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THE EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS OF P-LAPLACIAN BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we establish sufficient conditions to guarantee the existence of at least three or $2n-1$ positive solutions of nonlocal boundary value problems consisting of the second-order differential equation with p-Laplacian

$$[\phi_p(x'(t))]' + f(t, x(t)) = 0, \qquad t \in (0,1), \tag{1}$$

and one of following boundary conditions

$$x(0) = \int_0^1 x(s)\,\mathrm{d}h(s), \quad \phi_p(x'(1)) = \int_0^1 \phi_p(x'(s))\,\mathrm{d}g(s), \tag{2}$$

and

$$\phi_p(x'(0)) = \int_0^1 \phi_p(x'(s))\,\mathrm{d}h(s), \quad x(1) = \int_0^1 x(s)\,\mathrm{d}g(s). \tag{3}$$

Examples are presented to illustrate the main results.

1. Introduction

In this paper, we are concerned with the multiplicity of positive solutions to the nonlocal boundary value problems (BVP for short) consisting of the one-dimension p-Laplacian differential equation

$$[\phi_p(x'(t))]' + f(t, x(t)) = 0, \qquad t \in (0,1), \tag{1}$$

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associated with one of following boundary value conditions

$$x(0) - \int_0^1 x(s)\,\mathrm{d}h(s), \qquad \phi_p(x'(1)) = \int_0^1 \phi_p(x'(s))\,\mathrm{d}g(s), \qquad 2$$

and

$$\phi_p(x'(0)) = \int_0^1 \phi_p(x'(s))\,\mathrm{d}h(s), \qquad x(1) = \int_0^1 x(s)\,\mathrm{d}g(s), \qquad (3$$

where $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$, $h, g\colon [0,1] \to [0, +\infty)$ are given functions and the integrals in (2) or (3) are meant in the Riemann-Stieltjes sense. $\phi_p(s) - s|^{p-2}s$, $(p > 1)$ is called p-Laplacian.

The study of nonlocal boundary value problems of this form was initiated in the early 1960s by Bitsadze [4] and later studied by Bitsadze and Samarskii [5], Il'in and Moiseev [11] and Karakostas and Tsamatos [14]. This class of problem.s includes, as special cases, multi-point boundary value problems, which were considered by many authors (see e g., [2], [3], [9 , [10], [17] [20] and the references therein and the recent book by Agarwal and O'Regan [1] and [7], [8]).

The boundary value problems consisting of equation (1) and different two-point boundary value conditions have been studied extensively (see, for example, [1], [6], [11], [12], [14], [17], [21] [24]). When p 2 and $f(t, x) = a(t)g(x$, (1) becomes the following

$$x''(t) + a(t)g(x(t)) - 0, \qquad t \in (0,1). \qquad (4$$

Recently, Ma in [20] showed the existence of at least one po itive solution to (5) with the following boundary value conditions

$$x'(0) = \sum_{i=1}^{m-2} b_i x'(\xi_i), \qquad x(1) = \sum_{i\ 1}^{m-2} a_i x(\eta_i) \qquad (5$$

under the conditions that f is continuous, nonnegative and either super-linear or sub-linear. We note that BVP (5) and (6) is a special case of BVP (1) and (3). Gupta in [9], using Leray-Schauder fixed point theorem, studied the existence of solutions of the following BVP consisting of the equation $x''(t) = -f(t, x(t)$, $t \in [0,1]$, and boundary conditions

$$x(0) = \sum_{i\ 1}^{m} b_i x(\xi_i), \qquad x'(1) = \sum_{i\ 1}^{k} a_i x'(\eta_i). \qquad (6$$

The existence of at least one solution of the following BVP consisting of the equation $x''(t) = f(t, x(t))$, $t \in [0,1]$, and of boundary value conditions

$$x(0) = \sum_{i=1}^{m} b_i x(\xi_i), \qquad x(1) = \sum_{i=1}^{k} a_i x(\eta_i) \tag{7}$$

was studied by Liu and Yu in [18]. BVP (1) and (i) (i=2,3,4) contains BVP (5) and (j) (j=6,7,8) as special cases, respectively. In a recent paper, Gupta in [10], using Schauder fixed point theorem, established the existence results of solutions of BVP for the p-Laplacian differential equation

$$\begin{aligned} &[\phi_p(x'(t))]' = f(t, x(t), x'(t)) + e(t), \qquad t \in [0,1], \\ &x(0) = 0, \qquad \phi_p(x'(1)) = \sum_{i=1}^{k} a_i \phi_p(x'(\eta_i)). \end{aligned} \tag{8}$$

In [2], the authors studied the existence of positive solutions of boundary value problem (1) and (6). In [2], the operator A is defined by

$$\begin{aligned} Ax(t) &= -\int_0^t \phi_q\left(\int_0^s f(\tau, x(\tau))\,\mathrm{d}\tau\right) + t\frac{\sum_{i=1}^{m-2} b_i \phi_q\left(\int_0^{\xi_i} f(\tau, x(\tau))\,\mathrm{d}\tau\right)}{\sum_{i=1}^{m-2} b_i - 1} \\ &\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i}\left[\int_0^1 \phi_q\left(\int_0^s f(\tau, x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\right. \\ &\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\int_0^s f(\tau, x(\tau))d\tau\right)\mathrm{d}s \\ &\quad \left. - \frac{\sum_{i=1}^{m-2} b_i \phi_q\left(\int_0^{\xi_i} f(\tau, x(\tau))\,\mathrm{d}\tau\right)}{\sum_{i=1}^{m-2} b_i - 1}\left(1 - \sum_{i=1}^{m-2} a_i \xi_i\right)\right] \end{aligned}$$

for $x \in C[0,1]$. The authors claim that $x(t)$ is a solution of (1) and (6) if and only if x is a fixed point of the operator A. One can see that if $Ax = x$, then

$$x'(t) = -\phi_q\left(\int_0^t f(\tau, x(\tau))\,\mathrm{d}\tau\right) + \frac{\sum_{i=1}^{m-2} b_i \phi_q\left(\int_0^{\xi_i} f(\tau, x(\tau))\,\mathrm{d}\tau\right)}{\sum_{i=1}^{m-2} b_i - 1}.$$

We find that $[\phi_p(x'(t))]' \neq -f(t, x(t))$. The claim is false.

In recent papers, Liu and Ge [17], He and Ge [25] studied following boundary value problems

$$\begin{cases} [\phi_p(u'(t))]' + a(t)f(u(t)) = 0, & 0 < t < 1, \\ u(0) - B_0(u'(\eta)) = 0 = u'(1), \end{cases}$$

and

$$\begin{cases} u''(t) + a(t)f(u(t)) = 0, & 0 < t < 1, \\ u(0) = 0 = u(1) - \alpha u(\eta). \end{cases}$$

They proved the existence results for positive solutions by using fixed point theorems. In recent papers [26] and [27], the authors studied some similar two-point or three-point boundary value problems by using Leggett-Williams fixed point theorem or its generalized form. So it is interesting and valuable to establish the existence criteria for multiple positive solutions of equation (1) subject to different nonlocal boundary value conditions

In very recent papers [28], [29], Karakostas studied the existence of positive solutions for the Φ-Laplacian when Φ is a sup-multiplicative-like function

$$(\Phi(x'))' + p(t)f(t, x(g_1(t)), \ldots, x(g_n(t))) = 0$$

subject to one of the boundary value conditions

$$x(0) - B_0(x'(0)) = x(1) + B_1(x'(1)) = 0,$$

$$x(0) - B_0(x'(0)) = x'(1) = 0,$$

and

$$x'(0) = x(1) + B_1(x'(1)) = 0$$

by using Krasnoselskii's and Leggett-Williams fixed point theorems.

Inspired and motivated by the works mentioned above, especially, by [9], [20], our purpose here is to give some existence results for at least three positive solutions to BVP (1) and (2) and BVP (1) and (3) by using Leggett-Williams fixed-point theorem, which has been reported to be a successful technique for dealing with the existence of multiple positive solutions of BVPs for second order differential equations, however, there is no paper reported discussing the positive solutions of nonlocal boundary value problems for p-Laplacian equations by using it since the presence of p-Laplacian causes some difficulty. The results in this paper are new.

By a positive solution of BVP (1) and (i) (i=2,3) we mean a function $x(t)$ which is positive on $(0,1)$ and satisfies (1) and the boundary conditions (i) (i=2,3) respectively. In the sequel, we suppose the following:

(H_1) f is a real valued continuous function defined on $[0,1]\times\mathbb{R}$, and satisfies the inequality $f(t,x)\geq 0$ when $t\in[0,1]$ and $x\geq 0$, and $f(t,0)\not\equiv 0$ on any sub-interval of $[0,1]$.

(H_2) $g\colon [0,1]\to\mathbb{R}$ is non-decreasing function with $0=g(0)\leq g(1)<1$ and $h\colon [0,1]\to\mathbb{R}$ is a non-decreasing function with $0=h(0)\leq h(1)<1$ for all $t\in I$.

2. Preliminary lemmas

To obtain positive solutions of BVP (1) and (i) (i=2,3,4), we first present the following definitions, a fixed-point theorem in cones and preliminary lemmas. The main results will be given in Section 3.

Definition 2.1. Let X be a real Banach space, a non-empty closed convex set $P\ (\neq\{0\})\subset X$ is called a *cone* of X if it satisfies the following conditions:

(i) $x\in P$ and $\lambda\geq 0$ implies $\lambda x\in P$.

(ii) $x\in P$ and $-x\in P$ implies $x=0$.

Every cone $P\subset X$ induces an ordering in X which is given by $x\leq y$ if and only if $y-x\in P$.

Definition 2.2. A map $\psi\colon P\to[0,+\infty)$ is called a *concave* functional map provided ψ satisfies

$$\psi(tx+(1-t)y)\geq t\psi(x)+(1-t)\psi(y)$$

for all $x,y\in P$ and $t\in[0,1]$.

Definition 2.3. An operator is called *completely continuous* if it is continuous and maps bounded sets into pre-compact sets.

Definition 2.4. Let $0 < a < b$ and r be given and let ψ be a nonnegative continuous concave functional on the cone P. Define the convex set P_r and $P(\psi; a, b)$ by

$$P_r = \{y \in P : \ \|y\| < r\}, \qquad P(\psi; a, b) = \{y \in P : \ a \leq \psi(y), \ \|y\| < b\}.$$

Next, we state the Leggett-Williams fixed-point theorem. The proof of this theorem can be found in Guo and Lakshmikantham's book [8], Deimling's text [7].

Theorem 2.1 (Leggett-Williams Fixed-Point Theorem). *Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on P such that $\psi(y) \leq \|y\|$ for all $y \in \overline{P}_c$. Suppose that there exist $0 < a < b < d \leq c$ such that*

(C_1) $\{y \in P(\psi; b, d) : \ \psi(y) > b\} \neq \emptyset$ *and* $\psi(Ty) > b$ *for* $y \in P(\psi; b, d)$;

(C_2) $\|Ty\| < a$ *for* $\|y\| \leq a$;

(C_3) $\psi(Ty) > b$ *for* $y \in P(\psi; b, c)$ *with* $\|Ty\| > d$.

Then T has at least three fixed points y_1, y_2 and y_3 such that $\|y_1\| < a, b < \psi(y_2)$ and $\|y_3\| > a$ with $\psi(y_3) < b$.

Now, we give some preliminary lemmas. Suppose $x(t)$ is a solution of equation (1) associated with condition (2), integrating (1) from t to 1, we get, using (H_2),

$$\phi_p(x'(1)) - \phi_p(x'(t)) = -\int_t^1 f(s, x(s))\, ds, \tag{9}$$

then

$$\begin{aligned}
&\phi_p(x'(1)) \int_0^1 dg(s) - \int_0^1 \phi_p(x'(t))\, dg(s) \\
= \ & -\int_0^1 \int_s^1 f(\tau, x(\tau))\, d\tau\, dg(s) \\
= \ & -g(s) \int_s^1 f(\tau, x(\tau))\, d\tau \Big|_0^1 + \int_0^1 g(s)\, d\left(\int_s^1 f(\tau, x(\tau))\, d\tau \right) \\
= \ & -\int_0^1 g(s) f(s, x(s))\, ds.
\end{aligned}$$

Using (2), we find

$$\phi_p(x'(1)) = \frac{1}{1-g(1)} \int_0^1 g(s) f(s, x(s))\,\mathrm{d}s.$$

From (10), one gets

$$\phi_p(x'(t)) = \frac{1}{1-g(1)} \int_0^1 g(s) f(s, x(s))\,\mathrm{d}s + \int_t^1 f(s, x(s))\,\mathrm{d}s,$$

i.e.

$$x'(t) = \phi_q\left(\frac{1}{1-g(1)} \int_0^1 g(s) f(s, x(s))\,\mathrm{d}s + \int_t^1 f(s, x(s))\,\mathrm{d}s\right),$$

where q satisfies $1/p + 1/q = 1$. So we have

$$x(t) - x(0) = \int_0^t \phi_q\left(\frac{1}{1-g(1)} \int_0^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right) \mathrm{d}s.$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 x(s)\,\mathrm{d}h(s) - x(0) \int_0^1 \mathrm{d}h(s) \\
= &\int_0^1 \int_0^t \phi_q\left(\frac{1}{1-g(1)} \int_0^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right) \mathrm{d}s\,\mathrm{d}h(t) \\
= &\ h(1) \int_0^1 \phi_q\left(\frac{1}{1-g(1)} \int_0^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right) \mathrm{d}s \\
&- \int_0^1 h(t)\phi_q\left(\frac{1}{1-g(1)} \int_0^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_t^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right) \mathrm{d}t.
\end{aligned}$$

It follows from (2) that

$$x(0)=\frac{h(1)}{1-h(1)}\int_0^1\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$-\frac{1}{1-h(1)}\int_0^1 h(s)\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Then

$$x(t)=\int_0^t\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$+\frac{h(1)}{1-h(1)}\int_0^1\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$-\frac{1}{1-h(1)}\int_0^1 h(s)\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Let X be the set of all continuous functions on $[0,1]$ and be endowed with the norm $\|x\|=\max_{t\in[0,1]}|x(t)|$, then X is a Banach space. We note that $x(t)\geq\frac{1}{2}\|x\|$ for $t\in[1/2,1]$ if $x(t)$ is positive, concave and increasing on $[0,1]$. To apply Theorem 2.1, we define P_1 by

$$P_1=\bigl\{x\in X:\ x(t)\geq\tfrac{1}{2}\|x\|\ \text{ for } t\in\bigl[\tfrac{1}{2},1\bigr],\ x(t)\ \text{is positive, increasing and concave on}\ (0,1)\bigr\}.$$

We find that P_1 is a cone in X. Define an operator A_1 on cone P_1 by

$$A_1x(t)=\int_0^t\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$+\frac{h(1)}{1-h(1)}\int_0^1\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$-\frac{1}{1-h(1)}\int_0^1 h(s)\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s$$

for every $x\in P_1$. Define a functional ψ_1 by $\psi_1(x)=\min_{t\in[1/2,1]}x(t)=x(\frac{1}{2})$ for $x\in P_1$. Now we give some preliminary results.

Denote

$$\delta_1 = \frac{1}{1-h(1)} \int_0^{\frac{1}{2}} (1 - h(s))\phi_q \left(\frac{1}{1-g(1)} \int_{\frac{1}{2}}^1 g(\tau) \, \mathrm{d}\tau + \tfrac{1}{2} \right) \mathrm{d}s,$$

$$m_1 = \left[\frac{1}{1-h(1)} \int_0^1 (1 - h(s))\phi_q \left(\frac{1}{1-g(1)} \int_0^1 g(\tau) \, \mathrm{d}\tau + 1 - s \right) \mathrm{d}s \right]^{-1}.$$

P_1, A_1, ψ_1, δ_1 and m_1 will be used in Theorems 3.3 and 3.4.

Lemma 2.2. *Assume* (H_1) *and* (H_2). *Then the following results hold:*

(i) $A_1 P_1 \subset P_1$.
(ii) A_1 *is completely continuous.*
(iii) ψ_1 *is nonnegative and concave,* $\psi_1(x) \le \|x\|$ *for every* $x \in P_1$.
(iv) $(A_1 x)(0) = \int_0^1 (A_1 x)(s) \, \mathrm{d}h(s)$ *and* $\phi_p((A_1 x)'(1)) = \int_0^1 \phi_p((A_1 x)'(s)) \, \mathrm{d}g(s)$.
(v) $x \in P_1$ *is a fixed point of the operator* A_1 *on* P_1 *if and only if* $x(t)$ *satisfies equation* (1) *and conditions* (2).

Proof. The proof is standard and the similar proof can be find in [8], [16], [24] and [26] especially and is omitted. □

Similar to above discussion, we get, if $x(t)$ is a solution of BVP (1) and (3), that

$$\begin{aligned} x(t) = {} & \frac{1}{1-g(1)} \int_0^1 g(s)\phi_q \left(\int_0^s f(\tau, x(\tau)) \, \mathrm{d}\tau \right. \\ & \left. + \frac{1}{1-h(1)} \int_0^1 (h(1) - h(\tau)) f(\tau, x(\tau)) \, \mathrm{d}\tau \right) \mathrm{d}s \\ & + \int_t^1 \phi_q \left(\int_0^s f(\tau, x(\tau)) \, \mathrm{d}\tau + \frac{1}{1-h(1)} \int_0^1 (h(1) - h(\tau)) f(\tau, x(\tau)) \, \mathrm{d}\tau \right) \mathrm{d}s. \end{aligned}$$

To apply Theorem 2.1 we define a cone

$$\begin{aligned} P_2 = \big\{ x \in X : \ & x(t) \ge \tfrac{1}{2}\|x\| \ \text{ for } t \in \big[0, \tfrac{1}{2}\big], \\ & x(t) \ \text{is positive, decreasing, continuous and concave on } (0,1) \big\}. \end{aligned}$$

Define the operator A_2 by

$$A_2x(t) = \frac{1}{1-g(1)}\int_0^1 g(s)\phi_q\left(\int_0^s f(\tau,x(\tau))\,\mathrm{d}\tau + \frac{1}{1-h(1)}\int_0^1 (h(1)-h(\tau))f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s + \int_t^1 \phi_q\left(\int_0^s f(\tau,x(\tau))\,\mathrm{d}\tau + \frac{1}{1-h(1)}\int_0^1 (h(1)-h(\tau))f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Define the functional $\psi_2(x) = \min\limits_{t\in[0,1/2]} x(t)$ for $x \in P_2$. Let

$$\delta_2 = \frac{1}{1-g(1)}\int_{1/2}^1 g(s)\phi_q\left(s + \frac{1}{1-h(1)}\int_0^1 (h(1)-h(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s + \int_{1/2}^1 \phi_q\left(s + \frac{1}{1-h(1)}\int_0^1 (h(1)-h(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$m_2 = \frac{1}{1-g(1)}\int_0^1 g(s)\phi_q\left(s + \frac{1}{1-h(1)}\int_0^1 (h(1)-h(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s + \int_0^1 \phi_q\left(s + \frac{1}{1-h(1)}\int_0^1 (h(1)-h(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

P_2, A_2, ψ_2, δ_2 and m_2 will be used in Theorems 3.5 and 3.6.

Lemma 2.3. *Assume* (H_1) *and* (H_2). *Then the following results hold:*

(i) $A_2P_2 \subset P_2$.

(ii) A_2 *is completely continuous.*

(iii) ψ_2 *is nonnegative and concave,* $\psi_2(x) \le \|x\|$ *for every* $x \in P_2$.

(iv) $\phi_p((A_2x)'(0)) = \int_0^1 \phi_p((A_2x)'(s))\,\mathrm{d}h(s)$ *and* $(A_2x)(1) = \int_0^1 (A_2x)(s)\,\mathrm{d}g(s)$.

(v) $x \in P_2$ *is a fixed point of the operator* A_2 *on* P_2 *if and only if* $x(t)$ *satisfies equation* (1) *and conditions* (3).

Proof. The proof is standard and the similar proof can be find in [8], [16 , [24] and [26] especially and is omitted. □

3. Main results

In this section, we present the main results and their proofs.

Theorem 3.1. *Assume* (H_1) *and* (H_2). *In addition, there exist constants* $0 < a < b < 2b \le c$, $b < m_1\delta_1 c$ *such that*

(D_1) $f(t,x) < \phi_p(m_1 a)$ *for* $t \in [0,1]$ *and* $x \in [0,a]$;

(D_2) $f(t,x) > \phi_p(b/\delta_1)$ *for* $t \in \left[\frac{1}{2},1\right]$ *and* $x \in [b,2b]$;

(D_3) $f(t,x) < \phi_p(m_1 c)$ *for* $t \in [0,1]$ *and* $x \in [0,c]$.

Then BVP (1) *and* (2) *has at least three positive solutions* x_1, x_2 *and* x_3 *such that*

$$\|x_1\| < a, \quad b < \psi_1(x_2) \quad \text{and} \quad \|x_3\| > a \ \text{with}\ \psi_1(x_3) < b.$$

Proof. By Lemma 2.2(v), it suffices to show that conditions of Theorem 2.1 are satisfied. From the definition of ψ_1, $\psi_1(x) \le \|x\|$ for all $x \in P_1$. Now, if $x \in \overline{P_1}_c$, then $\|x\| \le c$ and (D_3) implies $f(t,x(t)) \le \phi_p(m_1 c)$ for all $t \in [0,1]$. Consequently,

$$\begin{aligned}
\|A_1x\| &= \\
&= \int_0^1 \phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{h(1)}{1-h(1)}\int_0^1 \phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad - \frac{1}{1-h(1)}\int_0^1 h(s)\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \frac{1}{1-h(1)}\int_0^1 (1-h(s))\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)f(\tau,x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&< \frac{1}{1-h(1)}\int_0^1 (1-h(s))\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)\phi_p(m_1c)\,\mathrm{d}\tau + \int_s^1 \phi_p(m_1c)\,\mathrm{d}\tau\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{m_1 c}{1-h(1)} \int_0^1 (1-h(s))\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau)\,\mathrm{d}\tau + 1 - s\right)\mathrm{d}s \\
&= c.
\end{aligned}
$$

Hence, $A_1\colon \overline{P_1}_c \to \overline{P_1}_c$. In the same way, if $x \in \overline{P_1}_a$, then assumption (D_1) yields $f(t,x(t)) \le m_1 a$ for $t \in [0,1]$. As in the same argument above, we can obtain that $A_1\colon \overline{P_1}_a \to \overline{P_1}_a$. Therefore, condition (C_2) of Theorem 2.1 is satisfied.

Check condition (C_1) of Theorem 2.1. It is easy to see that $\{y \in P_1(\psi_1; b, 2b),\ \psi_1(y) > b\} \ne \emptyset$. If $x \in P_1(\psi_1; b, 2b)$, then $\psi_1(x) = x\left(\frac{1}{2}\right) \ge b$ and $\|x\| \le 2b$. So

$$
b \le x(t) \le 2b \qquad \text{for} \quad t \in \left[\tfrac{1}{2}, 1\right].
$$

So, (D_2) implies $f(t,x) \ge \phi_p(b/\delta_1)$ for $t \in \left[\frac{1}{2}, 1\right]$ and $x \in [b, 2b]$. Then

$$
\begin{aligned}
&\psi(A_1 x) = A_1 x\left(\tfrac{1}{2}\right) \\
= &\int_0^{\frac{1}{2}} \phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&+ \frac{h(1)}{1-h(1)}\int_0^1 \phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&- \frac{1}{1-h(1)}\int_0^1 h(s)\phi_q\left(\frac{1}{1-g(1)}\int_0^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_s^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
\ge &\int_0^{\frac{1}{2}} \phi_q\left(\frac{1}{1-g(1)}\int_{\frac{1}{2}}^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_{\frac{1}{2}}^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&+ \frac{h(1)}{1-h(1)}\int_0^{\frac{1}{2}} \phi_q\left(\frac{1}{1-g(1)}\int_{\frac{1}{2}}^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau + \int_{\frac{1}{2}}^1 f(\tau, x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{1-h(1)} \int_0^{\frac{1}{2}} h(s)\phi_q \left(\frac{1}{1-g(1)} \int_{\frac{1}{2}}^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau \right.\\
&\left. + \int_{\frac{1}{2}}^1 f(\tau, x(\tau))\,\mathrm{d}\tau \right) \mathrm{d}s \\
= \;& \frac{1}{1-h(1)} \int_0^{\frac{1}{2}} (1-h(s))\phi_q \left(\frac{1}{1-g(1)} \int_{\frac{1}{2}}^1 g(\tau) f(\tau, x(\tau))\,\mathrm{d}\tau \right.\\
&\left. + \int_{\frac{1}{2}}^1 f(\tau, x(\tau))\,\mathrm{d}\tau \right) \mathrm{d}s \\
> \;& \frac{1}{1-h(1)} \int_0^{\frac{1}{2}} (1-h(s))\phi_q \left(\frac{1}{1-g(1)} \int_{\frac{1}{2}}^1 g(\tau)\phi_p\big(\tfrac{b}{\delta_1}\big)\,\mathrm{d}\tau + \int_{\frac{1}{2}}^1 \phi_p\big(\tfrac{b}{\delta_1}\big)\,\mathrm{d}\tau \right) \mathrm{d}s \\
= \;& \frac{b/\delta_1}{1-h(1)} \int_0^{\frac{1}{2}} (1-h(s))\phi_q \left(\frac{1}{1-g(1)} \int_{\frac{1}{2}}^1 g(\tau)\,\mathrm{d}\tau + \frac{1}{2} \right) \mathrm{d}s \\
= \;& b,
\end{aligned}
$$

i.e., $\psi_1(Ax) > b$ for $x \in P_1(\psi_1; b, 2b)$. This shows that (C_1) of Theorem 2.1 is satisfied. Finally, we show that (C_3) of Theorem 2.1 also holds. Suppose $x \in P_1(\psi_1; b, c)$ with $\|Ax\| > 2b$. Since $(A_1x)''(t) \le 0$ and $(A_1x)'(t) \ge 0$ for $t \in [0,1]$, we have $A_1x(t) \ge t\|A_1x\|$, then

$$\psi_1(A_1x) = A_1x\big(\tfrac{1}{2}\big) \ge \frac{1}{2}\|A_1x\| > b.$$

So condition (C_3) of Theorem 2.1 is satisfied. Therefore an application of Theorem 2.1 completes the proof. □

From Theorem 3.1, we see that when assumptions like (D_1)–(D_3) are appropriately imposed on f, we can obtain any number of positive solutions of (1) and (2). To be more precise, we have the following conclusion.

Theorem 3.2. *Suppose* (H_1) *and* (H_2) *hold and there exist constants* $0 < a_1 < b_1 < 2b_1 < a_2 < b_2 < 2b_2 < \cdots < a_n$ *for* $n \in \mathbb{N}$, *such that the following conditions are satisfied:*

(E_1) $f(t,x) < \phi_p(m_1 a_i)$ *for* $t \in [0,1]$ *and* $x \in [0,a_i]$ *with* $i = 1,\dots,n$.

(E_2) $f(t,x) > \phi_p(b_i/\delta_1)$ *for* $t \in [\frac{1}{2},1]$ *and* $x \in [b_i, 2b_i]$ *with* $i = 1,\dots,n$.
Then BVP (1) *and* (2) *has at least* $2n-1$ *positive periodic solutions.*

Proof. When $n = 1$, it follows from condition (E_1) that $A_1 \colon \overline{P_1}_{a_1} \to P_{1,a_1}$ which means that A_1 has at least one fixed point $x_1 \in \overline{P_1}_{a_1}$ by the Schauder fixed point theorem. When $n = 2$, it is clear that Theorem 3.1 holds with $c = a_2$. Then we can obtain at least three positive solutions x_1, x_2 and x_3 satisfying $\|x_1\| \le a_1$, $\psi_1(x_2) > b_1$, $\|x_3\| > a_1$ and $\psi_1(x_3) < b_1$. Following this way, we finish the proof by the induction method.

Similarly, we have the following theorems for existence of at least three positive solutions or $2n-1$ positive solutions of BVP (1) and (3), whose proofs are similar to those of Theorem 3.3 and Theorem 3.4 and hence are omitted. □

Theorem 3.3. *Assume* (H_1) *and* (H_2). *In addition, there exist constants* $0 < a < b < 2b \le c$, $b < m_2\delta_2 c$ *such that*

(G_1) $f(t,x) < \phi_p(m_2 a)$ *for* $t \in [0,1]$ *and* $x \in [0,a]$.

(G_2) $f(t,x) > \phi_p(b/\delta_2)$ *for* $t \in [\frac{1}{2},1]$ *and* $x \in [b,2b]$.

(G_3) $f(t,x) < \phi_p(m_2 c)$ *for* $t \in [0,1]$ *and* $x \in [0,c]$.

Then BVP (1) *and* (3) *has at least three positive solutions* x_1, x_2 *and* x_3 *such that*

$$\|x_1\| < a, \quad b < \psi_2(x_2) \quad \text{and} \quad \|x_3\| > a \ \text{ with } \ \psi_2(x_3) < b.$$

Theorem 3.4. *Suppose* (H_1) *and* (H_2) *hold and there exist constants* $0 < a_1 < b_1 < 2b_1 < a_2 < b_2 < 2b_2 < \cdots < a_n$ *for* $n \in \mathbb{N}$, *such that the following conditions are satisfied:*

(F_1) $f(t,x) < \phi_p(m_2 a_i)$ *for* $t \in [0,1]$ *and* $x \in [0,a_i]$ *with* $i = 1,\dots,n$.

(F_2) $f(t,x) > \phi_p(b_i/\delta_2)$ *for* $t \in [\frac{1}{2},1]$ *and* $x \in [b_i, 2b_i]$ *with* $i = 1,\dots,n$.

Then BVP (1) *and* (3) *has at least* $2n-1$ *positive periodic solutions.*

Now, we present some examples to illustrate the main results.

Example 3.1. Consider the following BVP

$$\begin{cases} x''(t) + f(x(t)) = 0, & t \in (0,1), \\ x(0) = 0, & x'(1) = \frac{1}{2}x'\left(\frac{1}{2}\right), \end{cases} \tag{10}$$

where

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 8 + \frac{7}{3}(x-4), & 1 \le x \le 4, \\ 8 + \frac{16}{x}(x-4), & x \ge 4. \end{cases}$$

We find that for $h(t) \equiv 0$ and

$$g(t) = \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{1}{2}, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

the boundary value conditions in (10) reduce to (2). It is easy to check $m_1 = 1$ and $\delta_1 = 1/2$. Choose $a = 1$, $b = 4$ and $c = 24$. Then $a < b < 2b < c$ and $b < m_1\delta_1 c$, furthermore,

$$f(x) \begin{cases} \le m_1 c = 24, & 0 \le x \le 24, \\ \ge b/\delta_1 = 8, & 4 \le x \le 8, \\ \le m_1 a = 1, & 0 \le x \le 1. \end{cases}$$

An application of Theorem 3.1 implies (10) has at least three positive solutions x_1, x_2 and x_3 such that $\|x_1\| \le 1$, $x\left(\frac{1}{2}\right) > 4$, $\|x_3\| > 1$ and $x_3\left(\frac{1}{2}\right) < 4$.

Example 3.2. Consider the BVP problem

$$\begin{cases} (\phi_3(y'))' + f(y) = 0, & t \in (0,1), \\ u(0) = \int\limits_0^1 s^2 u(s)\,\mathrm{d}s, & u'(1) = \int\limits_0^1 s^3 u'(s)\,\mathrm{d}s. \end{cases} \tag{11}$$

We observe that $p = 3$, $q = 3/2$, $g(s) = \frac{1}{4}s^4$ and $h(s) = \frac{1}{3}s^3$. It is easy to check that

$$\delta_1 = \frac{3}{2}\int\limits_0^{1/2} \left(1 - \frac{1}{3}s^3\right)\phi_{3/2}\left(\frac{4}{3}\int\limits_{1/2}^{1} \frac{1}{4}\tau^4\,\mathrm{d}\tau + \frac{1}{2}\right)\mathrm{d}s = \frac{95}{128}\sqrt{\frac{271}{480}}.$$

$$
\begin{aligned}
m_1 &= \left[\frac{3}{2}\int_0^1 \Bigl(1-\frac{1}{3}s^3\Bigr)\phi_{3/2}\left(\frac{4}{3}\int_0^1 \frac{1}{4}\tau^4\,\mathrm{d}\tau + 1 - s\right)\mathrm{d}s\right]^{-1} \\
&> \left[\frac{3}{2}\int_0^1 \Bigl(1-\frac{1}{3}s^3\Bigr)\phi_{3/2}\left(\frac{4}{3}\int_0^1 \frac{1}{4}\tau^4\,\mathrm{d}\tau + 1\right)\mathrm{d}s\right]^{1} \\
&= \frac{2\sqrt{15}}{11}.
\end{aligned}
$$

Choose $a = 1$, $b = 2$, $c = 60$, we find $a < b < 2b < c$ and $b < m_1\delta_1 c$. Hence if

$$
f(x)\begin{cases} \le \phi_3\left(\frac{2\sqrt{15}}{11}\right), & 0 \le x \le 1, \\ \ge \phi_3\left(\frac{256}{95}\sqrt{\frac{480}{271}}\right), & 2 \le x \le 4, \\ \le \phi_3\left(\frac{120\sqrt{15}}{11}\right), & 0 \le x \le 60, \end{cases}
$$

then by application of Theorem 3.1, (11) has at least three positive solutions.

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