}$ be a filter finer than $\mathcal{X}$. We need to show the existence of a $\mathbb{D}$-filter $\mathcal{J}$ such that $\mathcal{J}\#\mathcal{D}$ and $\mathcal{J}$ is in $\mathbb{K}$. As above, we proceed by induction on the ordinals less than μ.

Base step. The filter $(\mathcal{D}^-\mathcal{X}^1)\#\mathcal{X}_0$ and, as $\mathcal{X}^1 \in \mathbb{D}$, belongs to $\mathbb{D}$. As $\mathcal{X}_0$ is nearly $\mathbb{D}/\mathbb{D}\cap\mathbb{K}$-meshable, there exists $\mathcal{J}_0 \in \mathbb{D}(X_0)$ which is a $\mathbb{K}$-filter and meshes with $\mathcal{D}^-\mathcal{X}^1$. We define $\mathcal{K}_0 = \mathcal{X}$ and $\mathcal{K}_1 = \mathcal{J}_0 \times \mathcal{X}^1$; they are $\mathbb{D}$-filters meshing with $\mathcal{D}$.

Inductive step. Suppose that for all $\alpha < \beta$, where $\beta < \mu$, we have already defined the transfinite sequence of $\mathbb{D}$-filters.

$$\mathcal{K}_\alpha = \prod\{\mathcal{J}_\gamma : \gamma < \alpha\} \times \mathcal{X}^\alpha$$

satisfying

(1) $\mathcal{J}_\gamma \in \mathbb{K}$;

(2) $\mathcal{K}_\alpha\#\mathcal{D}$.

Suppose first that β is a limit ordinal. We check that $\mathcal{D}$ meshes with $\mathcal{K}_\beta$.

Let H be a basic set of $\mathcal{K}_\beta$. It can be taken to be of the form

$$J_{\alpha_1\alpha_2\ldots\alpha_n}$$

where the latter set denotes the X-cylinder determined by the sets

$$J_{\alpha_1} \in \mathcal{J}_{\alpha_1}, \ J_{\alpha_2} \in \mathcal{J}_{\alpha_2}, \ \ldots, \ J_{\alpha_n} \in \mathcal{J}_{\alpha_n}$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \beta$.

As β is a limit ordinal, there exists α larger than α_n and smaller than β. Then $J_{\alpha_1\alpha_2\ldots\alpha_n} \in \mathcal{K}_\alpha$ and therefore meshes with $\mathcal{D}$ by (2).

Suppose $\beta = \gamma+1$ and write $\mathcal{K}_\gamma = \mathcal{K}_\gamma^c \times X_\gamma$, where $\mathcal{K}_\gamma^c = \prod\{\mathcal{J}_\alpha : \alpha < \gamma\} \times X^\beta$. Note that $\mathcal{K}_\gamma^c$ is a $\mathbb{D}$-filter. As $\mathcal{D}\#\mathcal{K}_\gamma$, $\mathcal{D}\mathcal{K}_\gamma^c$ is a non-degenerate filter which is in the class $\mathbb{D}$ by composability of $\mathbb{D}$. Since $\mathcal{D} \ge \mathcal{X}$, it can be checked that $\mathcal{D}\mathcal{K}_\gamma^c$ is finer than $\mathcal{X}_\gamma$.

By assumption on $\mathcal{X}_\gamma$, there exists $\mathcal{J}_\gamma \in \mathbb{D} \cap \mathbb{K}$ meshing with $\mathcal{D}\mathcal{K}_\gamma^c$. Hence $\mathcal{D}$ meshes with $\mathcal{J}_\gamma \times \mathcal{K}_\gamma^c$. We define

$$\mathcal{K}_\beta = \prod\{\mathcal{J}_\alpha : \alpha < \gamma\} \times \mathcal{J}_\gamma \times X^\beta$$

and see that the conditions (1) and (2) are satisfied.

To finish the proof, observe that $\mathcal{K}_\nu$ is as required. □

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