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Dedicated to Professor B. Riečan on the occasion of his 70th birthday

AN ORDER FOR QUANTUM OBSERVABLES

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ABSTRACT. The set of bounded observables for a quantum system is represented by the set of bounded self-adjoint operators $\mathcal{S}(H)$ on a complex Hilbert space H. The usual order $A \leq B$ on $\mathcal{S}(H)$ is determined by assuming that the expectation of A is not greater than the expectation of B for every state of the system. We may think of $\leq$ as a numerical order on $\mathcal{S}(H)$. In this article we introduce a new order $\preceq$ on $\mathcal{S}(H)$ that may be interpreted as a logical order. This new order is determined by assuming that $A \preceq B$ if the proposition that A has a value in Δ implies the proposition that B has a value in Δ for every Borel set Δ not containing 0. We give various characterizations of this order and show that it is generated by an orthosum $\oplus$ that endows $\mathcal{S}(H)$ with the structure of a generalized orthoalgebra. We also show that the usual order $\leq$ cannot be generated by an orthosum. We demonstrate that if we restrict $\oplus$ to an interval $[0, A] \subseteq \mathcal{S}(H)$, then we obtain a structure that is isomorphic to an orthomodular lattice of projections on H. The lattice structure of $\mathcal{S}(H)$ is investigated and unlike $\big(\mathcal{S}(H), \leq\big)$ it is shown that $\big(\mathcal{S}(H), \preceq\big)$ is a near-lattice in the sense that if $A, B \preceq C$, then $A \wedge B$ and $A \vee B$ exist. Moreover, we show that if $\dim(H) < \infty$, then $A \wedge B$ always exists. We also consider the commutative case in which observables are represented by fuzzy random variables.

1. Introduction

In this article we introduce a new order for quantum observables. In some respects this order is more natural and has better properties than the usual order of observables while in other respects the usual order appears to be more suitable. In any case it is useful to compare the two orders and perhaps to employ one or the other depending on the applications or circumstances involved.

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The set of bounded observables for a quantum system is usually represented by the set $\mathcal{S}(H)$ of bounded self-adjoint operators on a complex Hilbert space H. In the traditional order for $A, B \in \mathcal{S}(H)$ we define $A \leq B$ if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for every $x \in H$. This order has the physical interpretation that the expectation of A is not greater than the expectation of B in every state of the system. We may think of $\leq$ as a numerical order on $\mathcal{S}(H)$. Under this order $(\mathcal{S}(H), \leq)$ becomes a partially ordered set or poset. However, a well-known theorem due to R. Kadison [8] shows that $(\mathcal{S}(H), \leq)$ is not a lattice. In fact, $(\mathcal{S}(H), \leq)$ is as far from being a lattice as possible in the sense that the greatest lower bound $A \wedge B$ exists if and only if $A \leq B$ or $B \leq A$ in which case $A \wedge B$ is the smaller of the two. This is unfortunate because lattices have a much stronger structure than posets and $A \wedge B$ and the least upper bound $A \vee B$ have physical significance. The new order $\preceq$ introduced in this paper remedies this situation in the sense that $A \wedge B$ and $A \vee B$ exist if there is a $C \in \mathcal{S}(H)$ such that $A, B \preceq C$. We call such a structure a near-lattice. In this way, $(\mathcal{S}(H), \preceq)$ becomes a near-lattice ordered generalized σ-orthoalgebra.

We present various characterizations of $\preceq$ and compare this partial order with the usual partial order. Physically, the most interesting characterization states that $A \preceq B$ if and only if $P^A(\Delta) \leq P^B(\Delta)$ for every Borel set Δ with $0 \notin \Delta$ where P^A is the spectral measure for A. This characterization may be interpreted as saying that the event (or proposition) $P^A(\Delta)$ implies the event $P^B(\Delta)$. We conclude that $\preceq$ is a logical order for observables. It is also observed that $\preceq$ is algebraic in the sense that $\preceq$ is generated by the orthosum $\oplus$ of an orthoalgebra whereas the usual order $\leq$ cannot be generated by an orthosum.

We demonstrate that if we restrict $\oplus$ to an interval $[0, A] \subseteq \mathcal{S}(H)$, then we obtain a structure that is isomorphic to an orthomodular lattice of projections on H. We also show that if $\dim(H) < \infty$, then $[0, A]$ is the cartesian product of projection lattices. Moreover, in the finite dimensional case it is shown that $A \wedge B$ always exists. We also consider the commutative case in which observables are represented by fuzzy random variables. This case further motivates our definition of $\preceq$ and provides intuition for results and proofs of the general noncommutative case. As we shall see, most of our results in the commutative case have noncommutative counterparts. A possible exception is that $f \wedge g$ always exists for random variables f and g while we do not know whether $A \wedge B$ always exists for $A, B \in \mathcal{S}(H)$.

Finally, we point out that in both the commutative and noncommutative cases the numerical order $\leq$ and the logical order $\preceq$ agree on sharp elements. In the commutative case the sharp elements are given by the measurable subsets (or events) of the sample space and we have the equivalent statements $A \subseteq B$, $A \leq B$ and $A \preceq B$. In the noncommutative case the sharp elements are given by the set of orthogonal projections (quantum events) $\mathcal{P}(H)$ and for $P, Q \in \mathcal{P}(H)$

we have the equivalent statements, $PQ = P$, $P \leq Q$, $P \preceq Q$.

2. Effect algebras

The study of measurements is an important part of any physical theory. The simplest type of measurement is a yes-no measurement or effect ([1], [2], [7], [9], [10]). More general measurements and observables can be constructed using these effects. The set of effects for a quantum system can be organized into a mathematical structure called an effect algebra, which has recently been introduced for foundational studies in quantum mechanics ([3], [4], [5], [6]). This section reviews the definition of an effect algebra and the related concepts of generalized effect algebra and orthoalgebra. The main algebraic operation in these structures is an *orthosum* $a \oplus b$, which is a partial binary operation on the set of effects. If $a \oplus b$ is defined, we write $a \perp b$ and say that a and b are *orthogonal*. Roughly speaking, $a \oplus b$ corresponds to a parallel combination of the two effects a and b.

A *generalized effect algebra* is an algebraic system $(E, 0, \oplus)$ where E is a set, $0 \in E$ and $\oplus$ is a partial binary operation on E that satisfies the following conditions.

(GEA1) If $a \perp b$, then $b \perp a$ and $b \oplus a = a \oplus b$.

(GEA2) If $b \perp c$ and $a \perp (b \oplus c)$, then $a \perp b$, $c \perp (a \oplus b)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

(GEA3) $0 \perp a$ for all $a \in E$ and $0 \oplus a = a$.

(GEA4) If $a \oplus b = a \oplus c$, then $b = c$.

(GEA5) If $a \oplus b = 0$, then $a = b = 0$.

A *generalized orthoalgebra* is a generalized effect algebra $(E, 0, \oplus)$ that also satisfies:

(OA) If $a \perp a$, then $a = 0$.

An *effect algebra* is an algebraic system $(E, 0, 1, \oplus)$ where E is a set, $0, 1 \in E$ with $0 \neq 1$ and $\oplus$ is a partial binary operation on E that satisfies (GEA1), (GEA2) and:

(EA1) For every $a \in E$ there exists a unique $a' \in E$ such that $a \perp a'$ and $a \oplus a' = 1$.

(EA2) If $a \perp 1$, then $a = 0$.

An *orthoalgebra* is an effect algebra that satisfies (OA).

LEMMA 2.1. ([3]) *Every effect algebra is a generalized effect algebra and every orthoalgebra is a generalized orthoalgebra.*

Proof. Let $(E, 0, 1, \oplus)$ be an effect algebra. To show that (GEA3) holds we apply (EA1) and (EA2) to obtain $0 = 1'$. Hence, by (GEA2) we have that

$$(0 \oplus a) \oplus a' = 0 \oplus (a \oplus a') = 0 \oplus 1 = 1\,.$$

Applying (EA1) gives that $0 \oplus a = a$. To show that (GEA4) holds, suppose that $a \oplus b = a \oplus c$. By (EA1) there exists a $d \in E$ such that

$$(a \oplus b) \oplus d = (a \oplus c) \oplus d = 1\,.$$

Applying (GEA1) and (GEA2) we obtain

$$(a \oplus d) \oplus b = (a \oplus d) \oplus c = 1\,.$$

By (EA1) we have that $b = c = (a \oplus d)'$. To show that (GEA5) holds, suppose that $a \oplus b = 0$. Then $a \perp b$ and $a \oplus b \perp 1$, so by (GEA2) we have that $a \perp 1$. Applying (EA2) gives that $a = 0$. Hence, by (GEA3) we have that $b = 0$. That every orthoalgebra is a generalized orthoalgebra now follows. □

For a generalized effect algebra E with $a, b \in E$, we define $a \le b$ if there exists a $c \in E$ such that $a \perp c$ and $a \oplus c = b$. This unique c is denoted by $c = b \ominus a$.

LEMMA 2.2. ([3]) *If E is a generalized effect algebra, then $(E \le)$ is a poset and $0 \le a$ for every $a \in E$.*

Proof. We have that $a \le a$ because $a \oplus 0 = a$. If $a \le b$ and $b \le a$, then there exist $c, d \in E$ such that $a \oplus c = b$ and $b \oplus d = a$. Hence,

$$b \oplus d \oplus c = a \oplus c = b = b \oplus 0\,.$$

Applying (GEA4) we obtain $d \oplus c = 0$, so by (GEA5), $d = c = 0$. Therefore, $a = b$. To prove transitivity, suppose that $a \le b$ and $b \le c$. Then there exist $d, e \in E$ such that $a \oplus d = b$ and $b \oplus e = c$. Hence, $a \oplus (d \oplus e) = c$, so that $a \le c$. Since $0 \oplus a = a$, we have that $0 \le a$ for every $a \in E$. □

Let E be a generalized effect algebra and let $a \in E$ with $a \ne 0$. In the interval $[0, a] = \{b \in E : \ b \le a\}$ define $b \oplus_a c = b \oplus c$ whenever $b \perp c$ and $b \oplus c \le a$. We thus have that $b \perp_a c$ whenever $b \perp c$ and $b \oplus c \in [0, a]$. The next result shows that $[0, a]$ has desirable properties.

THEOREM 2.3. *If E is a generalized effect algebra (orthoalgebra) and $a \in E$ with $a \ne 0$, then $\{[0, a], 0, a, \oplus_a\}$ is an effect algebra (orthoalgebra). Moreover, the order on $[0, a]$ is the restriction of the order on E to $[0, a]$.*

Proof. Of course, $\oplus_a$ satisfies conditions (GEA1) and (GEA2). To demonstrate (EA1), suppose that $b \in [0,a]$ and define $c = a \ominus b$. Then $b \perp_a c$ and $b \oplus_a c = a$. Moreover, c is unique by (GEA4). To demonstrate (EA2) suppose that $b \in [0,a]$ and $b \perp_a a$. Then $b \oplus a \leq a$ and $b \oplus a \geq a$, so that $b \oplus a = a = 0 \oplus a$. Applying (GEA4) we conclude that $b = 0$. Now suppose that E is a generalized orthoalgebra and $b \perp_a b$. Then $b \perp b$, which implies that $b = 0$. Hence, $\{[0,a], 0, a, \oplus_a\}$ is an orthoalgebra. Finally, it is clear that $b \leq_a c$ in $[0,a]$ implies that $b \leq c$ in E. Conversely, if $b, c \in [0,a]$ and $b \leq c$ in E, then $b \oplus d = c$ for some $d \in E$. But $d \leq b \oplus d = c \leq a$, so that $b \ominus_a d = c$. Therefore, $b \leq_a c$. □

It is well known that $(E, \leq, ')$ is a *bounded involution poset* for any effect algebra E. That is, for every $a, b \in E$ we have that $0 \leq a \leq 1$, $a'' = a$ and $a \leq b$ implies $b' \leq a'$. Also $a \leq b'$ if and only if $a \perp b$. Moreover, if E is an orthoalgebra, then $(E, \leq, ')$ is an *orthocomplemented poset.* That is, $(E, \leq, ')$ is a bounded involution poset and $a \wedge a' = 0$, $a \vee a' = 1$ for every $a \in E$. Finally, if E is an orthoalgebra and $a \perp b$ implies that $a \oplus b = a \vee b$, then E is an *orthomodular poset.*

We call an effect algebra E a *σ-effect algebra* if for any nondecreasing sequence $a_1 \leq a_2 \leq \cdots$ in E the least upper bound $\bigvee a_i$ exists in E. We define generalized σ-effect algebras, generalized σ-orthoalgebras and σ-orthoalgebras in similar ways.

Although there are many examples of effect algebras and orthoalgebras, we shall only consider a few of them here. Any Boolean algebra is an orthoalgebra where $a \perp b$ if $a \wedge b = 0$ and in this case $a \oplus b = a \vee b$. If X is a nonempty set, the collection of fuzzy subsets $[0,1]^X$ of X forms an effect algebra where $f \perp g$ if $f + g \leq 1$ and in this case $f \oplus g = f + g$. For a complex hilbert space H we define the set of *quantum effects* $\mathcal{E}(H)$ on H as the set of bounded linear operators on H satisfying $0 \leq A \leq I$ where $\leq$ is the usual order of self-adjoint operators. For $A, B \in \mathcal{E}(H)$, define $A \perp B$ if $A + B \leq I$ and in this case $A \oplus B = A + B$. Then $\big(E(H), 0, I, \oplus\big)$ becomes a σ-effect algebra. The set of orthogonal projections $\mathcal{P}(H) \subseteq \mathcal{E}(H)$ corresponds to the set of *quantum events* and forms a σ-orthoalgebra.

An example of a generalized effect algebra that is not an effect algebra is the set of nonnegative real numbers $\mathbb{R}^+ = [0, \infty)$. In this case, we define $a \perp b$ for all $a, b \in \mathbb{R}^+$ and $a \oplus b = a + b$. Similar examples are given by the set of all nonnegative functions on a nonempty set and the set of all positive operators $\mathcal{S}(H)^+$ on a complex Hilbert space. Examples of generalized orthoalgebras that are not orthoalgebras will be given in the next two sections.

A generalized effect algebra E is *lattice ordered* if E is a lattice relative to its usual order; that is, the greatest lower bound $a \wedge b$ and least upper bound $a \vee b$ exist for all $a, b \in E$. We say that E is *near-lattice ordered* if $a \wedge b$ and

$a \vee b$ exist whenever there is a $c \in E$ with $a, b \leq c$. Our previous examples except $\mathcal{E}(H)$ and $\mathcal{S}(H)^+$ are lattice ordered while $\mathcal{E}(H)$ and $\mathcal{S}(H)^+$ are not even near-lattice ordered. A lattice ordered orthoalgebra in which $a \oplus b = a \vee b$ is called an *orthomodular* lattice. An important example of an orthomodular lattice is $\mathcal{P}(H)$.

An element a of an effect algebra E is *sharp* if $a \wedge a' = 0$. It is easy to show that E is an orthoalgebra if and only if every $a \in E$ is sharp. In $[0,1] \subseteq \mathbb{R}$ the only sharp elements are 0 and 1 and in $[0,1]^X$ an element is sharp if and only if it is a characteristic function and hence a subset of X. The sharp elements of $\mathcal{E}(H)$ are precisely the projections $\mathcal{P}(H)$. An element a of a generalized effect algebra E is *principal* if $b, c \leq a$ with $b \perp c$ imply that $b \oplus c \leq a$. In the examples $\mathbb{R}^+$, $\mathcal{S}(H)^+$ the only principal element is 0. If E is an effect algebra, then every principal element is sharp, but the converse does not hold ([3]).

3. The commutative case

This section considers the case of classical commuting observables. These are represented by random variables on a probability space $(\Omega, \mathcal{A}, \mu)$. As usual, we think of $\mathcal{A}$ as the set of events for some statistical experiment. For $A, B \in \mathcal{A}$ we write $A \perp B$ if $A \cap B = \emptyset$ and define the orthosum $A \oplus B = A \cup B$ whenever $A \perp B$. Then $(\mathcal{A}, \emptyset, \Omega, \oplus)$ is a σ-orthoalgebra in which $A' = A^c$. In fact, $(\mathcal{A}, \emptyset, \Omega, \oplus)$ is a Boolean σ-algebra, which is a much stronger statement.

We identify an event $A \in \mathcal{A}$ with its characteristic function χ_A. Notice that $A \perp B$ if and only if $\chi_A \chi_B = 0$. We can think of characteristic functions as yes-no or $1-0$ measurements for our statistical system. That is, given an outcome $\omega \in \Omega$, $\chi_A(\omega)$ gives the values 1 or 0 depending on whether $\omega \in A$ or not. One of the main reasons that the orthosum is important is that $\mu(A \oplus B) = \mu(A) + \mu(B)$. In fact, considering $[0,1] \subseteq \mathbb{R}$ to be an effect algebra we have that $\mu(A \oplus B) = \mu(A) \oplus \mu(B)$ and $\mu(\Omega) = 1$, so μ becomes an effect algebra morphism. Note that the orthoalgebra order $\preceq$ on $(\mathcal{A}, \emptyset, \Omega, \oplus)$ coincides with the usual order $\leq$. That is $\chi_A \preceq \chi_B$ if and only if $\chi_A(\omega) \leq \chi_B(\omega)$ for every $\omega \in \Omega$.

We would now like to extend the orthosum to general measurements that can have more than two values so as to obtain a mathematical structure for the set of all measurements associated with $\mathcal{A}$. These measurements are represented by the set $\mathcal{M}(\mathcal{A})$ of random variables on $(\Omega, \mathcal{A}, \mu)$. A natural extension is obtained by defining $f \perp g$ if $fg = 0$ for $f, g \in \mathcal{M}(\mathcal{A})$. Defining the *support* of f by $\operatorname{supp}(f) = \{\omega \in \Omega : f(\omega) \neq 0\}$ and the *nullity* of f by $\operatorname{null}(f) = f^{-1}(0)$ we have that $f \perp g$ if and only if $\operatorname{supp}(g) \subseteq \operatorname{null}(f)$. Equivalently, $f \perp g$ if and only if $\operatorname{supp}(f) \perp \operatorname{supp}(g)$. If $f \perp g$, we define $f \oplus g = f + g$. It is straightforward to check that $(\mathcal{M}(\mathcal{A}), 0, \oplus)$ is a generalized orthoalgebra. Let $\preceq$ be the orthoalgebra order on $\mathcal{M}(\mathcal{A})$; that is, $f \preceq g$ if there is an $h \in \mathcal{M}(\mathcal{A})$ such that $f \perp h$ and $f \oplus h = g$. It follows from Lemma 2.2 that $(\mathcal{M}(\mathcal{A}), \preceq)$ is a poset and $0 \preceq f$ for every $f \in \mathcal{M}(\mathcal{A})$.

THEOREM 3.1. *The following statements are equivalent:*

(i) $f \preceq g$.
(ii) $f(\omega) = g(\omega)$ *for every* $\omega \in \operatorname{supp}(f)$.
(iii) $f = g\chi_{\operatorname{supp}(f)}$.
(iv) $fg = f^2$.
(v) $f^{-1}(\Delta) \subseteq g^{-1}(\Delta)$ *for every* $\Delta \in \mathcal{B}(\mathbb{R})$ *with* $0 \notin \Delta$ *where* $\mathcal{B}(\mathbb{R})$ *is the* σ*-algebra of Borel subsets of* $\mathbb{R}$.

Proof. To show that (i) implies (ii), suppose that (i) holds. Then there exists an $h \in \mathcal{M}(\mathcal{A})$ such that $fh = 0$ and $f + h = g$. If $\omega \in \operatorname{supp}(f)$, then $h(\omega) = 0$, so that $f(\omega) = g(\omega)$. It is clear that (ii), (iii) and (iv) are equivalent. We now show that (ii) and (v) are equivalent. Suppose that (ii) holds and that $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. If $\omega \in f^{-1}(\Delta)$, then $f(\omega) \in \Delta$ and $f(\omega) \neq 0$. Hence, $\omega \in \operatorname{supp}(f)$ so that $f(\omega) = g(\omega)$. Therefore, $g(\omega) \in \Delta$, so that $\omega \in g^{-1}(\Delta)$. It follows that $f^{-1}(\Delta) \subseteq g^{-1}(\Delta)$. Conversely, suppose that $f^{-1}(\Delta) \subseteq g^{-1}(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. If $\omega \in \operatorname{supp}(f)$, then $f(\omega) \neq 0$. Hence,

$$\{\omega\} \subseteq f^{-1}\bigl(f(\omega)\bigr) \subseteq g^{-1}\bigl(f(\omega)\bigr),$$

so that $g(\omega) = f(\omega)$. Finally, to show that (iv) implies (i), suppose that $fg = f^2$ and let $h = g - f$. Then $fh = fg - f^2 = 0$, so that $f \perp h$ and $f \oplus h = g$. □

It follows from Theorem 3.1 that $f \preceq g$ if and only if f is a truncation of g. Another interpretation is that $f \preceq g$ if and only if the event $f^{-1}(\Delta)$ is contained in the event $g^{-1}(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. Since this amounts to certain propositions implying other propositions, we can think of $\preceq$ as a logical order. Notice that $f \preceq g$ is not related to the usual order $f \leq g$ in general. However, if $g \geq 0$, then $f \preceq g$ implies that $f \leq g$. It also follows from Theorem 3.1 that every $f \in \mathcal{M}(\mathcal{A})$ is principal. Indeed, if $g, h \preceq f$ with $g \perp h$, then

$$(g + h)f = gf + hf = g + h,$$

so that $g \oplus h \preceq f$.

THEOREM 3.2. *If* $f_1 \preceq f_2 \preceq \cdots$, *then* $f = \bigvee f_i$ *exists in* $\mathcal{M}(\mathcal{A})$ *and* $f = \lim f_i$ *pointwise.*

Proof. Since $f_1 \preceq f_2 \preceq \cdots$, we have that $\operatorname{supp}(f_1) \subseteq \operatorname{supp}(f_2) \subseteq \cdots$. Now $\bigcup \operatorname{supp}(f_i) = A \in \mathcal{A}$. Define $f \in \mathcal{M}(\mathcal{A})$ as follows. If $\omega \in A$, then $\omega \in \operatorname{supp}(f_n)$ for some n and define $f(\omega) = f_n(\omega)$, and if $\omega \notin A$, define $f(\omega) = 0$. The function f is well-defined because if $\omega \in \operatorname{supp}(f_m)$, then either $f_m \preceq f_n$ or $f_n \preceq f_m$. In either case, $f_m(\omega) = f_n(\omega)$. For every $\omega \in \Omega$, if $\omega \notin A$, then $f_i(\omega) = f(\omega) = 0$ for every i. If $\omega \in A$, then there exists an n such that $f_n(\omega) = f_{n+1}(\omega) = \cdots = f(\omega)$. Hence, $\lim f_i = f$ pointwise. Clearly, $f_i \preceq f$ for every i. Suppose that $g \in \mathcal{M}(\mathcal{A})$ and $f_i \leq g$ for every i. If $\omega \in A = \operatorname{supp}(f)$, then $f(\omega) = g(\omega)$, so that $f \preceq g$. Therefore, $f = \bigvee f_i$. □

COROLLARY 3.3. *$(\mathcal{M}(\mathcal{A}), 0, \oplus)$ is a generalized σ-orthoalgebra and $(\mathcal{M}(\mathcal{A}), \preceq)$ is a σ-poset.*

The equation $\mu(A \oplus B) = \mu(A) \oplus \mu(B)$ can be rewritten as

$$\mu\big((\chi_A \oplus \chi_B)^{-1}(1)\big) = \mu\big(\chi_A^{-1}(1)\big) \oplus \mu\big(\chi_B^{-1}(1)\big).$$

We can also write this as

$$\mu\big((\chi_A \oplus \chi_B)^{-1}(\Delta)\big) = \mu\big(\chi_A^{-1}(\Delta)\big) \oplus \mu\big(\chi_B^{-1}(\Delta)\big)$$

for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. It will follow from the next result that we can extend this last equation to arbitrary random variables.

LEMMA 3.4. *If $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$ and $f, g \in \mathcal{M}(\mathcal{A})$ with $f \perp g$, then*

$$(f \oplus g)^{-1}(\Delta) = f^{-1}(\Delta) \oplus g^{-1}(\Delta).$$

Proof. If $\omega \in f^{-1}(\Delta) \cap g^{-1}(\Delta)$, then $f(\omega) \in \Delta$ and $g(\omega) \in \Delta$. Since $0 \notin \Delta$, $f(\omega), g(\omega) \neq 0$. Hence, $\omega \in \operatorname{supp}(f) \cap \operatorname{supp}(g)$, which contradicts the fact that $f \perp g$. Therefore, $f^{-1}(\Delta) \perp g^{-1}(\Delta)$ and we have that

$$f^{-1}(\Delta) \oplus g^{-1}(\Delta) = f^{-1}(\Delta) \cup g^{-1}(\Delta).$$

If $(f+g)(\omega) \in \Delta$, then $f(\omega) \in \Delta$ or $g(\omega) \in \Delta$. Hence, $(f \oplus g)^{-1}(\Delta) \subseteq f^{-1}(\Delta) \cup g^{-1}(\Delta)$. Conversely, if $f(\omega) \in \Delta$ or $g(\omega) \in \Delta$, then $(f \oplus g)(\omega) \in \Delta$, so that $f^{-1}(\Delta) \cup g^{-1}(\Delta) \subseteq (f \oplus g)^{-1}(\Delta)$. □

We conclude from Lemma 3.4 that if $f \perp g$, then

$$\mu\big[(f \oplus g)^{-1}(\Delta)\big] = \mu\big[f^{-1}(\Delta)\big] \oplus \mu\big[g^{-1}(\Delta)\big]$$

for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. We interpret $\mu\big[f^{-1}(\Delta)\big] = \mu(f \in \Delta)$ as the probability that the measurement f has a value in Δ. We can restrict our attention to $\Delta \subseteq \mathbb{R} \smallsetminus \{0\}$ because if we know $\mu(f \in \{0\}') = \mu(f \neq 0)$, then $\mu(f = 0) = 1 - \mu(f \neq 0)$ is determined. Another way of viewing this is that $0 \in \mathbb{R}$ has a special significance in the sense that $\mu(A) = \mu(\chi_A^{-1} \neq 0)$ is the probability that A occurs. We can interpret $\mu(f \neq 0)$ as the probability that measurement f occurs, so we disregard $A \in \mathcal{A}$ that are not a subset of $\operatorname{supp}(f)$.

It follows from Theorem 3.2 and Lemma 3.4 that if $f_i \perp f_j$ for $i \neq j$, then $f_1 \oplus f_2 \oplus \cdots$ exists and $\mu(\bigoplus f_i \in \Delta) = \bigoplus \mu(f_i \in \Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$ where $\bigoplus f_i = \bigvee g_j$ and $g_j = f_1 \oplus \cdots \oplus f_j$. Moreover, since

$$\lim_{n\to\infty}(f_1 + \cdots + f_n) = \bigoplus f_i$$

and

$$|f_1 + \cdots + f_n| = |f_1| + \cdots + |f_n| \leq \left|\bigoplus f_i\right|,$$

it follows from the dominated convergence theorem that if $\bigoplus f_i$ is integrable, then

$$\int \bigoplus f_i \, \mathrm{d}\mu = \sum \int f_i \, \mathrm{d}\mu .$$

We interpret $\int f \, \mathrm{d}\mu = E_\mu(f)$ as the expectation or average value of the measurement f. Hence, if $E_\mu(\bigoplus f_i)$ exists, then $E_\mu(\bigoplus f_i) = \sum E_\mu(f_i)$.

THEOREM 3.5. *Let $f, g \in \mathcal{M}(\mathcal{A})$ with the partial order $\preceq$ on $\mathcal{M}(\mathcal{A})$.*

(i) *$f \wedge g$ exists.*

(ii) *$f \vee g$ exists if and only if there is an $h \in \mathcal{M}(\mathcal{A})$ such that $f, g \preceq h$.*

Proof.

(i) Let $A = \{\omega : \ f(\omega) = g(\omega)\}$ and let $h = f\chi_A$. Then $h \preceq f, g$. Now suppose that $u \in \mathcal{M}(\mathcal{A})$ with $u \preceq f, g$. If $u(\omega) \neq 0$, then $u(\omega) = f(\omega) = g(\omega)$, so $\omega \in A$. Hence, if $u(\omega) \neq 0$, then $u(\omega) = h(\omega)$. Therefore, $u \preceq h$, so $h = f \wedge g$.

(ii) If $f \vee g$ exists, then $f, g \preceq f \vee g \in \mathcal{M}(\mathcal{A})$. Conversely, suppose there exists an $h \in \mathcal{M}(\mathcal{A})$ with $f, g \preceq h$. By Theorem 3.1, $f = h\chi_A$ and $g = h\chi_B$ for some $A, B \in \mathcal{A}$. Let $u = h\chi_{A\cup B}$. Then $f, g \preceq u$. If $f, g \preceq v$, then $h\chi_A, h\chi_B \preceq v$. It follows that $u = h\chi_{A\cup B} \preceq v$. Therefore, $u = f \vee g$. □

THEOREM 3.6. *If $f \in \mathcal{M}(\mathcal{A})$, then $[0, f]$ is a Boolean σ-algebra isomorphic to* $\operatorname{supp}(f) \cap \mathcal{A}$.

Proof. Every element of $[0, f]$ has the form $g = f\chi_A$ where $A = \operatorname{supp}(g) \subseteq \operatorname{supp}(f)$. For $g \in [0, f]$ define $g' = f - g$. Then $g' = f\chi_{A'}$ where $A' = \operatorname{supp}(f) \smallsetminus \operatorname{supp}(g)$. Define $\phi(f\chi_A) = A$ where $A \in \operatorname{supp}(f) \cap \mathcal{A}$. Then $\phi\colon [0, f] \to \operatorname{supp}(f) \cap \mathcal{A}$ is clearly bijective. If $g, h \in [0, f]$ with $g \preceq h$, then $\operatorname{supp}(g) \subseteq \operatorname{supp}(h)$, so that $\phi(g) \subseteq \phi(h)$. Conversely, if $\phi(g) \subseteq \phi(h)$, then $\operatorname{supp}(g) \subseteq \operatorname{supp}(h)$, so that $g \preceq h$. If $f_1 \preceq f_2 \preceq \cdots$, then as in the proof of Theorem 3.2 we have that $\phi(\bigvee f_i) = \bigcup \phi(f_i)$. Finally, $\phi(0) = \emptyset$ and for $g \in [0, f]$ we have that

$$\phi(g') = \operatorname{supp}(f) \smallsetminus \operatorname{supp}(g) = \phi(g)' .$$

Hence, ϕ is a σ-isomorphism from $[0, f]$ to the Boolean σ-algebra $\operatorname{supp}(f) \cap \mathcal{A}$. □

The next section shows that many of the results of this section carry over to the noncommutative case of quantum observables.

4. Quantum observables

As in Section 1 we denote the set of bounded self-adjoint operators on a complex Hilbert space H by $\mathcal{S}(H)$ and the set of orthogonal projections on H by $\mathcal{P}(H)$. We interpret $\mathcal{P}(H)$ as the set of events and $\mathcal{S}(H)$ as the set of bounded observables for some quantum system. If $A \in \mathcal{S}(H)$ and $P^A(\Delta)$, $\Delta \in \mathcal{B}(\mathbb{R})$, is the spectral measure for A, then $P^A(\Delta)$ is interpreted as the event that A has a value in Δ. If ρ is a density operator on H, then ρ corresponds to a state of the system and $\operatorname{tr}(\rho P^A(\Delta))$ gives the probability that A has a value in Δ and $\operatorname{tr}(\rho A)$ is the expectation of A in the state ρ.

For $P, Q \in \mathcal{P}(H)$ it is easy to show that $P \perp Q$ (that is, $P + Q \leq I$) if and only if $PQ = 0$. We extend this definition to $A, B \in \mathcal{S}(H)$ by defining $A \perp B$ if $AB = 0$ in which case $A \oplus B = A + B$. This definition is also motivated by the work in Section 3. We denote the closure of the range of A by $\overline{\operatorname{ran}}(A)$ and the projection onto $\overline{\operatorname{ran}}(A)$ by P_A. The proof of the next result is straightforward.

LEMMA 4.1. *For $A, B \in \mathcal{S}(H)$ the following statements are equivalent.*

(i) $A \perp B$.
(ii) $\overline{\operatorname{ran}}(A) \subseteq \operatorname{null}(B)$.
(iii) $\overline{\operatorname{ran}}(B) \subseteq \operatorname{null}(A)$.
(iv) $P_A P_B = 0$.
(v) $\overline{\operatorname{ran}}(A) \perp \overline{\operatorname{ran}}(B)$.

THEOREM 4.2. $(\mathcal{S}(H), 0, \oplus)$ *is a generalized orthoalgebra.*

Proof. The conditions (GEA1), (GEA3) and (GEA4) clearly hold. To demonstrate (GEA2) we first show that $AB^2 = 0$ implies that $AB = 0$ for all $A, B \in \mathcal{S}(H)$. If $AB^2 = 0$, then $AB^2A = 0$ so that

$$0 = \langle AB^2Ax, x\rangle = \langle BAx, BAx\rangle = \|BAx\|^2 .$$

Hence, $BAx = 0$ for every $x \in H$, so $AB = BA = 0$. If $B \perp C$ and $A \perp (B \oplus C)$, then $BC = 0$ and $A(B + C) = 0$. Hence, $AB + AC = 0$, which implies that $AB^2 + ACB = 0$, so that $AB^2 = 0$. By our previous work $AB = 0$, so that $A \perp B$. Moreover, since $AC = 0$, we have that $C \perp (A \oplus B)$. Finally,

$$(A \oplus B) \oplus C = A + B + C = A \oplus (B \oplus C) .$$

To demonstrate (GEA5), suppose that $A \oplus B = 0$. Then $AB = 0$ and $A + B = 0$. Multiplying this last equation by A we conclude that $A^2 = 0$, so by our previous work $A = 0$ and then $B = 0$. To demonstrate condition (OA), if $A \perp A$, then $A^2 = 0$, so again, $A = 0$. □

We denote the orthoalgebra order on $\mathcal{S}(H)$ by $\preceq$, that is, for $A, B \in \mathcal{S}(H)$ we have that $A \preceq B$ if there exists a $C \in \mathcal{S}(H)$ such that $A \perp C$ and $A \oplus C = B$.

It follows from Lemma 2.2 that $(\mathcal{S}(H), \preceq)$ is a poset and $0 \preceq A$ for every $A \in \mathcal{S}(H)$. We now give various characterizations of this order.

LEMMA 4.3. *For $A, B \in \mathcal{S}(H)$ the following statements are equivalent.*

(i) $A \preceq B$.
(ii) $Ax = Bx$ *for all* $x \in \overline{\mathrm{ran}}(A)$.
(iii) $A = BP_A$.
(iv) $AB = A^2$.

Proof.

(i) $\Longrightarrow$ (ii): If (i) holds, there exists a $C \in \mathcal{S}(H)$ such that $AC = 0$ and $A + C = B$. Since $\overline{\mathrm{ran}}(A) \subseteq \mathrm{null}(C)$, if $x \in \overline{\mathrm{ran}}(A)$, then $Cx = 0$. Hence,

$$Ax = (A + C)x = Bx .$$

(ii) $\Longrightarrow$ (iii): If (ii) holds, then for every $y \in H$ we have that $AP_A y = BP_A y$. Hence, $A = AP_A = BP_A$.

(iii) $\Longrightarrow$ (iv): If (iii) holds, then

$$P_A B = (BP_A)^* = A = BP_A .$$

Hence,

$$A^2 = ABP_A = AP_A B = AB .$$

(iv) $\Longrightarrow$ (i): If $A^2 = AB$, then $A(B - A) = 0$ and $A + (B - A) = B$. Hence, $A \preceq B$. □

COROLLARY 4.4. *In $\mathcal{S}(H)$, $[0, I] = \mathcal{P}(H)$ with their usual order.*

Proof. We have that $P \in \mathcal{P}(H)$ if and only if $P \in \mathcal{S}(H)$ and $PI = P^2$. By Lemma 4.3, $PI = P^2$ if and only if $P \in [0, I]$. Hence, $[0, I] = \mathcal{P}(H)$. Also, for $P, Q \in \mathcal{P}(H)$ we have that $P \preceq Q$ if and only if $PQ = P^2 = P$. But $PQ = P$ if and only if $P \leq Q$. □

COROLLARY 4.5. *Every $A \in \mathcal{S}(H)$ is principal.*

Proof. Suppose that $B, C \preceq A$ and $B \perp C$. Then by Lemma 4.3 we have that

$$(B + C)A = BA + CA = B^2 + C^2 = (B + C)^2 .$$

Again, by Lemma 4.3 we conclude that $B \oplus C \preceq A$. □

THEOREM 4.6. *For $A, B \in \mathcal{S}(H)$, $A \preceq B$ if and only if $P^A(\Delta) \leq P^B(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$.*

Proof. Suppose that $P^A(\Delta) \leq P^B(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. By the spectral theorem we have that

$$A = \int_{\{0\}'} \lambda P^A(\mathrm{d}\lambda) , \qquad B = \int_{\{0\}'} \lambda P^B(\mathrm{d}\lambda) .$$

By definition of the integrals there exist operators $A_i, B_i \in \mathcal{S}(H)$ given by

$$A_i = \sum_j \lambda_{j,i} P^A(\Delta_{j,i}), \qquad B_i = \sum_j \lambda_{j,i} P^B(\Delta_{j,i})$$

where $0 \notin \Delta_{j,i}$, $\Delta_{j,i} \cap \Delta_{k,i} = \emptyset$ for $j \neq k$ and $\lim A_i = A$, $\lim B_i = B$ in the strong operator topology. Then

$$A_i B_i = \sum_j \lambda_{j,i}^2 P^A(\Delta_{j.i}) = A_i^2 .$$

Letting $i \to \infty$ gives that $AB = A^2$, so by Lemma 4.3, $A \preceq B$. Conversely, suppose that $A \preceq B$, so that $AB = A^2$. We first prove that if $C, D \in \mathcal{S}(H)$ satisfy $CD = 0$, then for every bounded Borel function f with $f(0) = 0$ we have that $f(C+D) = f(C) + f(D)$. We can approximate f pointwise by a polynomial $\sum_{i=1}^{n} c_i \lambda^i$. We then have that

$$f(C+D) \approx \sum c_i (C+D)^i = \sum c_i C^i + \sum c_i D^i \approx f(C) + f(D)$$

where the approximations are in the strong operator topology. Taking limits, since the polynomials converge to f we obtain $f(C+D) = f(C) + f(D)$. Now $\chi_\Delta(0) = 0$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. Since $A(B-A) = 0$, we have by our previous work that for $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$

$$P^B(\Delta) = \chi_\Delta(B) = \chi_\Delta[A + (B-A)] = \chi_\Delta(A) + \chi_\Delta(B-A) \geq \chi_\Delta(A) = P^A(\Delta) .$$

□

We can interpret Theorem 4.6 as saying that $A \preceq B$ if and only if the event that A has a value in Δ implies the event that B has a value in Δ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$.

Corollary 4.7. *If $A \preceq B$ and f is a Borel function satisfying $f(0) = 0$, then $f(A) \preceq f(B)$.*

Proof. Since $A \preceq B$, there exists a $C \in \mathcal{S}(H)$ such that $A \perp C$ and $A \oplus C = B$. As in the proof of Theorem 4.6, we have that $f(A) \perp f(C)$ and

$$f(A) \oplus f(C) = f(A \oplus C) = f(B) .$$

Hence, $f(A) \preceq f(B)$. □

For example, it follows from Corollary 4.7 that if $A \preceq B$, then $A^2 \preceq B^2$. This property does not hold for the usual order $\leq$ even when $A \geq 0$, $B \geq 0$. For instance, letting

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

we have that $A \geq 0$, $B \geq 0$, $A \leq B$, but $A^2 \not\leq B^2$.

THEOREM 4.8. *If $A_1 \preceq A_2 \preceq \cdots \preceq B$, then $A = \bigvee A_i$ exists in $\mathcal{S}(H)$ and $A = \lim A_i$ in the strong operator topology.*

Proof. Applying Theorem 4.6 we have that

$$P_{A_i} = P^{A_i(\{0\}')} \leq P^{A_{i+1}(\{0\}')} = P_{A_{i+1}}\,.$$

Hence, $P_{A_1} \leq P_{A_2} \leq \cdots$. It follows that $P = \bigvee P_{A_i}$ exists in $\mathcal{P}(H)$ and $P = \lim P_{A_i}$ in the strong operator topology. Applying Lemma 4.3 we have that $A_n = BP_{A_n}$ and it follows that $BP = PB$. Define the operator $A \in \mathcal{S}(H)$ by $A = BP$. Since

$$A_n = BP_{A_n} = BPP_{A_n} = AP_n\,,$$

we conclude from Lemma 4.3 that $A_n \preceq A$. Suppose that $A_n \preceq C$ for all n where $C \in \mathcal{S}(H)$. Then $CP_{A_n} = A_n = BP_{A_n}$, so that $CP = BP = A$. Since

$$C^2P = C^2P^2 = (CP)^2\,,$$

we conclude by Lemma 4.3 that $A = CP \preceq C$. Hence, $A = \vee A_i$. Since $A_n = AP_n$, we conclude that $\lim A_n = A$ in the strong operator topology. □

COROLLARY 4.9. *For $A \in \mathcal{S}(H)$, $[0,A]$ is a σ-orthomodular poset.*

Proof. By Theorems 2.3 and 4.8, $[0,A]$ is a σ-orthoalgebra. If $B, C \preceq A$ with $B \perp_A C$, then $B, C \preceq B \oplus_A C$. Now suppose that $D \preceq A$ and $B, C \preceq D$. Then by Lemma 4.3, $BD = B^2$, $CD = C^2$. Hence,

$$(B+C)D = BD + CD = B^2 + C^2 = (B+C)^2\,.$$

Again, by Lemma 4.3 we have that $B \oplus_A C \preceq D$. We conclude that $B \oplus_A C = B \vee C$ in $[0,A]$ and the result follows. □

Corollary 4.9 could also be proved using Corollary 4.5. The next lemma generalizes Corollary 4.4.

LEMMA 4.10. *If $P \in \mathcal{P}(H)$, then $[0,P] = \{P_1 \in \mathcal{P}(H) :\ P_1 \leq P\}$.*

Proof. If $P_1 \in \mathcal{P}(H)$ and $P_1 \leq P$, then $P_1P = P_1 = P_1^2$, so $P_1 \preceq P$ and $P_1 \in [0,P]$. Suppose that $A \in \mathcal{S}(H)$ and $A \in [0,P]$. Then $A \preceq P$, so that $A = PP_A$. It follows that $A = A^2$. We conclude that $A \in \mathcal{P}(H)$. □

We now show that we need the condition $A_i \preceq B$ in Theorem 4.8. For example, let $H = L^2(\mathbb{R}, \mu)$ where μ is Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $B_i = [-i, i]$, $i = 1, 2, \dots$, and let $A_i f(\lambda) = \lambda\chi_{B_i}(\lambda)f(\lambda)$ for every $f \in H$. Then $A_i \in \mathcal{S}(H)$ and $A_i \preceq A_{i+1}$ because $A_iA_{i+1} = A_i^2$, $i = 1, 2, \dots$. However, $\bigvee A_i$ does not exist in $\mathcal{S}(H)$. Indeed, if $A = \vee A_i$ exists in $\mathcal{S}(H)$, then clearly $\|A_i\| \le \|A\|$ for every $i = 1, 2, \dots$. However, it is easy to show that $\|A_i\| = i$, which is a contradiction. This shows that $(\mathcal{S}(H), 0, \oplus)$ is not a generalized σ-orthoalgebra.

It is easy to show that the order $\preceq$ is not related to the order $\le$ on $\mathcal{S}(H)$. For example, we can have $A \le B$, but $AB \ne BA$, so that $A \not\preceq B$. Conversely, letting $A = \operatorname{diag}(1,0)$ and $B = \operatorname{diag}(1,-1)$ we have that $A \preceq B$, but $A \not\le B$. In fact, in this case $B \le A$. Nevertheless, the next result shows that $A \preceq B$ implies that $A \le B$ for the case of positive operators.

THEOREM 4.11. *If $A \preceq B$ and $B \ge 0$, then $A \le B$.*

P r o o f . Suppose that $A \preceq B$ and $B \ge 0$. Applying Lemma 4.3 we have that $Ax = Bx$ for all $x \in \overline{\operatorname{ran}}(A)$. Now for every $z \in H$ we have that $z = x + y$ for $x \in \overline{\operatorname{ran}}(A)$, $y \in \operatorname{null}(A)$. Hence,

$$\begin{aligned}\big\langle (B-A)z, z\big\rangle &= \big\langle (B-A)(x+y), x+y\big\rangle = \big\langle (B-A)y, x+y\big\rangle \\ &= \big\langle y, (B-A)(x+y)\big\rangle = \big\langle y, (B-A)y\big\rangle = \langle y, By\rangle \ge 0\,.\end{aligned}$$

Therefore, $A \le B$. □

We say that a partial order $\preceq$ on a set E is *algebraic* if E can be organized into a generalized effect algebra $(E, 0, \oplus)$ that generates $\preceq$; that is, $a \preceq b$ if and only if there exists a $c \in E$ such that $c \perp a$ and $a \oplus c = b$. Of course, the order $\preceq$ on $\mathcal{S}(H)$ is algebraic. We now show that the usual order $\le$ on $\mathcal{S}(H)$ is not algebraic. Suppose $\le$ is generated by a generalized effect algebra $(\mathcal{S}(H), 0, \boxplus)$; that is, $A \le B$ if and only if there exists a $C \in \mathcal{S}(H)$ such that $A \boxplus C = B$. Let $A \in \mathcal{S}(H)$ satisfy $A \le 0$ and $A \ne 0$ (for example, $A = -I$). Then there exists a $B \in \mathcal{S}(H)$ such that $A \boxplus B = 0$. By (GEA5) we have that $A = B = 0$, which is a contradiction.

Notice that if $A \in \mathcal{S}(H)$ is invertible, then A is a maximal element of $(\mathcal{S}(H), \preceq)$. Indeed, if A invertible, then $P_A = I$ and if $A \preceq B$, then $A = BP_A = B$. This shows that if $A, B \in \mathcal{S}(H)$ are invertible and $A \ne B$, then $A \vee B$ does not exist in $(\mathcal{S}(H), \preceq)$. We conclude that $(\mathcal{S}(H), \preceq)$ is not a lattice. We do not know whether $A \wedge B$ always exists in $(\mathcal{S}(H), \preceq)$. The next result shows that $(\mathcal{S}(H), \preceq)$ is a near-lattice.

THEOREM 4.12. *If $A \in \mathcal{S}(H)$, then $[0, A]$ is σ-isomorphic to the σ-orthomodular lattice $L_A = \{P \in \mathcal{P}(H) : \ P \le P_A,\ \ PA = AP\}$.*

P r o o f . By Lemma 4.3, $B \in [0, A]$ if and only if $B = AP_B$ where $P_B \in L_A$. Define $\phi\colon [0, A] \to L_A$ by $\phi(B) = P_B$. If $\phi(B) = \phi(C)$, then $P_B = P_C$ and $B = AP_B = AP_C = C$, so ϕ is injective. If $B = AP$ for $P \in L_A$, then

$$P_B = P^B(\{0\}') = P^A(\{0\}')P = P_A P = P\,.$$

Hence, if $P \in L_A$, letting $B = AP$ we have that $P = P_B$ and $\phi(B) = P$. Thus, ϕ is surjective. If $B, C \in [0, A]$ with $B \preceq C$, then $P_B \leq P_C$, so that $\phi(B) \leq \phi(C)$. Conversely, if $\phi(B) \leq \phi(C)$, then $P_B \leq P_C$ so that

$$B = AP_B \preceq AP_C = C.$$

Clearly, $\phi(0) = 0$ and for $B \in [0, A]$ we have that

$$A - B = A - AP_B = A(P_A - P_B)$$

where $P_A - P_B \in L_A$. Hence, $P_{A-B} = P_A - P_B$ and we conclude that

$$\phi(B') = \phi(A - B) = P_A - P_B = \phi(B)'.$$

It follows that ϕ is an isomorphism. If $A_1 \preceq A_2 \preceq \cdots \preceq A$, then as in the proof of Theorem 4.8,

$$\phi(\bigvee A_i) = \bigvee P_{A_i} = \bigvee \phi(A_i).$$

Hence, ϕ is a σ-isomorphism. □

Of course, it follows from Theorem 4.12 that $[0, A]$ is a σ-orthomodular lattice.

Corollary 4.13. *For $A, B \in \mathcal{S}(H)$, $A \wedge B$ and $A \vee B$ exist in $\{\mathcal{S}(H), \preceq\}$ if there exists a $C \in \mathcal{S}(H)$ such that $A, B \preceq C$. Hence, $\{\mathcal{S}(H), \preceq\}$ is near-lattice ordered.*

We now consider some finite-dimensional examples. In the sequel let $\dim H = n < \infty$. If $A, B \in \mathcal{S}(H)$ and $B \preceq A$, then it follows from Theorem 4.6 that A and B are simultaneously diagonalizable and can be represented by matrices $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $B = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i = \lambda_i$ whenever $\alpha_i \neq 0$. Denoting the spectrum of A by $\sigma(A)$, it follows that $\sigma(B) \subseteq \sigma(A) \cup \{0\}$. We say that $A \in \mathcal{S}(H)$ is *nondegenerate* if its nonzero eigenvalues are distinct.

Theorem 4.14. *If $\dim H = n < \infty$ and $A \in \mathcal{S}(H)$ is invertible and nondegenerate, then $[0, A]$ is a Boolean algebra isomorphic to the Boolean algebra 2^n.*

Proof. We can diagonalize A so that A has the representation $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \in \mathbb{R} \setminus \{0\}$ and $\lambda_i \neq \lambda_j$, $i \neq j$. Let $\mathcal{P}(A)$ be the power set of $\{\lambda_1, \ldots, \lambda_n\}$, so that $\mathcal{P}(A) = 2^n$. If $B \in [0, A]$, then B has the representation $B = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i = \lambda_i$ whenever $\alpha_i \neq 0$. Define $\phi\colon [0, A] \to 2^n$ by $\phi(B) = \{\alpha'_1, \ldots, \alpha'_m\}$ where $\alpha'_1, \ldots, \alpha'_m$ are the nonzero eigenvalues of B. Then $\phi(B) \in \mathcal{P}(A)$ and it is easy to check that ϕ is an isomorphism. □

The next result shows that we can get the isomorphism $[0, A] \approx 2^m$ for every $m \leq n$.

COROLLARY 4.15. *If* $\dim H = n < \infty$ *and* A *is nondegenerate with* $|\sigma(A) \setminus \{0\}| = m \leq n$, *then* $[0, A] \approx 2^m$.

In the general case we have the following result.

THEOREM 4.16. *If* $\dim H = n < \infty$ *and* $A \in \mathcal{S}(H)$ *has the spectral representation* $A = \sum_{i=1}^{j} \lambda_i P_i$ *where* λ_i *are nonzero and distinct with* $\dim(P_i) = n_i$, *then*

$$[0, A] \approx \mathcal{P}(\mathbb{C}^{n_1}) \times \mathcal{P}(\mathbb{C}^{n_2}) \times \cdots \times \mathcal{P}(\mathbb{C}^{n_j}).$$

Proof. If $B \preceq A$, then $B = \sum \lambda_{i_k} Q_{i_k}$ where $Q_{i_k} \leq P_{i_k}$. Let $Q_i = 0$ if $i \neq j_k$ for some k and define $\phi\colon [0, A] \to \mathcal{P}(\mathbb{C}^{n_1}) \times \cdots \times \mathcal{P}(\mathbb{C}^{n_j})$ by $\phi(B) = Q_1 \times Q_2 \times \cdots \times Q_j$. It is straightforward to show that ϕ is an isomorphism. □

Although we do not know whether $A \wedge B$ exists in general, we do have an affirmative answer when $\dim H < \infty$.

THEOREM 4.17. *If* $\dim H < \infty$, *then* $A \wedge B$ *exists for every* $A, B \in \mathcal{S}(H)$.

Proof. Let A and B have spectral representation $A = \sum_{i=1}^{r} \lambda_i P_i$, $B = \sum_{i=1}^{s} \mu_i Q_i$ where the λ_i are distinct, the μ_i are distinct and $\lambda_i, \mu_i \neq 0$. If there exists no $C \in \mathcal{S}(H)$ with $C \neq 0$ and $C \preceq A, B$, then $A \wedge B = 0$. Otherwise, suppose we have a $C \in \mathcal{S}(H)$ with $C \neq 0$ and $C \preceq A, B$. Then at least one λ_i and μ_j coincide and $P_i \wedge Q_j \neq 0$. We can then rearrange the spectral representation of B to have the form

$$B = \sum_{i=1}^{t} \lambda_i Q_i + \sum_{i=t+1}^{s} \mu_i Q_i,$$

where μ_j are distinct from any λ_i. Let $D \in \mathcal{S}(H)$ have the spectral representation

$$D = \sum_{i=1}^{t} \lambda_i Q_i \wedge P_i.$$

It is clear that $D \preceq A, B$ and it is straightforward to show that if $E \in \mathcal{S}(H)$ with $E \preceq A, B$, then $E \preceq D$. Hence, $D = A \wedge B$. □

REFERENCES

[1] BUSCH, P.—GRABOWSKI, M.—LAHTI, P. J.: *Operational Quantum Physics*, Springer-Verlag, Berlin, 1995.

[2] DAVIES, E. B.: *Quantum Theory of Open Systems*, Academic Press, New York, 1976.

[3] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer Acad. Publ., Dordrecht, 2000.

[4] FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[5] GIUNTINI, R.—GREULING, H.: *Toward a formal language for unsharp properties*, Found. Phys. **19** (1989), 931–945.

[6] GUDDER, S.: *A structure for quantum measurements*, Rep. Math. Phys. **55** (2005), 249 267.

[7] HOLEVO, A. S.: *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam, 1982.

[8] KADISON, R.: *Order properties of bounded self-adjoint operators*, Proc. Amer. Math. Soc. **34** (1951), 505–510.

[9] KRAUS, K.: *States, Effects and Operations*, Springer-Verlag, Berlin, 1983.

[10] LUDWIG, G.: *Foundations of Quantum Mechanics, Vols. I, II*, Springer, Berlin, 1983/1985.

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