Mathematica Slovaca

Lubomír Kubáček

Multivariate regression model with constraints

Mathematica Slovaca, Vol. 57 (2007), No. 3, [271]--296

Persistent URL: http://dml.cz/dmlcz/136954

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0022-7
Math. Slovaca **57** (2007), No. 3, 271–296

MULTIVARIATE REGRESSION MODEL WITH CONSTRAINTS

LUBOMÍR KUBÁČEK

(*Communicated by Gejza Wimmer*)

ABSTRACT. The aim of the paper is to present explicit formulae for parameter estimators and confidence regions in multivariate regression model with different kind of constraints and to give some comments to it. The covariance matrix of observation is either totally known, or some unknown parameters of it must be estimated, or the covariance matrix is totally unknown.

1. Introduction

A multivariate regression model is considered in the form ([1])

$$\underline{\mathbf{Y}} \sim_{nm} (\mathbf{XB}, \mathbf{\Sigma} \otimes \mathbf{I}), \tag{1}$$

where $\underline{\mathbf{Y}}$ is $n \times m$ random matrix (observation matrix), $\underline{\mathbf{Y}} = (\mathbf{Y}_1, \dots, \mathbf{Y}_m)$, $\mathbf{Y}_i \sim_n (\mathbf{X}\boldsymbol{\beta}_i, \sigma_{i,i}\mathbf{I}_{n,n})$, $i = 1, \dots, m$, $\mathbf{B} = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_m)$, $\operatorname{cov}(\mathbf{Y}_i, \mathbf{Y}_j) = \sigma_{i,j}\mathbf{I}_{n,n}$,

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{1,1}, & \sigma_{1,2}, & \dots, & \sigma_{1,m} \\ \sigma_{2,1}, & \sigma_{2,2}, & \dots, & \sigma_{2,m} \\ \dots & \dots & \dots & \dots \\ \sigma_{m,1}, & \sigma_{m,2}, & \dots, & \sigma_{m,m} \end{pmatrix},$$

$\mathbf{XB}$ is the mean value of the observation matrix $E(\underline{\mathbf{Y}}) = \mathbf{XB}$, $\mathbf{X}$ is an $n \times k$ given matrix and $\mathbf{B}$ is a $k \times m$ matrix of unknown parameters. $\mathbf{\Sigma} \otimes \mathbf{I}$ is the covariance matrix of the observation vector $\operatorname{vec}(\underline{\mathbf{Y}}) = (\mathbf{Y}_1', \mathbf{Y}_2', \dots, \mathbf{Y}_m')'$ and the constraints can be given in different forms, e.g. $\mathbf{GBH} + \mathbf{G}_0 = \mathbf{0}$, $\mathbf{GB} + \mathbf{G}_0 = \mathbf{0}$, $\mathbf{BH} + \mathbf{G}_0 = \mathbf{0}$,

2000 Mathematics Subject Classification: Primary 62H12, 62J05.
Keywords: multivariate regression model with constraints, parameter estimator, confidence region, variance component, Wishart matrix.
Supported by the Council of Czech Government J14/98:153100011.

Springer

$\mathrm{Tr}(\mathbf{G}_i\mathbf{B}) + g_i = 0$, $i = 1,\dots,q$, etc. Here the matrices $\mathbf{G},\mathbf{H},\mathbf{G}_0,\mathbf{G}_i$ are known and also the vector $\boldsymbol{g} = (g_1, g_2, \dots, g_q)'$ is known.

The constraints of the type $\mathbf{GB}+\mathbf{G}_0 = \mathbf{0}$ is considered mainly in the literature (e.g. cf. [18]). In the case of modelling deformation measurement this kind of constraints is typical. Let a triangle network covering a part of the Earth surface characterize a state of the investigated area at some time. Measurement of distances and angles in his network is realized at the times $t_1 < \dots < t_m$. From the obtained results a geophysical research of the area (recent crustal movement can be made. However measured distances β_1, β_2, β_3 and angles β_4, β_5, β_6 in a plane triangle must satisfy constraints

$$\beta_4+\beta_5+\beta_6=\pi/2,\quad \beta_1\sin\beta_6=\beta_3\sin\beta_4,\quad \beta_2\sin\beta_4 \quad \beta_2\sin\beta_5.$$

After a linearization of the constraints and some technical adaptation we obtain constraints of the type $\mathbf{GB}+\mathbf{G}_0 = \mathbf{0}$. Constraints $\mathbf{GBH}+\mathbf{G}_0 = \mathbf{0}$ are a moderate generalization useful in other structures of multivariate models, e.g. in growth curve models.

The model considered in the paper is regular if $r(\mathbf{X}_{n,k}) = k < n$, and $\boldsymbol{\Sigma}$ is positive definite (p.d.). The constraints $\mathbf{GBH}+\mathbf{G}_0 = \mathbf{0}$ are regular if $r(\mathbf{G}_{q,k}) = q < k$ & $r(\mathbf{H}_{m,r}) = r < m$. The constraints $\mathrm{Tr}(\mathbf{G}_i\mathbf{B}) + g_i = 0$, $i = 1,\dots,q$ are regular if $r(\tilde{\mathbf{G}}_{q,mk}) = q < km$, where

$$\tilde{\mathbf{G}} = \Big(\big[\mathrm{vec}(\mathbf{G}_1')\big],\dots,\big[\mathrm{vec}(\mathbf{G}_q')\big]\Big)'.$$

The covariance matrix $\boldsymbol{\Sigma}$ can be either totally known, or it is of the form $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, where $\sigma^2\in(0,\infty)$ is an unknown parameter and the $m\times m$ positive definite matrix $\mathbf{V}$ is known, or $\boldsymbol{\Sigma}$ is of the form $\boldsymbol{\Sigma} = \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$, where $\boldsymbol{\vartheta} = (\vartheta_1,\dots,\vartheta_p)'$ is an unknown vector, $\boldsymbol{\vartheta}\in\underline{\vartheta}\subset\mathbb{R}^p$, $\underline{\vartheta}$ is an open set and the $m\times m$ symmetric matrices $\mathbf{V}_1,\dots,\mathbf{V}_p$ are known, or $\boldsymbol{\Sigma}$ is totally unknown.

The aim of the paper is to find explicit formulae for parameter estimator and confidence regions for the parameters, respectively.

2. Parameter estimators

2.1. The matrix $\boldsymbol{\Sigma}$ is known

Lemma 2.1.1. *Let the model and the constraints*

$$\underline{\boldsymbol{Y}} \sim_{nm} (\mathbf{XB}, \boldsymbol{\Sigma}\otimes\mathbf{I}),\qquad \mathbf{G}_{q,k}\mathbf{B}_{k,m}\mathbf{H}_{m,r}+\mathbf{G}_0 = \mathbf{0}_{q,r},$$

be regular. Then the best linear unbiased estimator (BLUE) of the matrix $\mathbf{B}$ *is*

$$\hat{\hat{\mathbf{B}}} = \hat{\mathbf{B}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\hat{\mathbf{B}}\mathbf{H}+\mathbf{G}_0)(\mathbf{H}'\boldsymbol{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{\Sigma}$$

and

$$\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}})] = \mathbf{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1} \\ - [\mathbf{\Sigma}\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{\Sigma}] \otimes \{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\}.$$

Here $\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\boldsymbol{Y}}$.

Proof. In the univariate regular model $\boldsymbol{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \Sigma), \boldsymbol{b}_{q,1} + \mathbf{B}_{q,k}\boldsymbol{\beta} = \boldsymbol{0}$, the BLUE of $\boldsymbol{\beta}$ is

$$\hat{\hat{\boldsymbol{\beta}}} = \hat{\boldsymbol{\beta}} - (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}(\mathbf{B}\hat{\boldsymbol{\beta}} + \boldsymbol{b})$$

and

$$\mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}) = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1} - (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1},$$

where $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\boldsymbol{Y}$ (cf., e.g. [4]). Now it suffices to write the multivariate model in the form

$$\mathrm{vec}(\underline{\boldsymbol{Y}}) \sim_{nm} [(\mathbf{I} \otimes \mathbf{X})\,\mathrm{vec}(\mathbf{B}), \mathbf{\Sigma} \otimes \mathbf{I}], \qquad (\mathbf{H}' \otimes \mathbf{G})\,\mathrm{vec}(\mathbf{B}) + \mathrm{vec}(\mathbf{G}_0) = \boldsymbol{0}$$

and to use the equality $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{B})$. □

Corollary 2.1.2. *Let in the regular model the regular constraints are of the form* $\mathbf{G}_{q,k}\mathbf{B}_{k,m} + \mathbf{G}_{0,(q,m)} = \mathbf{0}$. *Then*

$$\widehat{\widehat{\mathbf{B}}} = \widehat{\mathbf{B}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}} + \mathbf{G}_0),$$

$$\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}})] = \Sigma \otimes \Big\{(\mathbf{X}'\mathbf{X})^{-1} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\Big\} \\ = \mathbf{\Sigma} \otimes \Big[\mathbf{M}_{G'}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'}\Big]^{+},$$

where $\mathbf{M}_{G'} = \mathbf{I} - \mathbf{P}_{G'}$, $\mathbf{P}_{G'} = \mathbf{G}'(\mathbf{G}\mathbf{G}')^{-1}\mathbf{G}$. *(The notation* $^{+}$ *means the Moore-Penrose generalized inverse of the matrix (cf. in more detail* [19]*).*

Remark 2.1.3. In Corollary 2.1.2 the BLUE of $\mathbf{B}$ does not require the knowledge of $\mathbf{\Sigma}$.

Corollary 2.1.4. *If the regular constraints are of the form* $\mathbf{B}_{k,m}\mathbf{H}_{m,r} + \mathbf{G}_{0,(k,r)} - \boldsymbol{0}_{k,r}$, *then*

$$\widehat{\widehat{\mathbf{B}}} = \widehat{\mathbf{B}} - (\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{\Sigma},$$

$$\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}})] = [\mathbf{\Sigma} - \mathbf{\Sigma}\mathbf{H}(\mathbf{H}'\mathbf{\Sigma}\mathbf{H})^{-1}\mathbf{H}'\mathbf{\Sigma}] \otimes (\mathbf{X}'\mathbf{X})^{-1} \\ = (\mathbf{M}_H\mathbf{\Sigma}^{-1}\mathbf{M}_H)^{+} \otimes (\mathbf{X}'\mathbf{X})^{-1}.$$

Here the knowledge of $\mathbf{\Sigma}$ *is essential.*

Lemma 2.1.5. *Let the constraints be of the form*

$$\mathrm{Tr}(\mathbf{G}_1\mathbf{B}) + g_1 = 0, \ \ldots, \ \mathrm{Tr}(\mathbf{G}_q\mathbf{B}) + g_q = 0,$$

and the matrix $\tilde{\mathbf{G}} = \Big(\mathrm{vec}(\mathbf{G}_1'), \ldots, \mathrm{vec}(\mathbf{G}_q')\Big)'$ *is of the full rank in rows, i.e.* $r(\tilde{\mathbf{G}}) = q < km$. *Then*

$$\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}) = \mathrm{vec}(\widehat{\mathbf{B}}) - [\boldsymbol{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1}]\tilde{\mathbf{G}}'\Big\{\tilde{\mathbf{G}}\big[\boldsymbol{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1}\big]\tilde{\mathbf{G}}'\Big\}^{-1}\big[\mathbf{G}\,\mathrm{vec}(\widehat{\mathbf{B}}) + \boldsymbol{g}\big],$$

$$\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}})] = \Big\{\mathbf{M}_{\tilde{G}}[\boldsymbol{\Sigma}^{-1} \otimes (\mathbf{X}'\mathbf{X})]\mathbf{M}_{\tilde{G}}\Big\}^{+}.$$

P r o o f. Proof is analogous as in Lemma 2.1.1.

2.2. The matrix $\boldsymbol{\Sigma}$ is of the form $\sigma^2\mathbf{V}$

Lemma 2.2.1. *Under the assumption of* Lemma 2.1.1 *the estimator of* σ^2 *is*

$$\hat{\sigma}_I^2 = \mathrm{Tr}(\underline{\boldsymbol{v}}_I'\underline{\boldsymbol{v}}_I\mathbf{V}^{-1})/[m(n-k) + qr],$$

where

$$\begin{aligned} \underline{\boldsymbol{v}}_I &= \underline{\boldsymbol{Y}} - \mathbf{X}\widehat{\widehat{\mathbf{B}}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{k}}_I, \quad \underline{\boldsymbol{v}} = \mathbf{M}_X\underline{\boldsymbol{Y}}, \\ \underline{\boldsymbol{k}}_I &= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}, \\ \widehat{\mathbf{B}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\boldsymbol{Y}}. \end{aligned}$$

The underlined symbols are used in order to emphasize that matrices (not vectors) are under consideration.

If the observation matrix is normally distributed, then

$$\hat{\sigma}_I^2 \sim \sigma^2\chi^2_{m(n-k)+qr}/[m(n-k) + qr].$$

P r o o f. It is a consequence of the analogous statement on the estimator in the univariate regular regression model $\boldsymbol{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V}), \boldsymbol{b} + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{0}$, where

$$\sigma_I^2 = \boldsymbol{v}_I'\mathbf{V}^{-1}\boldsymbol{v}_I/(n + q - k),$$

$$\boldsymbol{v}_I = \boldsymbol{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}} = \boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} + \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}(\mathbf{B}\boldsymbol{\beta} + \boldsymbol{b}),$$

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\boldsymbol{Y}.$$

□

Lemma 2.2.2. *Let* $\boldsymbol{v}, \underline{\boldsymbol{v}}_I$ *and* $\underline{\boldsymbol{k}}_I$ *be matrices from* Lemma 2.2.1. *If* $\underline{\boldsymbol{Y}}$ *is normally distributed, then* $\underline{\boldsymbol{v}}$ *and* $\underline{\boldsymbol{k}}_I$ *are stochastically independent and*

$$\underline{\boldsymbol{k}}_I'\underline{\boldsymbol{k}}_I \sim W_m[q, \sigma^2\mathbf{V}\mathbf{H}(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}], \qquad \underline{\boldsymbol{v}}_I'\underline{\boldsymbol{v}}_I = \underline{\boldsymbol{v}}'\underline{\boldsymbol{v}} + \underline{\boldsymbol{k}}_I'\underline{\boldsymbol{k}}_I,$$

where $\underline{\boldsymbol{v}}'\underline{\boldsymbol{v}} \sim W_m[(n-k), \sigma^2\mathbf{V}]$. *Here* $W_m(f, \mathbf{U})$ *means the m-dimensional Wishart distribution with* f *degrees of freedom and variance matrix* $\mathbf{U}$.

Proof. Since $\underline{\mathbf{v}}$ and $\widehat{\mathbf{B}}$ are stochastically independent, also the matrices $\underline{\mathbf{v}}$ and $\underline{\mathbf{k}}_I$ are stochastically independent. Further the implication

$$\mathbf{U}_{q,r} \sim N_{qr}(\mathbf{0}, \mathbf{T}_{r,r} \otimes \mathbf{S}_{q,q}) \implies \mathbf{U}'\mathbf{S}^{-}\mathbf{U} \sim W_r[r(\mathbf{S}), \mathbf{T}]$$

will be utilized. Since

$$(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0) \sim N_{qr}\Big\{\mathbf{0}, (\mathbf{H}'\Sigma\mathbf{H}) \otimes \big[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'\big]\Big\},$$

we have

$$(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0) \sim W_r[q, (\mathbf{H}'\Sigma\mathbf{H})]$$

and because of

$$\underline{\mathbf{k}}_I'\underline{\mathbf{k}}_I = \mathbf{V}\mathbf{H}(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)'[\mathbf{G}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V},$$

it is valid $\underline{\mathbf{k}}_I'\underline{\mathbf{k}}_I \sim W_m[q, \sigma^2\mathbf{V}\mathbf{H}(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}]$.

Further the equality $\underline{\mathbf{k}}_I'\underline{\mathbf{v}} = \mathbf{0}$ can be easily verified and therefore

$$\underline{\mathbf{v}}_I'\underline{\mathbf{v}}_I = \underline{\mathbf{v}}'\underline{\mathbf{v}} + \underline{\mathbf{k}}_I'\underline{\mathbf{k}}_I.$$

□

Remark 2.2.3. The relationships

$$\begin{aligned} \mathrm{vec}(\underline{\mathbf{v}}) &\sim N_{nm}[\mathbf{0}, \sigma^2(\mathbf{V} \otimes \mathbf{M}_X)], \\ \mathrm{vec}(\underline{\mathbf{k}}_I) &\sim N_{nm}\Big[\mathbf{0}, \sigma^2\Big([\mathbf{V}\mathbf{H}(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}] \otimes \\ &\otimes \Big\{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big\}\Big)\Big], \end{aligned}$$

and

$$\begin{aligned} \mathrm{vec}(\underline{\mathbf{v}}_I) &\sim N_{nm}\Big[\mathbf{0}, \sigma^2\Big(\mathbf{V} \otimes \mathbf{M}_X + [\mathbf{V}\mathbf{H}(\mathbf{H}'\mathbf{V}\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}] \otimes \\ &\otimes \Big\{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big\}\Big)\Big], \end{aligned}$$

are implied by Lemma 2.2.1. It can be easily verified that the matrix

$$\sigma^{-2}(\mathbf{V}^{-1} \otimes \mathbf{I})$$

is generalized inverse of all matrices $\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}})]$, $\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{k}}_I)]$ and $\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}}_I)]$, respectively.

Since $\mathbf{W} \sim W_m(f, \sigma^2\mathbf{T}) \implies \mathrm{Tr}(\mathbf{W}\mathbf{T}^-) = \sigma^2\chi^2_{r(T)f}$, we have

$$\begin{aligned} \hat{\sigma}^2 &= \mathrm{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{V}^{-1})/[m(n-k)], \\ \hat{\sigma}^2_{\text{corr}} &= \mathrm{Tr}(\underline{\boldsymbol{k}}_I'\underline{\boldsymbol{k}}_I\mathbf{V}^{-1})/(qr), \\ \hat{\sigma}^2_I &= \mathrm{Tr}(\underline{\boldsymbol{v}}_I'\underline{\boldsymbol{v}}_I\mathbf{V}^{-1})/[m(n-k)+qr], \end{aligned}$$

(cf. also Lemma 2.2.1).

Here $\hat{\sigma}^2$ is the best estimator (i.e. it is unbiased and its dispersion is smallest in the class of unbiased estimators of σ^2) in the model (without constraints) $\underline{\mathbf{Y}} \sim N_{nm}(\mathbf{XB}, \sigma^2\mathbf{V}\otimes\mathbf{I})$. The symbol $\hat{\sigma}^2_I$ denotes the best estimator of σ^2 in the model (with constraints) $\underline{\mathbf{Y}} \sim N_{nm}(\mathbf{XB}, \sigma^2\mathbf{V}\otimes\mathbf{I})$, $\mathbf{GB}+\mathbf{G}_0 = \mathbf{0}$. The symbol $\hat{\sigma}^2_{\text{corr}}$ denotes a correction term (due to constraints) which must be used in order to obtain the estimator $\hat{\sigma}^2_I$ of σ^2.

Thus

$$\hat{\sigma}^2_I = [m(n-k)\hat{\sigma}^2 + qr\hat{\sigma}^2_{\text{corr}}]/[m(n-k)+qr]$$

and we can judge the influence of the constraints $\mathbf{GBH}+\mathbf{G}_0 = \mathbf{0}$ on the estimator of σ^2.

Remark 2.2.4. If the matrix $\boldsymbol{\Sigma}$ is of the form $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, there is no problem to write directly expressions for the estimators considered, since they do not depend on the parameter σ^2. The parameter σ^2 occurs in their covariance matrices only and thus it must be estimated by the help of $\hat{\sigma}^2_I$ from Lemma 2.2.1.

2.3. The matrix $\boldsymbol{\Sigma}$ is of the form $\sum_{i=1}^{p} \vartheta_i\mathbf{V}_i$

Lemma 2.3.1. *In the univariate regular model*

$$\boldsymbol{Y} \sim_n \left(\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\right), \qquad \boldsymbol{b}+\mathbf{B}\boldsymbol{\beta} = \boldsymbol{0},$$

the function $h(\boldsymbol{\vartheta}) = \boldsymbol{h}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, can be estimated by MINQUE iff

$$\boldsymbol{h} \in \mathcal{M}\left(\mathbf{S}_{(M_{XM_{B'}}\Sigma_0 M_{XM_{B'}})^+}\right),$$

where

$$\mathcal{M}\left(\mathbf{S}_{(M_{XM_{B'}}\Sigma_0 M_{XM_{B'}})^+}\right) = \left\{\mathbf{S}_{(M_{XM_{B'}}\Sigma_0 M_{XM_{B'}})^+}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^p\right\},$$

$$\left\{\mathbf{S}_{(M_{XM_{B'}}\Sigma_0 M_{XM_{B'}})^+}\right\}_{i,j} = \mathrm{Tr}\Big[\left(\mathbf{M}_{XM_{B'}}\boldsymbol{\Sigma}_0\mathbf{M}_{XM_{B'}}\right)^+\mathbf{V}_i \times$$
$$\times \left(\mathbf{M}_{XM_{B'}}\boldsymbol{\Sigma}_0\mathbf{M}_{XM_{B'}}\right)^+\mathbf{V}_j\Big], \quad i,j = 1,\ldots,p.$$

If this condition is satisfied, then the $\boldsymbol{\vartheta}^{(0)}$*-MINQUE is*

$$\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i \boldsymbol{v}_I' \boldsymbol{\Sigma}_0^{-1} \mathbf{V}_i \boldsymbol{\Sigma}_0^{-1} \boldsymbol{v}_I,$$

where

$$\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_i^{(0)} \mathbf{V}_i, \quad \boldsymbol{\vartheta}^{(0)} = (\vartheta_1^{(0)}, \dots, \vartheta_p^{(0)})',$$

$$\mathbf{S}_{(M_{XM_{B'}}\Sigma_0 M_{XM_{B'}})^+} \boldsymbol{\lambda} = \boldsymbol{h}, \quad \boldsymbol{v}_I = \boldsymbol{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}},$$

$$\hat{\hat{\boldsymbol{\beta}}} = \hat{\boldsymbol{\beta}} - (\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}(\mathbf{B}\hat{\boldsymbol{\beta}} + \boldsymbol{b}),$$

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{Y}$$

and $\boldsymbol{\vartheta}^{(0)}$ *is an approximate value of the vector* $\boldsymbol{\vartheta}$*.*

P r o o f. The considered model can be rewritten as

$$\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0 \sim_n \left(\mathbf{X}\mathbf{K}_B\gamma, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right), \qquad \gamma \in \mathbb{R}^{k-q}, \ \mathcal{M}(\mathbf{K}_B) = \mathcal{K}er(\mathbf{B}),$$

where $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta}_0 = \boldsymbol{0}$, i.e. $\boldsymbol{\beta}_0$ is any solution of the equation $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{0}$. Then the $\boldsymbol{\vartheta}_0$-MINQUE can be written in the form ([20])

$$\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0)'(\mathbf{M}_{XK_B}\boldsymbol{\Sigma}_0\mathbf{M}_{XK_B})^+ \mathbf{V}_i (\mathbf{M}_{XK_B}\boldsymbol{\Sigma}_0\mathbf{M}_{XK_B})^+ (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0).$$

Since

$$\begin{aligned}
&(\mathbf{M}_{XK_B}\boldsymbol{\Sigma}_0\mathbf{M}_{XK_B})^+(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0)\\
&= \left[\boldsymbol{\Sigma}_0^{-1} - \boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{M}_{B'}(\mathbf{M}_{B'}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{M}_{B'})^+\mathbf{M}_{B'}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\right](\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0)\\
&= \boldsymbol{\Sigma}_0^{-1}\Big[\boldsymbol{Y} - \mathbf{X}\Big(\boldsymbol{\beta}_0 + \Big\{\mathbf{I} - (\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}\mathbf{B}\Big\} \times\\
&\quad \times (\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{Y}\Big) + \Big\{\mathbf{X}\boldsymbol{\beta}_0 + \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{B}' \times\\
&\quad \times [\mathbf{B}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}(-\boldsymbol{b})\Big\}\Big] = \boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}) = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{v}_I,
\end{aligned}$$

we have the explicit expression for $\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}}$. □

Corollary 2.3.2. *In the regular multivariate model*

$$\underline{\boldsymbol{Y}} \sim_{nm} \Big(\mathbf{XB}, \sum_{i=1}^{p} \vartheta_i (\mathbf{V}_i \otimes \mathbf{I})\Big), \qquad \mathbf{G}_{q,k}\mathbf{B}_{k,m}\mathbf{H}_{m,r} + \mathbf{G}_{0\,(q,r)} = \mathbf{0},$$

the $\boldsymbol{\vartheta}_0$-MINQUE of the function $h(\boldsymbol{\vartheta}) = \boldsymbol{h}'\boldsymbol{\vartheta}, \boldsymbol{\vartheta} \in \underline{\vartheta}$, exists iff

$$\boldsymbol{h} \in \mathcal{M}\Big[(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'}\Big],$$

where

$$\Big\{\mathbf{S}_{\Sigma_0^{-1}}\Big\}_{i,j} = \mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j),$$

$$\Big\{\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'}\Big\}_{i,j} = \mathrm{Tr}\Big[\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}_0\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}_0\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_j\Big],$$

$$i, j = 1, \dots, p.$$

The $\boldsymbol{\vartheta}_0$-MINQUE is

$$\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i \,\mathrm{Tr}(\underline{\boldsymbol{v}}_I'\boldsymbol{v}_I\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}),$$

where

$$\Big[(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'}\Big]\boldsymbol{\lambda} = \boldsymbol{h}.$$

P r o o f. With respect to Lemma 2.3.1

$$\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i [\mathrm{vec}(\underline{\boldsymbol{v}}_I)]'(\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{I})(\mathbf{V}_i \otimes \mathbf{I})(\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{I})\,\mathrm{vec}(\boldsymbol{v}_I)$$

$$= \sum_{i=1}^{p} \lambda_i \,\mathrm{Tr}(\boldsymbol{v}_I'\underline{\boldsymbol{v}}_I\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}),$$

where

$$\boldsymbol{h} \in \mathcal{M}(\mathbf{S}_A), \qquad \mathbf{A} = \Big[\mathbf{M}_{(I\otimes X)K_{(H'\otimes G)}}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\mathbf{M}_{(I\otimes X)K_{(H'\otimes G)}}\Big]^+.$$

The matrix $\mathbf{A}$ can be rewritten as follows

$$\mathbf{A} = (\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{I}) - (\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{X})\Big[\mathbf{M}_{H\otimes G'}(\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{I})\mathbf{M}_{H\otimes G'}\Big]^+(\mathbf{I} \otimes \mathbf{X}')(\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{I})$$

$$= \boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{M}_X + [\mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}_0\mathbf{H})^{-1}\mathbf{H}'] \otimes \Big\{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1} \times$$

$$\times\, \mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big\} = \boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{M}_X + \mathbf{A}_1 \otimes \mathbf{A}_2, \quad \mathbf{A}_1 = \mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}_0\mathbf{H})^{-1}\mathbf{H}',$$

$$\mathbf{A}_2 = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \quad \text{(idempotent matrix).}$$

Further

$$\mathrm{Tr}[\mathbf{A}(\mathbf{V}_i \otimes \mathbf{I})\mathbf{A}(\mathbf{V}_j \otimes \mathbf{I})] = \mathrm{Tr}\Big[(\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{M}_X + \mathbf{A}_1 \otimes \mathbf{A}_2)(\mathbf{V}_i \otimes \mathbf{I})(\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{M}_X$$
$$+ \mathbf{A}_1 \otimes \mathbf{A}_2)(\mathbf{V}_j \otimes \mathbf{I})\Big] = \mathrm{Tr}\Big((\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) \otimes \mathbf{M}_X$$
$$+ \Big[\mathbf{H}(\mathbf{H}'\Sigma_0\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}(\mathbf{H}'\Sigma_0\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_j\Big] \otimes \Big\{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}' \times$$
$$\times [\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Big\}\Big) = \mathrm{Tr}(\mathbf{M}_X)\,\mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{V}_j)$$
$$+ \mathrm{Tr}\Big\{[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'\Big\} \times$$
$$\times \mathrm{Tr}\Big[\mathbf{H}(\mathbf{H}'\Sigma_0\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}(\mathbf{H}'\Sigma_0\mathbf{H})^{-1}\mathbf{H}'\mathbf{V}_j\Big]$$
$$= \Big\{(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'}\Big\}_{i,j}.$$

□

Remark 2.3.3. If the matrix $(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'}$ is regular, then the vector $\boldsymbol{\vartheta}$ can be estimated and

$$\hat{\boldsymbol{\vartheta}} = \Big[(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'}\Big]^{-1} \begin{pmatrix} \mathrm{Tr}(\underline{\mathbf{v}}_I'\underline{\mathbf{v}}_I\Sigma_0^{-1}\mathbf{V}_1\Sigma_0^{-1}) \\ \vdots \\ \mathrm{Tr}(\underline{\mathbf{v}}_I'\underline{\mathbf{v}}_I\Sigma_0^{-1}\mathbf{V}_1\Sigma_0^{-1}) \end{pmatrix}.$$

In the case of normality

$$\mathrm{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = 2\Big[(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'}\Big]^{-1}.$$

Remark 2.3.4. If $\mathbf{H}_{m,r} = \mathbf{I}_{m,m}$, then

$$(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'} = (n+q-k)\mathbf{S}_{\Sigma_0^{-1}}.$$

If $\mathbf{G}_{q,k} - \mathbf{I}_{k,k}$, then

$$(n-k)\mathbf{S}_{\Sigma_0^{-1}} + q\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'} = n\mathbf{S}_{\Sigma_0^{-1}} + k\Big(\mathbf{S}_{H(H'\Sigma_0 H)^{-1}H'} - \mathbf{S}_{\Sigma_0^{-1}}\Big).$$

Lemma 2.3.5. *Let the model*

$$\mathrm{vec}(\boldsymbol{Y}) \sim_{nm} \Big[(\mathbf{I} \otimes \mathbf{X})\,\mathrm{vec}(\mathbf{B}), \boldsymbol{\Sigma} \otimes \mathbf{I})\Big], \qquad \tilde{\mathbf{G}}\,\mathrm{vec}(\mathbf{B}) + \boldsymbol{g} = \boldsymbol{0},$$
$$\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i(\mathbf{V}_i \otimes \mathbf{I}),$$

be regular. Then the $\boldsymbol{\vartheta}_0$*-MINQUE exists for the function* $h(\boldsymbol{\vartheta}) \quad \boldsymbol{h}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \boldsymbol{\vartheta}$, *iff* $\boldsymbol{h} \in \mathcal{M}(\mathbf{S}_*)$,

$$\{\mathbf{S}_*\}_{i,j} = \mathrm{Tr}\Big(\Big\{\tilde{\mathbf{G}}[\boldsymbol{\Sigma}_0 \quad (\mathbf{X}'\mathbf{X})^{-1}]\mathbf{G}'\Big\}^{\ 1}\mathbf{G}[(\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) \quad (\mathbf{X}'\mathbf{X})^{\ 1}]\tilde{\mathbf{G}}'\Big),$$
$$i, j = 1, \dots, p,$$

and then the $\boldsymbol{\vartheta}_0$*-MINQUE of* $h(\cdot)$ *is* $\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}} - \sum_{\imath \ 1}^{p} \lambda_\imath \,\mathrm{Tr}(\underline{\boldsymbol{v}}_I'\underline{\boldsymbol{v}}_I\boldsymbol{\Sigma}_0^{\ 1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{\ 1})$. *Here*

$$\underline{\boldsymbol{v}}_I = \underline{\boldsymbol{Y}} - \mathbf{X}\widehat{\widehat{\mathbf{B}}}, \quad \mathbf{S}_*\boldsymbol{\lambda} = \boldsymbol{h},$$
$$\widehat{\widehat{\mathbf{B}}} - \widehat{\mathbf{B}} - \mathrm{devec}\Big([\boldsymbol{\Sigma}_0 \otimes (\mathbf{X}'\mathbf{X})^{-1}]\mathbf{G}'\Big\{\tilde{\mathbf{G}}[\boldsymbol{\Sigma}_0 \otimes (\mathbf{X}'\mathbf{X})^{-1}]\tilde{\mathbf{G}}'\Big\}^{\ 1}\Big(\mathbf{G}\,\mathrm{vec}(\mathbf{B}) + \boldsymbol{g}\Big)\Big).$$

*The operation "*devec*" creates the* $k \times m$ *matrix from the* mk*-dimensional vector* $[\boldsymbol{\Sigma}_0 \otimes (\mathbf{X}'\mathbf{X})^{-1}]\tilde{\mathbf{G}}'\Big\{\tilde{\mathbf{G}}[\boldsymbol{\Sigma}_0 \otimes (\mathbf{X}'\mathbf{X})^{-1}]\tilde{\mathbf{G}}'\Big\}^{\ 1}\Big(\mathbf{G}\,\mathrm{vec}(\mathbf{B}) + \boldsymbol{g}\Big)$.

If the matrix $\mathbf{S}_*$ *is regular, then*

$$\boldsymbol{\vartheta} = \mathbf{S}_*^{-1}\begin{pmatrix} \mathrm{Tr}(\underline{\boldsymbol{v}}_I'\underline{\boldsymbol{v}}_I\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}) \\ \vdots \\ \mathrm{Tr}(\underline{\boldsymbol{v}}_I'\underline{\boldsymbol{v}}_I\boldsymbol{\Sigma}_0^{\ 1}\mathbf{V}_p\boldsymbol{\Sigma}_0^{-1}) \end{pmatrix}.$$

If $\underline{\boldsymbol{Y}}$ *is normally distributed, then* $\mathrm{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = 2\mathbf{S}_*^{-1}$.

P r o o f. With respect to Lemma 2.3.1

$$\{\mathbf{S}_*\}_{i,j} - \mathrm{Tr}\Big\{\Big[\mathbf{M}_{(I\otimes X)M_{G'}}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\mathbf{M}_{(I\otimes X)M_{\tilde{G}'}}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\Big] \quad (\mathbf{V}_\imath \quad \mathbf{I})$$
$$\times \Big[\mathbf{M}_{(I\otimes X)M_{G'}}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\mathbf{M}_{(I\otimes X)M_{G'}}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\Big]^+(\mathbf{V}_j \quad \mathbf{I})\Big\}.$$

Here

$$\Big[\mathbf{M}_{(I\otimes X)M_{\tilde{G}'}}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\mathbf{M}_{(I\otimes X)M_{G'}}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\Big]^+$$
$$= \boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{I} - (\boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{X})\big\{\mathbf{M}_{\tilde{G}}[\boldsymbol{\Sigma}_0^{-1} \quad (\mathbf{X}'\mathbf{X})]\mathbf{M}_{\tilde{G}}\big\}^+(\boldsymbol{\Sigma}_0^{\ 1} \otimes \mathbf{X}'),$$

and thus

$$\{\mathbf{S}_*\}_{i,j} = n\,\mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) - 2\,\mathrm{Tr}\Big(\Big\{\mathbf{M}_{\tilde{G}'}\big[\boldsymbol{\Sigma}_0^{-1}\otimes(\mathbf{X}'\mathbf{X})\big]\mathbf{M}_{\tilde{G}'}\Big\}^+ \times$$
$$\times\big[(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1})\otimes(\mathbf{X}'\mathbf{X})\big]\Big) + \mathrm{Tr}\Big(\Big\{\mathbf{M}_{\tilde{G}'}\big[\boldsymbol{\Sigma}_0^{-1}\otimes(\mathbf{X}'\mathbf{X})\big]\mathbf{M}_{\tilde{G}'}\Big\}^+ \times$$
$$\times\big[(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1})\otimes(\mathbf{X}'\mathbf{X})\big]\Big\{\mathbf{M}_{\tilde{G}'}\big[\boldsymbol{\Sigma}_0^{-1}\otimes(\mathbf{X}'\mathbf{X})\big]\mathbf{M}_{\tilde{G}'}\Big\}^+ \times$$
$$\times\big[(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1})\otimes(\mathbf{X}\mathbf{X}')\big]\Big).$$

Further

$$-2\,\mathrm{Tr}\Big(\Big\{\mathbf{M}_{\tilde{G}'}\big[\boldsymbol{\Sigma}_0^{-1}\otimes(\mathbf{X}'\mathbf{X})\big]\mathbf{M}_{\tilde{G}'}\Big\}^+\big[(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1})\otimes(\mathbf{X}'\mathbf{X})\big]\Big)$$
$$= -2n\,\mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) + 2\,\mathrm{Tr}\Big(\Big\{\tilde{\mathbf{G}}\big[\boldsymbol{\Sigma}_0\otimes(\mathbf{X}'\mathbf{X})^{-1}\big]\tilde{\mathbf{G}}'\Big\}^{-1} \times$$
$$\times\,\tilde{\mathbf{G}}\big[(\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j)\otimes(\mathbf{X}'\mathbf{X})^{-1}\big]\tilde{\mathbf{G}}'\Big),$$

$$\mathrm{Tr}\Big(\Big\{\mathbf{M}_{\tilde{G}'}\big[\boldsymbol{\Sigma}_0^{-1}\otimes(\mathbf{X}'\mathbf{X})\big]\mathbf{M}_{\tilde{G}'}\Big\}^+\big[(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1})\otimes(\mathbf{X}'\mathbf{X})\big] \times$$
$$\times\Big\{\mathbf{M}_{\tilde{G}'}\big[\boldsymbol{\Sigma}_0^{-1}\otimes(\mathbf{X}'\mathbf{X})\big]\mathbf{M}_{\tilde{G}'}\Big\}^+\big[(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1})\otimes(\mathbf{X}\mathbf{X}')\big]\Big)$$
$$= n\,\mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) - \mathrm{Tr}\Big(\Big\{\tilde{\mathbf{G}}\big[\boldsymbol{\Sigma}_0\otimes(\mathbf{X}'\mathbf{X})^{-1}\big]\tilde{\mathbf{G}}'\Big\}^{-1} \times$$
$$\times\,\tilde{\mathbf{G}}\big[(\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j)\otimes(\mathbf{X}'\mathbf{X})^{-1}\big]\tilde{\mathbf{G}}'\Big).$$

The rest of the proof is obvious. □

Remark 2.3.6. If the constraints are given in the form $\mathbf{GBH} + \mathbf{G}_0 = \mathbf{0}$, $\mathbf{H} \neq \mathbf{I}$ and $\tilde{\mathbf{G}}\,\mathrm{vec}(\mathbf{B}) + \boldsymbol{g} = \boldsymbol{0}$, the the $\boldsymbol{\vartheta}_0$-locally best linear estimator of $\mathbf{B}$ is known only. However if the estimator of $\boldsymbol{\vartheta}$ is sufficiently precise, then the estimator $\hat{\boldsymbol{\vartheta}}$ can be used instead of the actual value $\boldsymbol{\vartheta}^*$ of the parameter $\boldsymbol{\vartheta}$. What means "sufficiently precise" is commented in Section 3.

2.4. The matrix $\boldsymbol{\Sigma}$ is totally unknown

Analogously as in Remark 2.3.6 the constraints $\mathbf{GBH} + \mathbf{G}_0 = \mathbf{0}$, $\mathbf{H} \neq \mathbf{I}$ and $\tilde{\mathbf{G}}\,\mathrm{vec}(\mathbf{B}) + \boldsymbol{g} = \boldsymbol{0}$, respectively, make problems in the estimation of $\mathbf{B}$ when the matrix $\boldsymbol{\Sigma}$ is totally unknown. A $\boldsymbol{\Sigma}_0$-locally best estimator of $\mathbf{B}$ can be obtained easily, however an investigation of statistical properties of a plug-in estimator, i.e. the estimator of $\mathbf{B}$ with an estimated covariance matrix, is difficult. One possibility offers the following Lemma 2.4.1. However it is necessary to say something in advance.

Analogously as in [17] and [3] let an univariate regular model

$$\boldsymbol{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \Sigma), \qquad \boldsymbol{b} + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{0},$$

be under consideration. Let $\tilde{\boldsymbol{\beta}}$ be any unbiased estimator of $\boldsymbol{\beta}$, e.g. $\tilde{\boldsymbol{\beta}} = \mathbf{M}_{B'}\mathbf{X}^{-}\boldsymbol{Y} - \mathbf{B}'(\mathbf{B}\mathbf{B}')^{-1}\boldsymbol{b}$, and let $\tilde{\boldsymbol{v}} = \mathbf{Z}\boldsymbol{Y} + \boldsymbol{z} \sim N_n(\boldsymbol{0}, \mathbf{W})$. The class of all linear unbiased estimators of zero function of the parameter $\boldsymbol{\beta}$ is

$$\mathcal{L}_0 = \{\boldsymbol{u}'\mathbf{M}_{XM_{B'}}\boldsymbol{Y} + \boldsymbol{u}'\mathbf{M}_{XM_{B'}}\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^{-1}\boldsymbol{b}: \ \boldsymbol{u} \in \mathbb{R}^n\},$$

i.e. $\hat{\hat{\boldsymbol{\beta}}}$ is the BLUE of $\boldsymbol{\beta}$ iff $\forall\{\boldsymbol{u} \in \mathbb{R}^n\}\operatorname{cov}(\boldsymbol{u}'\mathbf{M}_{XM_{B'}}, \hat{\hat{\boldsymbol{\beta}}}) = \mathbf{0}$ (in more detail cf. [4, Chap. 10]). This class is created of all linear combination of the components of the vector $\mathbf{Z}\boldsymbol{Y} + \boldsymbol{z} \sim N_n(\boldsymbol{0}, \mathbf{W})$. Further

$$\begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{v}} \end{pmatrix} \sim N_{k+n}\left[\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{0} \end{pmatrix}, \begin{pmatrix} \mathbf{U}, & \mathbf{V} \\ \mathbf{V}', & \mathbf{W} \end{pmatrix}\right].$$

If $f\mathbf{S} = \sum_{\alpha=1}^{f} \boldsymbol{u}_\alpha \boldsymbol{u}'_\alpha \sim W_n(f, \Sigma)$ (Wishart distribution with f degrees of freedom), i.e. $\boldsymbol{u}_\alpha \sim N_n(\boldsymbol{0}, \Sigma)$, $\alpha = 1, \dots, f$, and $\boldsymbol{u}_1, \dots, \boldsymbol{u}_f$, are stochastically independent, then

$$f\begin{pmatrix} \widehat{\mathbf{U}}, & \widehat{\mathbf{V}} \\ \widehat{\mathbf{V}}', & \widehat{\mathbf{W}} \end{pmatrix} = \sum_{\alpha=1}^{f}(\mathbf{V}'_{\alpha,1}, \mathbf{V}'_{\alpha,2})'(\mathbf{V}'_{\alpha,1}, \mathbf{V}'_{\alpha,2}) \sim W_{k+n}\left[f, \begin{pmatrix} \mathbf{U}, & \mathbf{V} \\ \mathbf{V}', & \mathbf{W} \end{pmatrix}\right],$$

where $(\mathbf{V}'_{\alpha,1}, \mathbf{V}'_{\alpha,2})' = \begin{pmatrix} \mathbf{M}_{B'}\mathbf{X}^{-} \\ \mathbf{Z} \end{pmatrix} \boldsymbol{u}_\alpha$, $\alpha = 1, \dots, f$.

In the following text the symbol $^{(p)}$ means

"conditioned by $(\mathbf{V}_{1,2}, \dots, \mathbf{V}_{f,2}, \tilde{\boldsymbol{v}})$".

If the matrix $\mathbf{S}$ is substituted into the BLUE $\hat{\hat{\boldsymbol{\beta}}}$ of $\boldsymbol{\beta}$ instead of Σ, then we obtained the plug-in estimator denoted as $\tilde{\tilde{\boldsymbol{\beta}}}$.

Lemma 2.4.1. *If the Wishart matrix $f\mathbf{S} \sim W_n(f, \Sigma)$, $f > n+1$, is independent of the observation vector $\boldsymbol{Y}$, then*

$$\tilde{\tilde{\boldsymbol{\beta}}}^{(p)} \sim N_k\left[\boldsymbol{\beta}, \operatorname{Var}(\hat{\hat{\boldsymbol{\beta}}})\left(1 + \frac{\tilde{\boldsymbol{v}}'\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}}}{f}\right)\right],$$

where

$$\begin{aligned}\operatorname{Var}(\hat{\hat{\boldsymbol{\beta}}}) &= \mathbf{U} - \mathbf{V}\mathbf{W}^{-}\mathbf{V}' \\ &= (\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1} - (\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\end{aligned}$$

and

$$f(\widehat{\mathbf{U}} - \widehat{\mathbf{V}}\widehat{\mathbf{W}}^{-}\widehat{\mathbf{V}}') \sim W_k(f - r(\mathbf{W}), \operatorname{Var}(\hat{\hat{\boldsymbol{\beta}}})].$$

P r o o f. The plug-in estimator is $\tilde{\hat{\boldsymbol{\beta}}} = \tilde{\boldsymbol{\beta}} - \widehat{\mathbf{V}}\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}}$ and $\tilde{\hat{\boldsymbol{\beta}}}^{(p)} = \tilde{\boldsymbol{\beta}}^{(p)} - \widehat{\mathbf{V}}^{(p)}\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}}$. Here

$$\tilde{\boldsymbol{\beta}}^{(p)} \sim N_k[\boldsymbol{\beta} - \mathbf{V}\mathbf{W}^{-}\tilde{\boldsymbol{v}}, \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}})],$$

$$E(\widehat{\mathbf{V}}^{(p)}) = (1/f)E\Big(\sum_{\alpha=1}^{f} \mathbf{V}_{\alpha,1}^{(p)}\mathbf{V}_{\alpha,2}'\Big) = (1/f)\sum_{\alpha=1}^{p} \mathbf{V}\mathbf{W}^{-}\mathbf{V}_{\alpha,2}\mathbf{V}_{\alpha,2}' = \mathbf{V}\mathbf{W}^{-}\widehat{\mathbf{W}}.$$

Thus

$$E(\tilde{\hat{\boldsymbol{\beta}}}^{(p)}) = \boldsymbol{\beta} - \mathbf{V}\mathbf{W}^{-}\tilde{\boldsymbol{v}} + \mathbf{V}\mathbf{W}^{-}\widehat{\mathbf{W}}\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}} = \boldsymbol{\beta},$$

since $P\{\tilde{\boldsymbol{v}} \in \mathcal{M}(\widehat{\mathbf{W}})\} = 1 \implies \widehat{\mathbf{W}}\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$.

The vectors $\tilde{\boldsymbol{\beta}}^{(p)}$ and $\widehat{\mathbf{V}}^{(p)}\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}}$ are stochastically independent, thus

$$\mathrm{Var}(\tilde{\hat{\boldsymbol{\beta}}}^{(p)}) = \mathrm{Var}(\tilde{\boldsymbol{\beta}}^{(p)}) + \mathrm{Var}(\widehat{\mathbf{V}}^{(p)}\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}})$$

$$= \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}) + \mathrm{Var}\Big(\sum_{\alpha=1}^{f} \mathbf{V}_{\alpha,1}^{(p)}\mathbf{V}_{\alpha,2}'\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}}\Big) = \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}) + \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}})\sum_{\alpha=1}^{f}(\mathbf{V}_{\alpha,2}'\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}})^2$$

$$= \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}})\left(1 + \frac{\tilde{\boldsymbol{v}}'\widehat{\mathbf{W}}^{-}\tilde{\boldsymbol{v}}}{f}\right).$$

The other statement is well known (cf, e.g. [18]). □

Remark 2.4.2. The residual vector

$$\boldsymbol{v} = \boldsymbol{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}} = \boldsymbol{Y} - \mathbf{X}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol{Y} + \mathbf{X}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{B}' \times$$
$$\times [\mathbf{B}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}[\mathbf{B}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol{Y} + \boldsymbol{b}]$$

depends on the matrix Σ, however $\tilde{\boldsymbol{v}} = \mathbf{Z}\boldsymbol{Y} + \boldsymbol{z}$ does not depend on it. Thus there exists a regular matrix $\mathbf{R}_\Sigma$ with the property $\boldsymbol{v} = \mathbf{R}_\Sigma\tilde{\boldsymbol{v}}$ and

$$\mathrm{Var}(\boldsymbol{v}) = \Sigma - \mathbf{X}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1} \times$$
$$\times \mathbf{B}(\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X}' = \mathbf{R}_\Sigma \mathrm{Var}(\tilde{\mathbf{V}})\mathbf{R}_\Sigma' = \mathbf{R}_\Sigma\mathbf{W}\mathbf{R}_\Sigma'.$$

Let $\hat{\boldsymbol{v}} = \boldsymbol{Y} - \mathbf{X}\tilde{\hat{\boldsymbol{\beta}}}$. Then $\hat{\boldsymbol{v}} = \mathbf{R}_S\tilde{\boldsymbol{v}}$ and

$$\tilde{\boldsymbol{v}}'\widehat{\mathbf{W}}\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}'(\mathbf{Z}\mathbf{S}\mathbf{Z}')^{-}\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}'\mathbf{R}_S'(\mathbf{R}_S\widehat{\mathbf{W}}^{-}\mathbf{R}_S')^{-}\mathbf{R}_S\tilde{\boldsymbol{v}} = \hat{\boldsymbol{v}}'(\mathbf{R}_S\widehat{\mathbf{W}}\mathbf{R}_S')^{-}\hat{\boldsymbol{v}}.$$

However

$$\mathbf{R}_S\widehat{\mathbf{W}}\mathbf{R}_S' = \mathbf{R}_S(\mathbf{Z}\mathbf{S}\mathbf{Z}')\mathbf{R}_S' = \mathbf{S} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{B}' \times$$
$$\times [\mathbf{B}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'.$$

Since one version of the g-inverse of the matrix $\mathbf{R}_S\widehat{\mathbf{W}}\mathbf{R}_S'$ is $\mathbf{S}^{-1}$, we have

$$\tilde{\hat{\boldsymbol{\beta}}}^{(p)} \sim N_k\left[\boldsymbol{\beta}, \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}})\left(1 + \frac{\hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}}}{f}\right)\right],$$

where $\hat{\boldsymbol{v}} = \boldsymbol{Y} - \mathbf{X}\tilde{\hat{\boldsymbol{\beta}}}$.

THEOREM 2.4.3. *Let the multivariate model and constraints*

$$\underline{\boldsymbol{Y}} \sim N_{nm}(\mathbf{XB}, \boldsymbol{\Sigma}\otimes\mathbf{I}), \qquad \mathbf{GBH} + \mathbf{G}_0 = \mathbf{0},$$

be regular. If $f\hat{\mathbf{T}} \sim W_{nm}(f, \boldsymbol{\Sigma}\otimes\mathbf{I} = \mathbf{T})$, $f > nm+1$, *which is stochastically independent of* $\underline{\boldsymbol{Y}}$, *is at our disposal, then*

$$\tilde{\widehat{\mathbf{B}}}^{(p)} \sim N_{km}\left[\mathbf{B}, \mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}})]\left(1 + \frac{[\mathrm{vec}(\hat{\boldsymbol{v}}_I)]'\hat{\mathbf{T}}^{-1}\,\mathrm{vec}(\hat{\boldsymbol{v}}_I)]}{f}\right)\right],$$

where

$$\tilde{\widehat{\mathbf{B}}} = \mathrm{vec}(\widehat{\mathbf{B}}) - \left[(\mathbf{I}\otimes\mathbf{X}')\widehat{\mathbf{T}}^{-1}(\mathbf{I}\otimes\mathbf{X})\right]^{-1}(\mathbf{H}\otimes\mathbf{G}') \times$$
$$\times\left\{(\mathbf{H}'\otimes\mathbf{G})\left[(\mathbf{I}\otimes\mathbf{X}')\widehat{\mathbf{T}}^{-1}(\mathbf{I}\otimes\mathbf{X})\right]^{-1}(\mathbf{H}\otimes\mathbf{G}')\right\}^{-1}\left[(\mathbf{H}'\otimes\mathbf{G})\,\mathrm{vec}(\mathbf{B}) + \mathrm{vec}(\mathbf{G}_0)\right],$$

$$\underline{\hat{\boldsymbol{v}}}_I = \underline{\boldsymbol{Y}} - \mathbf{X}\tilde{\widehat{\mathbf{B}}}, \qquad \widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\boldsymbol{Y}}.$$

P r o o f. Regarding Lemma 2.4.1 and Remark 2.4.2 we have

$$\boldsymbol{u}_\alpha \sim N_{nm}(\mathbf{0}, \mathbf{T}), \qquad \alpha = 1, \dots, f,$$
$$\mathbf{V}_\alpha = \begin{pmatrix} \mathbf{I}\otimes\mathbf{X}^- - \mathbf{P}_H\otimes(\mathbf{P}_{G'}\mathbf{X}^-) \\ \mathbf{Z} \end{pmatrix}\boldsymbol{u}_\alpha \sim N_{km+nm}\left[\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{U}, & \mathbf{V}\\ \mathbf{V}', & \mathbf{W}\end{pmatrix}\right],$$
$$\sum_{\alpha=1}^{f}\begin{pmatrix}\mathbf{V}_{\alpha,1}\\ \mathbf{V}_{\alpha,2}\end{pmatrix}(\mathbf{V}_{\alpha,1}', \mathbf{V}_{\alpha,2}') = f\begin{pmatrix}\widehat{\mathbf{U}}, & \widehat{\mathbf{V}}\\ \widehat{\mathbf{V}}', & \widehat{\mathbf{W}}\end{pmatrix} \sim W_{km+nm}\left[f, \begin{pmatrix}\mathbf{U}, & \mathbf{V}\\ \mathbf{V}', & \mathbf{W}\end{pmatrix}\right].$$

Now it can be proceeded as in Lemma 2.4.1 and Remark 2.4.2. □

Remark 2.4.4. Even if the conditioned estimator $\hat{\tilde{\mathbf{B}}}^{(p)}$ has only restricted application, it can be suitable utilized in the determination of a confidence region.

Since

$$f(\hat{\mathbf{U}} - \hat{\mathbf{V}}\widehat{\mathbf{W}}^{-}\hat{\mathbf{V}}') \sim W_{km}[f - r(\mathbf{W}), \mathrm{Var}[\mathrm{vec}(\hat{\hat{\mathbf{B}}})],$$
$$r(\mathbf{W}) = nm + qr - km,$$
$$\hat{\tilde{\mathbf{B}}}^{(p)} \sim N_{km}\left[\mathbf{B}, \mathrm{Var}[\mathrm{vec}(\hat{\hat{\mathbf{B}}})]\left(1 + \frac{[\mathrm{vec}(\hat{\boldsymbol{v}}_I)]'\hat{\mathbf{T}}^{-1}\,\mathrm{vec}(\hat{\boldsymbol{v}}_I)]}{f}\right)\right]$$

and $\hat{\mathbf{U}} - \hat{\mathbf{V}}\widehat{\mathbf{W}}^{-}\hat{\mathbf{V}}'$ and $\hat{\tilde{\mathbf{B}}}^{(p)}$ are stochastically independent, the Hotelling theorem ([18]) can be used. With respect to it

$$\frac{[\mathrm{vec}(\mathbf{B} - \hat{\tilde{\mathbf{B}}})]'[f(\hat{\mathbf{U}} - \hat{\mathbf{V}}\widehat{\mathbf{W}}^{-}\hat{\mathbf{V}}')]\,\mathrm{vec}(\mathbf{B} - \hat{\tilde{\mathbf{B}}})}{1 + \frac{[\mathrm{vec}(\hat{\boldsymbol{v}}_I)]'\hat{\mathbf{T}}^{-1}\,\mathrm{vec}(\hat{\boldsymbol{v}}_I)}{f}} \sim \frac{\chi^2_{r\{\mathrm{Var}[\mathrm{vec}(\hat{\hat{B}}]\}}}{\chi^2_{f-r(W)-r\{\mathrm{Var}[\mathrm{vec}(\hat{\hat{\mathbf{B}}})]\}+1}}$$

and thus this random variable does not depend on the condition $^{(p)}$ and it can be used for a determination of an exact $(1-\alpha)$-confidence region. Since one version of $(\hat{\mathbf{U}} - \hat{\mathbf{V}}\widehat{\mathbf{W}}^{-}\hat{\mathbf{V}}')^{-}$ is $(\mathbf{I} \otimes \mathbf{X}')\hat{\mathbf{T}}^{-1}(\mathbf{I} \otimes \mathbf{X})$, this region can be written in the form

$$\mathcal{E} = \left\{\mathbf{B}:\ \mathbf{GBH} + \mathbf{G}_0 = \mathbf{0},\ \frac{[\mathrm{vec}(\mathbf{B} - \hat{\tilde{\mathbf{B}}})]'[(\mathbf{I} \otimes \mathbf{X}')\hat{\mathbf{T}}^{-1}(\mathbf{I} \otimes \mathbf{X})]\,\mathrm{vec}(\mathbf{B} - \hat{\tilde{\mathbf{B}}})}{1 + \frac{[\mathrm{vec}(\hat{\boldsymbol{v}}_I)]'\hat{\mathbf{T}}^{-1}\,\mathrm{vec}(\hat{\boldsymbol{v}}_I)}{f}} \right.$$
$$\left. \leq \frac{f(km - qr)}{f - nm + 1} F_{km-qr, f-nm+1}(1-\alpha)\right\}.$$

Remark 2.4.5. Unfortunately a realization of the matrix $\hat{\mathbf{T}}$ cannot be written in the form $\mathbf{S} \otimes \mathbf{I}$. Until now author has not been able to find a matrix of the form $\hat{\Sigma} \otimes \mathbf{I}$ with properties necessary for the validity of Theorem 2.4.3. Thus the explicit formulae are rather rough. Except this the degrees of freedom f must be larger than the number $nm + 1$ in order the matrix $\mathbf{T}$ can be inverted ($f < nm + 1 \implies \mathbf{T}$ is singular) and thus f could be huge number. Therefore the plug-in estimator given in Theorem 2.4.3 will be used rarely.

It seems that the approach given in the next section has a greater chance to be used in applications. This approach is demonstrated for a determination of a variance and a confidence region.

3. Sensitivity approach

Let in the model (1) an estimator of the function $f(\mathbf{B}) = \mathrm{Tr}(\mathbf{AB})$, $\mathbf{B} \in \{\mathbf{U} : \mathbf{GUH} + \mathbf{G}_0 = \mathbf{0}\}$, be considered. If $\mathbf{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, $\boldsymbol{\vartheta} \in \underline{\vartheta} \subset \mathbb{R}^p$, then the $\boldsymbol{\vartheta}_0$-locally best estimator of the function $f(\cdot)$ can be determined only, i.e.

$$\begin{aligned}
\widehat{\widehat{\mathbf{B}}}_{\vartheta} &= \widehat{\mathbf{B}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}[\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta}) \\
&\quad -(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}_0[\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta}) \\
&= \widehat{\mathbf{B}} - \mathbf{K} = \widehat{\mathbf{B}} - \left(\mathbf{P}_{G'}^{(X'X)^{-1}}\right)'\widehat{\mathbf{B}}\mathbf{P}_H^{\Sigma(\vartheta)} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}' \\
&\quad \times [\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}_0[\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta}), \\
\mathbf{K} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)[\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta}).
\end{aligned}$$

Lemma 3.1.

(i) $\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_{\vartheta+\delta\vartheta}) \approx \mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_{\vartheta}) + \boldsymbol{k}'\delta\boldsymbol{\vartheta}$,

$$\{\boldsymbol{k}\}_i = -\mathrm{Tr}\left[\mathbf{AK}\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta)}\right], \qquad i = 1,\dots,p$$

(ii) $\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_{\vartheta})$ *and* $\boldsymbol{k}$ *are uncorrelated and* $E(\boldsymbol{k}) = \boldsymbol{0}$,

$$\{\mathrm{Var}(\boldsymbol{k})\}_{i,j} = \mathrm{Tr}\left[\mathbf{P}_H^{\Sigma(\vartheta)}\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta)}\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{P}_{G'}^{(X'X)^{-1}}\mathbf{A}'\mathbf{V}_j\mathbf{\Sigma}(\boldsymbol{\vartheta})\right],$$
$$i, j = 1,\dots,p.$$

Proof.

(i)

$$\begin{aligned}
\{\boldsymbol{k}\}_i &= \frac{\partial\,\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_{\vartheta})}{\partial\vartheta_i} = -\mathrm{Tr}\Big(\mathbf{A}\big\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0) \times \\
&\qquad \times \frac{\partial}{\partial\vartheta_i}[\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\big\}\Big) \\
&= -\mathrm{Tr}\Big(\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}} + \mathbf{G}_0)\Big\{-[\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H} \times \\
&\quad \times [\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta}) + [\mathbf{H}'\mathbf{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\Big\}\Big) = -\mathrm{Tr}\left[\mathbf{AK}\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta)}\right].
\end{aligned}$$

(ii) The equality $E(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}+\mathbf{G}_0)=\mathbf{0}$ implies $E(\boldsymbol{k})=\boldsymbol{0}$. Since $\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_\vartheta)=\mathrm{vec}(\widehat{\mathbf{B}})-\mathrm{vec}(\mathbf{K})$ and

$$\mathrm{vec}(\mathbf{K})=\left[\left(\mathbf{P}_H^{\Sigma(\vartheta)}\right)'\otimes\left(\mathbf{P}_{G'H}^{(X'X)^{-1}}\right)'\right]\mathrm{vec}(\widehat{\mathbf{B}})+\Big(\{\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\}\otimes$$
$$\otimes\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\}\Big)\,\mathrm{vec}(\mathbf{G}_0),$$

we have

$$\mathrm{cov}\left[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_\vartheta),\mathrm{vec}(\mathbf{K})\right]=\mathrm{cov}\Big\{\mathbf{I}\otimes\mathbf{I}-\left[\left(\mathbf{P}_H^{\Sigma(\vartheta)}\right)'\otimes\left(\mathbf{P}_{G'H}^{(X'X)^{-1}}\right)'\right]\mathrm{vec}(\widehat{\mathbf{B}}),$$
$$\left[\left(\mathbf{P}_H^{\Sigma(\vartheta)}\right)'\otimes\left(\mathbf{P}_{G'H}^{(X'X)^{-1}}\right)'\right]\mathrm{vec}(\widehat{\mathbf{B}})\Big\}=\mathbf{0}.$$

Since $\boldsymbol{k}$ is a function of the matrix $\mathbf{K}$ and $\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}})$ is a function of the estimator $\widehat{\widehat{\mathbf{B}}}$, $\mathrm{cov}[\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}),\boldsymbol{k}]=\mathbf{0}$.

Let

$$\mathbf{U}=\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G},$$
$$\mathbf{W}_i=\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta)}.$$

Then

$$\mathrm{cov}\big(\{\boldsymbol{k}\}_i,\{\boldsymbol{k}\}_j\big)=\mathrm{cov}[\mathrm{Tr}(\mathbf{U}\widehat{\mathbf{B}}\mathbf{W}_i),\mathrm{Tr}(\mathbf{U}\widehat{\mathbf{B}}\mathbf{W}_j)]$$
$$=[\mathrm{vec}(\mathbf{U}'\mathbf{W}_i')]'\,\mathrm{Var}[\mathrm{vec}(\widehat{\mathbf{B}})]\,\mathrm{vec}(\mathbf{U}'\mathbf{W}_j')=\mathrm{Tr}[\mathbf{W}_i\mathbf{U}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{U}'\mathbf{W}_j\boldsymbol{\Sigma}(\boldsymbol{\vartheta})]$$

and

$$\{\mathrm{Var}(\boldsymbol{k})\}_{i,j}=\mathrm{Tr}\Big\{\mathbf{P}_H^{\Sigma(\vartheta)}\boldsymbol{\Sigma}(\boldsymbol{\vartheta})^{-1}\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta)}\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})\mathbf{G}']^{-1}\times$$
$$\times\,\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}'\Big\}$$
$$=\mathrm{Tr}\left[\mathbf{P}_H^{\Sigma(\vartheta)}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta)}\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{P}_{G'}^{(X'X)^{-1}}\mathbf{A}'\mathbf{V}_j\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right].$$

□

Let the variance of the estimator $\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}})$ of the function

$$f(\mathbf{B})=\mathrm{Tr}(\mathbf{A}\mathbf{B}),\qquad \mathbf{G}\mathbf{B}\mathbf{H}+\mathbf{G}_0=\mathbf{0},$$

be under consideration. The problem is whether the a priori unknown parameters $\vartheta_1,\dots,\vartheta_p$, can be substituted by their estimates from Remark 2.3.3. We have:

Theorem 3.2. *If the observation matrix* $\underline{\mathbf{Y}}$ *is normally distributed and*

$$\delta\boldsymbol{\vartheta} \in \left\{\boldsymbol{u}:\ \boldsymbol{u}'\,\mathrm{Var}(\boldsymbol{k})\boldsymbol{u} < c^2\right\},$$

then

$$\left|\mathrm{Var}\left[\mathrm{Tr}(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta+\delta\vartheta})\right] - \mathrm{Var}\left[\mathrm{Tr}(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta})\right]\right| \approx \delta\boldsymbol{\vartheta}'\,\mathrm{Var}(\boldsymbol{k})\delta\boldsymbol{\vartheta} < c^2.$$

P r o o f. Proof is a direct consequence of the assumption and Lemma 3.1. □

Corollary 3.3. *If the realization of the estimator* $\mathrm{Tr}(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta})$ *is given, then also the vector* $\boldsymbol{k}$ *is given and the implication*

$$|\boldsymbol{k}'\delta\boldsymbol{\vartheta}| < \varepsilon \implies |\,\mathrm{Tr}(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta+\delta\vartheta}) - \mathrm{Tr}(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta})| < \varepsilon$$

is obvious.

If it is known that with sufficiently high probability the actual value $\delta\boldsymbol{\vartheta}$ $\boldsymbol{\vartheta}^* - \boldsymbol{\vartheta}$ *lies in the domain* $\{\delta\boldsymbol{\vartheta}:\ |\boldsymbol{k}'\delta\boldsymbol{\vartheta}| < \varepsilon\}$ *(this fact can be verified by* Remark 2.3.3*), then the best estimate* $\mathrm{Tr}\big(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta^*}\big)$ *differs from the estimate* $\mathrm{Tr}\big(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta}\big)$ *less than* ε.

Moreover, if $\boldsymbol{\vartheta}^* = \boldsymbol{\vartheta} + \delta\boldsymbol{\vartheta}$, *where* $\delta\boldsymbol{\vartheta}'\,\mathrm{Var}(\boldsymbol{k})\delta\boldsymbol{\vartheta} < c^2$, *then also the variance of the best estimator* $\mathrm{Tr}(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta^*})$ *differs from the variance of the estimator* $\mathrm{Tr}(\mathbf{A}\hat{\hat{\mathbf{B}}}_{\vartheta}$ *less than* c^2.

Let now a confidence region for $\mathbf{B}$ *in the model* (1) *must be determined in the case* $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$.

Since the estimator

$$\hat{\hat{\mathbf{B}}}(\vartheta^*) = \hat{\mathbf{B}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\hat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)$$

depends on $\boldsymbol{\vartheta}^*$ *(the actual value of the parameter* $\boldsymbol{\vartheta}$*) and it can be easily proved that*

$$\left\{\mathrm{Var}_{\vartheta^*}\left[\mathrm{vec}\big(\hat{\hat{\mathbf{B}}}_{\vartheta^*}\big)\right]\right\}^{-} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes (\mathbf{X}'\mathbf{X}),$$

the $(1-\alpha)$*-confidence region for the parameter* $\mathbf{B}$ *can be written in the form*

$$\mathcal{E} = \Big\{\mathbf{U}:\ \mathbf{G}\mathbf{U}\mathbf{H} + \mathbf{G}_0 = \mathbf{0},\ \mathrm{Tr}\Big[\big(\mathbf{U} - \hat{\hat{\mathbf{B}}}_{\vartheta^*}\big)'\mathbf{X}'\mathbf{X}(\mathbf{U} - \hat{\hat{\mathbf{B}}}_{\vartheta^*})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\Big] < \\ \le \chi^2_{km-qr}(0, 1-\alpha)\Big\}.$$

Let

$$k(\boldsymbol{\vartheta}) = \mathrm{Tr}\Big[\big(\mathbf{B} - \hat{\hat{\mathbf{B}}}_{\vartheta}\big)'\mathbf{X}'\mathbf{X}\big(\mathbf{B} - \hat{\hat{\mathbf{B}}}_{\vartheta}\big)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\Big].$$

(Obviously $k(\boldsymbol{\vartheta}^*) \sim \chi^2_{km-qr}(0)$*.)*

Lemma 3.4.

$$E\left(\frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_i}\Big|_{\vartheta=\vartheta^*}\right) = -k\,\mathrm{Tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i] + q\,\mathrm{Tr}\Big\{\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\Big\},$$

$$\mathrm{cov}\left[\left(\frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_i}\Big|_{\vartheta=\vartheta^*}\right), \left(\frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_j}\Big|_{\vartheta=\vartheta^*}\right)\right] = -2k\,\mathrm{Tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_j]$$
$$-2q\,\mathrm{Tr}\{\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_j\}.$$

P r o o f. Since

$$\frac{\partial \widehat{\widehat{\mathbf{B}}}_\vartheta}{\partial \vartheta_i}\Big|_{\vartheta=\vartheta^*} = -(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}]^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0) \times$$
$$\times [\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta^*)},$$

we have

$$\frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_i}\Big|_{\vartheta=\vartheta^*} = 2\,\mathrm{Tr}\Big\{\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1} \times$$
$$\times (\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)[\mathbf{H}'\Sigma(\boldsymbol{\vartheta})\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{M}_H^{\Sigma(\vartheta^*)}\Sigma^{-1}(\boldsymbol{\vartheta}^*)(\mathbf{B} - \widehat{\widehat{\mathbf{B}}}_{\vartheta^*})'\Big\}$$
$$-\,\mathrm{Tr}\Big[(\mathbf{B} - \widehat{\widehat{\mathbf{B}}}_{\vartheta^*})'\mathbf{X}'\mathbf{X}(\mathbf{B} - \widehat{\widehat{\mathbf{B}}}_{\vartheta^*})\Sigma^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\Sigma^{-1}(\boldsymbol{\vartheta}^*)\Big].$$

Now the substitution

$$\widehat{\widehat{\mathbf{B}}}_{\vartheta^*} = \widehat{\mathbf{B}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)$$

is used and thus

$$\frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_i}\Big|_{\vartheta=\vartheta^*} = \mathrm{Tr}\Big\{\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H} \times$$
$$\times [\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}+\mathbf{G}_0)'\Big\} - \mathrm{Tr}[\mathbf{X}'\mathbf{X}(\mathbf{B}-\widehat{\mathbf{B}})\Sigma^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\Sigma^{-1}(\boldsymbol{\vartheta}^*)^{-1}(\mathbf{B}-\widehat{\mathbf{B}})'].$$

In the following consideration the equality

$$E[\mathrm{Tr}(\mathbf{U}\boldsymbol{\Xi}\mathbf{V}\boldsymbol{\Upsilon}')] = \{E[\mathrm{vec}(\boldsymbol{\Xi})]\}'(\mathbf{V}\otimes\mathbf{U}')E[\mathrm{vec}(\boldsymbol{\Upsilon})] + \mathrm{Tr}\{(\mathbf{V}\otimes\mathbf{U}')\,\mathrm{cov}[\mathrm{vec}(\boldsymbol{\Upsilon}),\mathrm{vec}(\boldsymbol{\Xi})]\},$$

which is valid for any random matrices $\boldsymbol{\Xi}$ and $\boldsymbol{\Upsilon}$, is used. Thus

$$\begin{aligned}
E\left(\frac{\partial k(\vartheta)}{\partial\vartheta_i}\Big|_{\vartheta=\vartheta^*}\right) &= \mathrm{Tr}\Big[\Big(\Big\{[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\Big\}\otimes \\
&\quad\otimes[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\Big)\,\mathrm{Var}[(\mathbf{H}'\otimes\mathbf{G})\,\mathrm{vec}(\widehat{\mathbf{B}})]\Big] \\
&\quad - \mathrm{Tr}\Big(\Big\{[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)]\otimes(\mathbf{X}'\mathbf{X})\Big\}\,\mathrm{Var}(\mathbf{B}-\widehat{\mathbf{B}})\Big) \\
&= \mathrm{Tr}\Big(\Big\{[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}\Big\}\otimes\Big\{[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'\Big\}\Big) \\
&\quad - \mathrm{Tr}\Big\{[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i]\otimes(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}\Big\} \\
&= -k\,\mathrm{Tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i] + q\,\mathrm{Tr}\Big\{\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\Big\}.
\end{aligned}$$

In order to obtain a formula for $\mathrm{cov}\left[\left(\frac{\partial k(\boldsymbol{\vartheta})}{\partial\vartheta_i}\Big|_{\vartheta=\vartheta^*}\right),\left(\frac{\partial k(\boldsymbol{\vartheta})}{\partial\vartheta_j}\Big|_{\vartheta=\vartheta^*}\right)\right]$, the following notation is used

$$\begin{aligned}
A_1 &= \mathrm{Tr}\Big\{[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}+\mathbf{G}_0)\times \\
&\quad\times\big[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}\big]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}\big[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}\big]^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}+\mathbf{G}_0)'\Big\}, \\
-A_2 &= -\mathrm{Tr}\big[\mathbf{X}'\mathbf{X}(\mathbf{B}-\widehat{\mathbf{B}})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)(\mathbf{B}-\widehat{\mathbf{B}})'\big], \\
B_1 &= \mathrm{Tr}\Big\{[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}+\mathbf{G}_0)\times \\
&\quad\times[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_j\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H}+\mathbf{G}_0)'\Big\}, \\
-B_2 &= -\mathrm{Tr}\big[\mathbf{X}'\mathbf{X}(\mathbf{B}-\widehat{\mathbf{B}})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)(\mathbf{B}-\widehat{\mathbf{B}})'\big].
\end{aligned}$$

Thus

$$\operatorname{cov}\left[\left(\frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_i}\Big|_{\vartheta=\vartheta^*}\right), \left(\frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_j}\Big|_{\vartheta=\vartheta^*}\right)\right] = \operatorname{cov}(A_1, -A_2, B_1 - B_2) = \\ = \operatorname{cov}(A_1, B_1) - \operatorname{cov}(A_1, B_2) - \operatorname{cov}(A_2, B_1) + \operatorname{cov}(A_2, B_2).$$

Let $\Xi = \mathbf{B} - \widehat{\mathbf{B}}, \Upsilon = \mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0$. Then

$$\operatorname{vec}(\Upsilon) = -(\mathbf{H}' \otimes \mathbf{G}) \operatorname{vec}(\Xi) + (\mathbf{H}' \otimes \mathbf{G}) \operatorname{vec}(\mathbf{B}) + \operatorname{vec}(\mathbf{G}_0).$$

Now the relationships

$$\operatorname{cov}\Big[\operatorname{Tr}(\mathbf{A}_i\Xi\mathbf{B}_i\Xi'), \operatorname{Tr}(\mathbf{A}_j\Xi\mathbf{B}_j\Xi')\Big] = 2\operatorname{Tr}\Big\{(\mathbf{B}_i' \otimes \mathbf{A}_i) \operatorname{Var}[\operatorname{vec}(\Xi)](\mathbf{B}_j' \otimes \mathbf{A}_j) \times \\ \times \operatorname{Var}[\operatorname{vec}(\Xi)]\Big\} + 4\Big\{E[\operatorname{vec}(\Xi)]\Big\}'(\mathbf{B}_i' \otimes \mathbf{A}_i) \operatorname{Var}[\operatorname{vec}(\Xi)](\mathbf{B}_j' \otimes \mathbf{A}_j)E[\operatorname{vec}(\Xi)], \\ \operatorname{cov}\big[\operatorname{Tr}(\mathbf{A}\Xi\mathbf{B}\Xi'), \operatorname{Tr}(\mathbf{C}\Upsilon\mathbf{D}\Upsilon')\big] = \operatorname{Tr}\Big\{(\mathbf{B}' \otimes \mathbf{A}) \operatorname{Var}[\operatorname{vec}(\Xi)][(\mathbf{U}'\mathbf{D}'\mathbf{U}) \otimes \\ \otimes (\mathbf{V}'\mathbf{C}\mathbf{V})] \operatorname{Var}[\operatorname{vec}(\Xi)]\Big\} + 4E[\operatorname{vec}(\Xi)]\big(\mathbf{B}' \otimes \mathbf{A}\big) \operatorname{Var}[\operatorname{vec}(\Xi)][(\mathbf{U}'\mathbf{D}'\mathbf{U}) \otimes \\ \otimes (\mathbf{V}'\mathbf{C}\mathbf{V})]E[\operatorname{vec}(\Xi)],$$

where $\operatorname{vec}(\Upsilon) = (\mathbf{U} \otimes \mathbf{V}) \operatorname{vec}(\Xi)$, will be utilized in the following calculation. Thus we obtain

$$\operatorname{cov}(A_1, B_1) = 2\operatorname{Tr}\Big[\Big(\Big\{[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\Big\} \otimes \\ \otimes [\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\Big)\Big\{[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}] \otimes [\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']\Big\}\Big(\Big\{[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1} \times \\ \times \mathbf{H}'\mathbf{V}_j\mathbf{H}[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\Big\} \otimes [\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\Big)\Big\{[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}] \otimes \\ \otimes [\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']\Big\}\Big] = 2q\operatorname{Tr}\Big\{\mathbf{H}[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_j\Big\}, \\ -\operatorname{cov}(A_2, B_1) = -2\operatorname{Tr}\Big[[\Sigma(\boldsymbol{\vartheta}^*)^{-1}\mathbf{V}_i\Sigma^{-1}(\boldsymbol{\vartheta}^*)] \otimes (\mathbf{X}'\mathbf{X})][\Sigma(\boldsymbol{\vartheta}^*) \otimes (\mathbf{X}'\mathbf{X})^{-1}] \times \\ \times \Big(\Big\{\mathbf{H}[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_j\mathbf{H}[\mathbf{H}'\Sigma(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\Big\} \otimes \Big\{\mathbf{G}'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{G}\Big\}\Big) \times \\ \times [\Sigma(\boldsymbol{\vartheta}^*) \otimes (\mathbf{X}'\mathbf{X})^{-1}]\Big]$$

$$= -2q\,\mathrm{Tr}\Big\{\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_j\Big\} = -\operatorname{cov}(A_1, B_2)$$
$$= -\operatorname{cov}(A_1, B_1),$$
$$\operatorname{cov}(A_2, B_2) = 2\,\mathrm{Tr}\Big(\Big\{[\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)] \otimes (\mathbf{X}'\mathbf{X})]\Big\}[\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*) \otimes (\mathbf{X}'\mathbf{X})^{-1}] \times$$
$$\times \Big\{[\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)^{-1}\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)] \otimes (\mathbf{X}'\mathbf{X})]\Big\}[\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*) \otimes (\mathbf{X}'\mathbf{X})^{-1}]\Big)$$
$$= 2k\,\mathrm{Tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_j].$$

□

In the following text the notation (cf. also Corollary 2.3.2)

$$\{\mathbf{S}_{\Sigma^{-1}}\}_{i,j} = \mathrm{Tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_j], \quad i, j = 1, \dots, p,$$

and

$$\big\{\mathbf{S}_{H(H'\Sigma H)^{-1}H'}\big\}_{i,j} = \mathrm{Tr}\Big\{\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_j\Big\},$$
$$i, j = 1, \dots, p,$$

will be used. Thus

$$\operatorname{Var}\left(\frac{\partial \boldsymbol{k}(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}}\right) = 2k\mathbf{S}_{\Sigma^{-1}} - 2q\mathbf{S}_{H(H'\Sigma H)^{-1}H'}.$$

COROLLARY 3.5. *Let*

$$\boldsymbol{a}' = (a_1, a_2, \dots, a_p),$$
$$a_i = k\,\mathrm{Tr}\Big\{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\Big\} - q\,\mathrm{Tr}\Big\{\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}']^{-1}\mathbf{H}'\mathbf{V}_i\Big\}, \qquad i - 1, \dots, p,$$
$$\mathbf{A} = 2k\mathbf{S}_{\Sigma^{-1}} - 2q\mathbf{S}_{H(H'\Sigma H)^{-1}H'}.$$

Then

$$k(\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \approx k(\boldsymbol{\vartheta}^*) + \kappa(\delta\boldsymbol{\vartheta}),$$
$$\kappa(\delta\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \frac{\partial k(\boldsymbol{\vartheta})}{\partial \vartheta_i}\Big|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}^*} \delta\vartheta_i$$

$$= \sum_{i=1}^{p}\Big(\mathrm{Tr}\Big\{[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)[\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{H}]^{-1}\mathbf{H}'\mathbf{V}_i\mathbf{H}[\mathbf{H}'\boldsymbol{\Sigma}\ \boldsymbol{\vartheta}^*)\mathbf{H}]^{-1} \times$$
$$\times (\mathbf{G}\widehat{\mathbf{B}}\mathbf{H} + \mathbf{G}_0)'\Big\} - \mathrm{Tr}[\mathbf{X}'\mathbf{X}(\mathbf{B} - \widehat{\mathbf{B}})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)(\mathbf{B} - \widehat{\mathbf{B}})']\Big)\delta\vartheta_i$$

and

$$E[\kappa(\delta\boldsymbol{\vartheta})] = -\boldsymbol{a}'\delta\boldsymbol{\vartheta}, \quad \operatorname{Var}[\kappa(\delta\boldsymbol{\vartheta})] = \delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta}.$$

Theorem 3.6. *If*

$$\delta\boldsymbol{\vartheta} \in \left\{ \boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^p, \ [\boldsymbol{u} - c_t(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')^+\boldsymbol{a}]'(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')[\boldsymbol{u} - c_t(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')^+\boldsymbol{a}] \le \frac{c_t^2 t^2}{t^2 - \boldsymbol{a}'\mathbf{A}^+\boldsymbol{a}} \right\},$$

then

$$P\Big\{\mathbf{B} \in \mathcal{E} = \Big\{\mathbf{U}:\ \mathbf{U} \in \mathcal{M}^{k\times m},\ \mathbf{GUH} + \mathbf{G}_0 = \mathbf{0},\ \mathrm{Tr}\big[(\mathbf{U} - \widehat{\widehat{\mathbf{B}}}_{\vartheta^* + \delta\vartheta})'\mathbf{X}'\mathbf{X} \times (\mathbf{U} - \widehat{\widehat{\mathbf{B}}}_{\vartheta^* + \delta\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta})\big] \le \chi^2_{km-qr}(0, 1-\alpha)\Big\}\Big\} \ge 1 - \alpha - \varepsilon,$$

where $\mathcal{M}^{k\times m}$ is the class of all $k \times m$ matrices, t is sufficiently large number satisfying the relationship

$$P\Big\{\kappa(\delta\boldsymbol{\vartheta}) \le -\boldsymbol{a}'\delta\boldsymbol{\vartheta} + t\sqrt{\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta}}\ \Big\} \approx 1$$

and c_t satisfies the equation

$$P\Big\{\chi^2_{km-qr}(0) \le \chi^2_{km-qr}(0, 1-\alpha) - c_t\Big\} = 1 - \alpha - \varepsilon,$$

i.e. $c_t = \chi^2_{km-qr}(0, 1-\alpha) - \chi^2_{km-qr}(0, 1-\alpha-\varepsilon)$.

Proof. Regarding Theorem 3.2

$$k(\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \approx k(\boldsymbol{\vartheta}^*) + \kappa(\delta\boldsymbol{\vartheta}) \quad \text{and} \quad P\Big\{k(\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \le \chi^2_{km-qr}(0, 1-\alpha)\Big\}$$
$$\approx P\Big\{\chi^2_{km-qr}(0) + \kappa(\delta\boldsymbol{\vartheta}) \le \chi^2_{km-qr}(0, 1-\alpha)\Big\} = P\Big\{\chi^2_{km-qr}(0) + \kappa(\delta\boldsymbol{\vartheta}) \le \chi^2_{km-qr}(0, 1-\alpha) | \kappa(\delta\boldsymbol{\vartheta}) < c\Big\} P\{\kappa(\delta\boldsymbol{\vartheta}) < c\} + P\Big\{\chi^2_{km-qr}(0) + \kappa(\delta\boldsymbol{\vartheta}) \le \chi^2_{km-qr}(0, 1-\alpha) | \kappa(\delta\boldsymbol{\vartheta}) \ge c\Big\} P\big\{\kappa(\delta\boldsymbol{\vartheta}) \ge c\big\}.$$

If $\kappa(\delta\boldsymbol{\vartheta}) < c$ occurs with probability near to 1, then $P\{k(\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \leq \chi^2_{km-qr}(0, 1-\alpha) - c\} \geq 1 - \alpha - \varepsilon$. Now the number c can be found for a given ε and the number t must bo chosen such that $\kappa(\delta\boldsymbol{\vartheta}) \leq -\boldsymbol{a}'\delta\boldsymbol{\vartheta} + t\sqrt{\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta}}$ occurs with probability near to 1.

If the numbers c and t are given, the nonsensitivity region can be found as a set $\{\delta\boldsymbol{\vartheta} : t^2\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta} \leq (c + \boldsymbol{a}'\delta\boldsymbol{\vartheta})^2\}$. The equality

$$t^2\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta} = c^2 + \delta\boldsymbol{\vartheta}'\boldsymbol{a}\boldsymbol{a}'\delta\boldsymbol{\vartheta} + 2c\boldsymbol{a}'\delta\boldsymbol{\vartheta}$$

can be rewritten as

$$(\delta\boldsymbol{\vartheta} - \boldsymbol{u}_0)'(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')(\delta\boldsymbol{\vartheta} - \boldsymbol{u}_0) = c^2 + c^2\boldsymbol{a}'(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')^+\boldsymbol{a}$$

where $\boldsymbol{u}_0 = c_t(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')^+\boldsymbol{a}$. Here it is necessary to remark that $\boldsymbol{a} \in \mathcal{M}(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')$. Since

$$\begin{aligned}(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')^+ &= (t^2\mathbf{A})^+ + (t^2\mathbf{A})^+\boldsymbol{a}[1 - \boldsymbol{a}'(t^2\mathbf{A})^+\boldsymbol{a}]^{-1}\boldsymbol{a}'(t^2\mathbf{A})^+ \\ &= \frac{1}{t^2}\mathbf{A}^+ + \frac{\mathbf{A}^+\boldsymbol{a}\boldsymbol{a}'\mathbf{A}^+}{t^2(t^2 - \boldsymbol{a}'\mathbf{A}^+\boldsymbol{a})},\end{aligned}$$

we have

$$c^2 + c^2\boldsymbol{a}'(t^2\mathbf{A} - \boldsymbol{a}\boldsymbol{a}')^+\boldsymbol{a} = \frac{c^2t^2}{t^2 - \boldsymbol{a}'\mathbf{A}^+\boldsymbol{a}}.$$

□

Sensitivity approach for other statistical inference is used in [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16].

Acknowledgement. Author would like to express many thanks to the referee for his thorough reading of the manuscript and advices. He discovered many typing errors and other mistakes which author missed.

REFERENCES

[1] ANDERSON, T.W.: *Introduction to Multivariate Statistical Analysis*, J. Wiley, New York, 1958.

[2] FIŠEROVÁ, E.—KUBÁČEK, L.: *Sensitivity analysis in singular mixed linear models with constraints*, Kybernetika (Prague) **39** (2003), 317–332.

[3] KUBÁČEK, L.: *Regression model with estimated covariance matrix*, Math. Slovaca **33** (1983), 395–408.

[4] KUBÁČEK, L.—KUBÁČKOVÁ, L.—VOLAUFOVÁ, J.: *Statistical Models with Linear Structures*, Veda, Bratislava, 1995.

[5] KUBÁČEK, L.: *Criterion for an approximation of variance components in regression models*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **34** (1995), 91–108.

[6] KUBÁČEK, L.: *Linear model with inaccurate variance components*, Appl. Math. **41** (1996), 433 445.

[7] KUBÁČEK, L.—KUBÁČKOVÁ, L.: *Nonsensitiveness regions in models with variance components*. In: 4th World Congress of the Bernoulli Society, Vienna, Austria, August 26 31, 1996, p. 281.

[8] KUBÁČEK, L.—KUBÁČKOVÁ, L.—TESAŘÍKOVÁ, E.—MAREK, J.: *How the design of an experiment influences the nonsensitiveness regions in models with variance components*, Appl. Math. **43** (1998) 439–460.

[9] KUBÁČEK, L. KUBÁČKOVÁ, L.: *Nonsensitiveness regions in universal models*, Math. Slovaca **50** (2000), 219 240.

[10] KUBÁČEK, L.—FIŠEROVÁ, E.: *Problems of sensitiveness and linearization in a determination of isobestic points*, Math. Slovaca **53** (2003), 407 426.

[11] KUBÁČEK, L.—FIŠEROVÁ, E.: *Isobestic points: sensitiveness and linearization*, Tatra Mt. Math. Publ. **26** (2003), 1–10.

[12] LEŠANSKÁ, E.: *Insensitivity regions for estimators of mean value parameters in mixed models with constraints*, Tatra Mt. Math. Publ. **22** (2001) 37 49.

[13] LEŠANSKÁ, E.: *Insensitivity regions for testing hypotheses in mixed models with constraints*, Tatra Mt. Math. Publ. **22** (2001), 209–222.

[14] LEŠANSKÁ, E.: *Optimization of the size of nonsensitiveness regions*, Appl. Math. **47** (2002) 9 23.

[15] LEŠANSKÁ, E.: *Nonsensitiveness regions for threshold ellipsoids*, Appl. Math. **47** (2002), 411 426.

[16] LESANSKÁ, E.: *Effect of inaccurate variance components in mixed models with constraints.* In: Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math. 11, Masaryk Univ., Brno, 2002, pp. 163 172.

[17] RAO, C. R.: *Least squares theory using an estimated dispersion matrix and its application to measurement in signal.* In.: Proc. 5th Berkeley Symposium on Mathematical Statistics and Probability, Vol. 1. Theory of Statistics. University of California Press, Berkeley-Los Angeles, 1967, pp. 355–372.

[18] RAO, C. R.: *Linear Statistical Inference and Its Applications* (2nd ed.), J. Wiley, New York, 1973.

[19] RAO, C. R.—MITRA, S. K.: *Generalized Inverse of Matrices and Its Applications*, J. Wiley, New York, 1971.

[20] RAO, C. R. KLEFFE, J.: *Estimation of Variance Components and Applications* North-Holland, Amsterdam, 1988.

Received 22. 2. 2005
Revised 31. 5. 2005

Katedra matematické analyzy
a aplikovane matematiky
PF Univerzita Palackeho
Tomkova 40
CZ 779 00 Olomouc
ČESKÁ REPUBLIKA
E-mail: kubacekl@ris .upol.cz