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BOOLEAN TRANSFORMATIONS WITH UNIQUE FIXED POINTS

Sergiu Rudeanu

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ABSTRACT. We determine the Boolean transformations $F\colon B^n \to B^n$ which have unique fixed points.

Boolean transformations are mappings of the form $F = (f_1, \dots, f_m)\colon B^n \to B^m$, where $(B, \vee, \cdot, ', 0, 1)$ is a Boolean algebra and $f_i\colon B^n \to B$ $(i = 1, \dots, n)$ are Boolean functions, that is, they are built up from variables and constants by superpositions of the basic operations $\vee$, $\cdot$, $'$ of B. The study of Boolean transformations goes back to Whitehead [11] and Löwenheim [3]. Since then Boolean transformations have been studied both *per se* [2], [7], [8], [9] and in view of applications to switching theory [5] and to the relationships between the consistency of a Boolean equation, the consequences of a Boolean equality and the functional dependence of a system of Boolean functions, including a characterization of the Moore-Marczewski independence in this case [1]. See also [6], [10].

The concept of fixed point makes sense for Boolean transformations with $m = n$. In the Boolean part of the paper [9] we gave a necessary and sufficient condition for the existence of fixed points of a Boolean transformation and we proved that the least fixed point of an isotone Boolean transformation can be reached by the Kleene well-known procedure. In the present paper we determine those Boolean transformations $F = (f_1, \dots, f_n)\colon B^n \to B^n$ which have a unique fixed point.

We assume the reader has some familiarity with computation in a Boolean algebra, which in fact is the same as computation with sets, in view of the well-known representation theorem for Boolean algebras. We denote conjunction by concatenation rather than $\cdot$. The ring operation

$$x + y = xy' \vee x'y \tag{1}$$

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is very useful; for instance, it has the property

$$x = y \iff x + y = 0 . \tag{2}$$

The symbols $\bigvee$ and $\prod$ will stand for iterated disjunction and conjunction, respectively. Greek subscripts $\alpha_1, \alpha_2, \ldots$ run over the subset $\{0,1\}$ of B. Thus e.g. the canonical disjunctive form of a Boolean funtion f is

$$f(x_1, \ldots, x_n) = \bigvee_{\alpha_1,\ldots,\alpha_n} f(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n} , \tag{3}$$

where we have set $x^1 = x$ and $x^0 = x'$. Recall also that

$$\Big(\bigvee_{\alpha_1,\ldots,\alpha_n} c_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \Big)' = \bigvee_{\alpha_1,\ldots,\alpha_n} (c_{\alpha_1\ldots\alpha_n})' x_1^{\alpha_1} \ldots x_n^{\alpha_n} . \tag{4}$$

We first settle the case $n := 1$, when Boolean transformations reduce to Boolean functions.

Proposition. *A Boolean function $f\colon B \to B$ has a unique fixed point if and only if it is a constant function, $f(x) = c$, in which case the unique fixed point is c.*

Proof. Let $f(x) = ax \vee bx'$ be a Boolean function. The equation $ax \vee bx' = x$ of fixed point is equivalent to

$$(ax \vee bx')'x \vee (ax \vee bx')x' = 0 \tag{5}$$

and using (4) we get

$$(5) \iff a'x \vee bx' = 0 \iff a'x = bx' = 0 \iff b \le x \le a .$$

Therefore the equation has a unique solution if and only if $b = a$, in which case the function reduces to $f(x) = a$ and the unique fixed point is a. $\square$

Before proving the main result, we exemplify it for the case $n := 2$, that is, $F = (f_1, f_2)$, where $f_i\colon B^2 \to B$ $(i = 1,2)$. We recall that δ_{ij} stands for the Kronecker symbol. Formulas (6) below reduce to

$$\prod_{\alpha_2} \bigvee_{j=1}^{2} f_j^{\delta_{1j} \vee \alpha_j'}(0, \alpha_2) = \bigvee_{\alpha_2} \prod_{j=1}^{2} f_j^{\delta_{1j} \vee \alpha_j}(1, \alpha_2) ,$$

$$\prod_{\alpha_1} \bigvee_{j=1}^{2} f_j^{\delta_{2j} \vee \alpha_j'}(\alpha_1, 0) = \bigvee_{\alpha_1} \prod_{j=1}^{2} f_j^{\delta_{2j} \vee \alpha_j}(\alpha_1, 1) ,$$

that is,

$$\prod_{\alpha_2}\bigl(f_1(0,\alpha_2)\vee f_2^{\alpha_2'}(0,\alpha_2)\bigr)=\bigvee_{\alpha_2} f_1(1,\alpha_2)f_2^{\alpha_2}(1,\alpha_2)\,,$$
$$\prod_{\alpha_1}\bigl(f_1^{\alpha_1'}(\alpha_1,0)\vee f_2(\alpha_1,0)\bigr)=\bigvee_{\alpha_1} f_1^{\alpha_1}(\alpha_1,1)f_2(\alpha_1,1)\,,$$

which quite explicitly means

$$\bigl(f_1(0,0)\vee f_2(0,0)\bigr)\bigl(f_1(0,1)\vee f_2'(0,1)\bigr)=f_1(1,0)f_2'(1,0)\vee f_1(1,1)f_2(1,1)\,,$$
$$\bigl(f_1(0,0)\vee f_2(0,0)\bigr)\bigl(f_1'(1,0)\vee f_2(1,0)\bigr)=f_1'(0,1)f_2(0,1)\vee f_1(1,1)f_2(1,1)\,.$$

Now we proceed with the main result.

Theorem. *Let $F=(f_1,\dots,f_n)\colon B^n\to B^n$ be a Boolean transformation, where $n\ge 2$. Then F has a unique fixed point if and only if conditions*

$$\begin{aligned}&\prod_{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n}\ \bigvee_{j=1}^{n} f_j^{\delta_{ij}\vee\alpha_j'}(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n)\\ =&\bigvee_{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n}\ \prod_{j=1}^{n} f_j^{\delta_{ij}\vee\alpha_j}(\alpha_1,\dots,\alpha_{i-1},1,\alpha_{i+1},\dots,\alpha_n)\end{aligned}\tag{6.i}$$

hold for $i=1,\dots,n$, in which case the unique fixed point $(\xi_1,\dots,\xi_n)$ is given by the prescription: ξ_i is either side of condition (6.i), for $i=1,\dots,n$.

Proof. Taking into account (1)–(4), the equation $F(x_1,\dots,x_n)=(x_1,\dots,x_n)$ of fixed points can be written in the following equivalent successive forms:

$$f_j(x_1,\dots,x_n)=x_j\qquad (j=1,\dots,n)\,,$$
$$\bigvee_{\alpha_1,\dots,\alpha_n} f_j(\alpha_1,\dots,\alpha_n)x_1^{\alpha_1}\dots x_n^{\alpha_n}=x_j\qquad (j=1,\dots,n)\,,$$

$$\Bigl(\bigvee_{\alpha_1,\dots,\alpha_n} f_j'(\alpha_1,\dots,\alpha_n)x_1^{\alpha_1}\dots x_n^{\alpha_n}\Bigr)x_j\vee\Bigl(\bigvee_{\alpha_1,\dots,\alpha_n} f_j(\alpha_1,\dots,\alpha_n)x_1^{\alpha_1}\dots x_n^{\alpha_n}\Bigr)x_j'=0$$
$$(j=1,\dots,n)\,,$$

$$\begin{aligned}\bigvee_{j=1}^{n}\Bigl(&\bigvee_{\substack{\alpha_1,\dots,\alpha_{j-1},\\ \alpha_{j+1},\dots,\alpha_n}} f_j'(\alpha_1,\dots,\alpha_{j-1},1,\alpha_{j+1},\dots,\alpha_n)x_1^{\alpha_1}\dots x_{j-1}^{\alpha_{j-1}}x_jx_{j+1}^{\alpha_{j+1}}\dots x_n^{\alpha_n}\\ &\vee\bigvee_{\substack{\alpha_1,\dots,\alpha_{j-1},\\ \alpha_{j+1},\dots,\alpha_n}} f_j(\alpha_1,\dots,\alpha_{j-1},0,\alpha_{j+1},\dots,\alpha_n)x_1^{\alpha_1}\dots x_{j-1}^{\alpha_{j-1}}x_j'x_{j+1}^{\alpha_{j+1}}\dots x_n^{\alpha_n}\Bigr)=0\,,\end{aligned}$$

$$\bigvee_{j=1}^{n} \bigvee_{\alpha_j} \bigvee_{\substack{\alpha_1,\dots,\alpha_{j-1},\\ \alpha_{j+1},\dots,\alpha_n}} f_j^{\alpha_j'}(\alpha_1,\dots,\alpha_n) x_1^{\alpha_1} \dots x_n^{\alpha_n} = 0\,,$$

$$f(x_1,\dots,x_n) \equiv \bigvee_{j=1}^{n} \bigvee_{\alpha_1,\dots,\alpha_n} f_j^{\alpha_j'}(\alpha_1,\dots,\alpha_n) x_1^{\alpha_1} \dots x_n^{\alpha_n} = 0\,. \tag{7}$$

Now the original problem has been reduced to that of finding a necessary and sufficient condition for equation (7) to have a unique solution. But Parker and Bernstein [4] (cf. [6, Theorems 6.7, 6.6]) have proved that a Boolean equation $f(x_1,\dots,x_n) = 0$ has a unique solution if and only if conditions

$$\begin{aligned} &\prod_{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n} f\big(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n\big) \\ = &\bigvee_{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n} f'\big(\alpha_1,\dots,\alpha_{i-1},1,\alpha_{i+1},\dots,\alpha_n\big) \qquad (i = 1,\dots,n) \end{aligned} \tag{8.i}$$

hold, in which case the unique solution $(\xi_1,\dots,\xi_n)$ is obtained by taking ξ_i to be either side of conditions (8.i), for $i = 1,\dots,n$. Therefore it remains to prove that conditions (8) for equation (7) become (6).

But it follows from (7) that for every $\alpha_1,\dots,\alpha_n \in \{0,1\}$ we have

$$\begin{aligned} f(\alpha_1,\dots,\alpha_n) &= \bigvee_{j=1}^{n} f_j^{\alpha_j'}(\alpha_1,\dots,\alpha_n) \\ &= f_i^{\alpha_i'}(\alpha_1,\dots,\alpha_n) \vee \bigvee_{j\neq i}^{1,n} f_j^{\alpha_j'}(\alpha_1,\dots,\alpha_n)\,, \end{aligned}$$

therefore

$$f'(\alpha_1,\dots,\alpha_n) = f_i^{\alpha_i}(\alpha_1,\dots,\alpha_n) \prod_{j\neq i}^{1,n} f_j^{\alpha_j}(\alpha_1,\dots,\alpha_n)\,,$$

hence for each $i \in \{1,\dots,n\}$,

$$\begin{aligned} &f\big(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n\big) \\ &= f_i(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n) \vee \bigvee_{j\neq i}^{1,n} f_j^{\alpha_j'}\big(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n\big) \\ &= \bigvee_{j=1}^{n} f_j^{\delta_{ij}\vee\alpha_j'}\big(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n\big) \end{aligned}$$

and

$$\begin{aligned}
& f'(\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_n) \\
&= f_i(\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_n) \prod_{j \neq i}^{1,n} f_j^{\alpha_j}(\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_n) \\
&= \prod_{j=1}^{n} f_j^{\delta_{ij} \vee \alpha_j}(\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_n) ,
\end{aligned}$$

completing the proof. □

Remark. As a matter of fact, [6, Theorem 6.7] provides 5 conditions equivalent to the fact that equation $f(x_1, \ldots, x_n) = 0$ has a unique solution; one of these conditions is system (8). The reader may wish to use the other 4 conditions in order to obtain several equivalent formulations of conditions (6). To do this it seems advantageous to use the notation in [6]. Thus the present formula (7) has the equivalent forms

$$(\forall X \in B^n) \Big(f(X) = \bigvee_{j=1}^{n} \bigvee_{A} f_j^{\alpha_j'}(A) X^A \Big) ,$$

$$(\forall A \in \{0,1\}^n) \Big(f(A) = \bigvee_{j=1}^{n} f_j^{\alpha_j'}(A) \Big) ,$$

$$(\forall A \in \{0,1\}^n) \Big(f'(A) = \prod_{j=1}^{n} f_j^{\alpha_j}(A) \Big) ,$$

and condition (ii) in Theorem 6.7 says that the system $\{f'(A) : \ A \in \{0,1\}^n\}$ is orthonormal; etc.

So, given a Boolean transformation F, we find out whether F has a unique fixed point by simply checking conditions (6), which provide as well the unique fixed point. But the Theorem also enables one to construct all the Boolean transformations with unique fixed points by solving equations (6) with respect to the unknowns $f_j(\alpha_1, \ldots, \alpha_n)$. We illustrate this point in the case $n := 2$.

To simplify notation, we set $F = (f, g)$ with

$$f(x,y) = axy \vee bxy' \vee cx'y \vee dx'y' , \tag{9.1}$$

$$g(x,y) = mxy \vee nxy' \vee px'y \vee qx'y' , \tag{9.2}$$

so that conditions (6) for $n := 2$, which we have written down explicitly before the Theorem, become

$$(d \vee q)(c \vee p') = bn' \vee am\,, \tag{10.1}$$

$$(d \vee q)(b' \vee n) = c'p \vee am\,, \tag{10.2}$$

and we are going to solve equations (10) with respect to the unknowns a, b, c, d, m, n, p, q.

Setting

$$am = A\,, \quad bn' = B\,, \quad c'p = C\,, \quad d'q' = D\,, \tag{11}$$

system (10) becomes

$$D'C' = B \vee A\,,$$

$$D'B' = C \vee A\,,$$

and is equivalent to the single equation

$$D'C'B'A' \vee (D \vee C)(B \vee A) \vee D'B'C'A' \vee (D \vee B)(C \vee A) = 0\,,$$

which, after simplifications, reduces to condition (ii) in the Remark:

$$A'B'C'D' \vee AB \vee AC \vee AD \vee BC \vee BD \vee CD = 0\,. \tag{12}$$

We split equation (12) into

$$AB = 0\,,$$

$$(A \vee B)C = 0\,,$$

$$A'B'C'D' \vee (A \vee B \vee C)D = 0\,,$$

and write the latter equations in the form

$$B \le A'\,, \tag{13.1}$$

$$C \le A'B'\,, \tag{13.2}$$

$$D = A'B'C'\,. \tag{13.3}$$

Thus formulas (13) provide a tree-like construction of all the solutions to equation (12): the unknown A is arbitrary in the Boolean algebra, then for each value of A the unknown B takes all the values satisfying (13.1), then for each couple (A, B) determined in this way the unknown C takes all the values satisfying (13.2) and finally for each triple (A, B, C) determined in this way the unknown D is uniquely determined by (13.3). Alternatively, the set of solutions to (12) can be described by formulas

$$A = r\,, \qquad B = r's\,, \qquad C = r's't\,, \qquad D = r's't'\,, \tag{13'}$$

where r, s, t are arbitrary parameters.

Now equation (12) is in fact a transcription of system (10) via the transformations (11). Therefore *we find all the transformations* $F = (f, g)$ *given by* (9)*and with unique fixed points by solving the system* (11) *with respect to the unknowns* a, m, b, n, c, p, d, q *for all the solutions* (A, B, C, D) *of equation* (12).

Each system of the form (11) is very easy to solve. It consists of four equations with separate variables. Each equation is of the form $\xi^\alpha \eta^\beta = k$, where ξ, η are the unknowns and k, α, β are constants with $\alpha, \beta \in \{0, 1\}$. The reader is referred e.g. to [6] for the solution of such equations.

The transformations F we have obtained depend in general on the Boolean algebra we are working in. If we want to obtain Boolean transformations whose expressions make sense in any Boolean algebra, we must impose the supplementary condition $a, b, c, d, m, n, p, q \in \{0, 1\}$; in other words, according to the terminology of [6] and [10], F must be a *simple Boolean transformation.* In this case $A, B, C, D \in \{0, 1\}$, hence equation (12) has only four solutions: $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Hence there are four systems of the form (11):

$$am = 1\,, \quad bn' = 0\,, \quad c'p = 0\,, \quad d'q' = 0\,, \tag{14}$$

$$am = 0\,, \quad bn' = 1\,, \quad c'p = 0\,, \quad d'q' = 0\,, \tag{15}$$

$$am = 0\,, \quad bn' = 0\,, \quad c'p = 1\,, \quad d'q' = 0\,, \tag{16}$$

$$am = 0\,, \quad bn' = 0\,, \quad c'p = 0\,, \quad d'q' = 1\,. \tag{17}$$

Now each equation $\xi^\alpha \eta^\beta = 1$ has the unique solution $\xi^\alpha = \eta^\beta = 1$, while each equation $\xi^\alpha \eta^\beta = 0$ has 3 solutions. Hence each of the systems (14)–(17) has 9 solutions, therefore *there exist* 36 *simple Boolean transformations* $F = (f, g)$ *with unique fixed points* out of the total number of $2^8 = 256$ simple Boolean transformations.

In particular we may wish to determine the *isotone* simple Boolean transformations $F(x, y)$ with unique fixed points. The isotony of the transformation F refers to the componentwise order and is therefore equivalent to the property that both f and g are isotone. It is well known that f given by (9.1) is isotone if and only if

$$d \le b, c \le a\,, \tag{18.1}$$

and similarly, the condition for g given by (9.2) is

$$q \le n, p \le m\,. \tag{18.2}$$

So the problem is now to find the 0–1 solutions of the systems (14)–(17) under the supplementary conditions (18).

System (14) implies $a = m = 1$. If $d = 0$, we infer $q = 1$, hence $g = 1$ by (18.2); thus system (14) reduces to $c' = 0$, therefore $f(x, y) = xy \vee bxy' \vee x'y = y \vee bx$

and we have obtained the solutions

$$F = (y, 1)\,, \qquad F = (x \vee y, 1)\,.$$

If $d = 1$, we infer $f = 1$ by (18.1), hence system (14) reduces to $n' = 0$, therefore $g(x, y) = xy \vee xy' \vee px'y \vee qx'y' = x \vee py \vee qy'$, where $q \leq p$ by (18.2), hence we have obtained the solutions

$$F = (1, x)\,, \qquad F = (1, x \vee y)\,, \qquad F = (1, 1)\,.$$

System (15) implies $b = 1$, $n = 0$. It follows that $a = 1$ by (18.1) and $q = 0$ by (18.2). Hence system (15) reduces to $m = 0$, $c'p = 0$, $d' = 0$. Therefore $g = 0$ by (18.2) and $d = 1$, hence $f = 1$ by (18.1) and we have obtained the solution

$$F = (1, 0)\,.$$

System (16) implies $c = 0$, $p = 1$. It follows that $d = 0$ by (18.1) and $m = 1$ by (18.2). Hence system (16) reduces to $a = 0$, $bn' = 0$, $q' = 0$. Therefore $f = 0$ by (18.1) and $q = 1$, hence $g = 1$ by (18.2) and we have obtained the solution

$$F = (0, 1)\,.$$

System (17) implies $d = q = 0$. If $a = 1$, then $m = 0$, hence $g = 0$ by (18.2) and system (17) reduces to $b = 0$, therefore $f(x, y) = xy \vee cx'y$ and we have obtained the solutions

$$F = (xy, 0)\,, \qquad F = (y, 0)\,.$$

If $a = 0$, then $f = 0$ by (18.1) and system (17) reduces to $p = 0$, therefore $g(x, y) = mxy \vee nxy'$, where $n \leq m$ by (18.2) and we have obtained the solutions

$$F = (0, 0)\,, \qquad F = (0, xy)\,, \qquad F = (0, x)\,.$$

So there are 12 *isotone simple Boolean transformations* $F = (f, g)$ *with unique fixed points.* See also [9].

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University of Bucharest
Faculty of Mathematics
Bucharest
ROMANIA
E-mail: srudeanu@yahoo.com