Mathematica Slovaca

Pekka Lahti; Kari Ylinen

Dilations of positive operator measures and bimeasures related to quantum mechanics

Mathematica Slovaca, Vol. 54 (2004), No. 2, [169]--189

Persistent URL: http://dml.cz/dmlcz/136901

Math. Slovaca, **54** (2004), No. 2, 169–189

Mathematica Slovaca

Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

DILATIONS OF POSITIVE OPERATOR MEASURES AND BIMEASURES RELATED TO QUANTUM MECHANICS

PEKKA LAHTI* — KARI YLINEN**

(Communicated by Anatolij Dvurečenskij)

ABSTRACT. In this largely expository paper, dilations of positive operator measures and bimeasures are studied in view of applications to quantum mechanics. Integration with respect to positive operator measures is developed to the extent needed in studying the moment problem of quantum observables. The minimal dilation of the canonical phase observable is related to the Toepliz measure, and the connection of a Schrödinger couple to a pair of coordinate operators on a phase space is worked out. Sequential combinations of instruments and the problem of the coexistence of quantum observables are studied as an application of dilations of operator bimeasures.

1. Introduction

The modern view of a quantum observable as a positive operator (valued) measure as opposed to the more traditional approach using only spectral measures has added a great deal to our understanding of the mathematical structure and conceptual foundations of quantum mechanics, and it has widely expanded the domain of applicability of quantum mechanics. The celebrated dilation theorem of Naimark, however, closely links projection valued measures with positive operator measures and, specifically in the normalized situation, spectral measures with semispectral ones. It has even been argued that semispectral measures are merely "shadows" of spectral measures. To elaborate on and evaluate this rather vague statement, we consider various aspects of semispectral and

2000 Mathematics Subject Classification: Primary 81P15; Secondary 47B15.
Keywords: operator measure, operator bimeasure, dilation, moment operator, canonical phase observable, Schrödinger couple, coexistence, sequential instrument.

spectral measures, their role in quantum mechanics, and their interrelations especially in terms of dilation theory. In keeping with this overall aim, most of the mathematical results we mention can be found in the literature, and we give proofs only in isolated cases where a reference does not seem to be easily available or we can offer some novelty or streamlining in existing approaches.

In Section 2 the stage is set, and the motivation for and implications of dilating operator measures which are already projection valued are considered. A short survey of the theory of integration of unbounded functions with respect to semispectral measures followed by applications to the operator moment problem is given in Section 3. A physically important situation where semispectral measures of necessity take the traditional role of spectral ones appears in the study of covariant phase observables. In Section 4 the role of dilation theory in this context is discussed, and its connection to the theory of Toeplitz operators is pointed out. Section 5 contains a discussion of further physical examples. The construction of the common minimal dilation of the phase space observables related to unit vectors of the relevant Hilbert space is recalled. Considering, in particular, the case where the unit vector is associated with a number state, one finds that the totally noncommutative Schrödinger pair (Q, P) can be obtained as (extensions of) certain projections of two mutually commuting selfadjoint coordinate operators. Section 6 deals with operator bimeasures and their dilations which are applied in Section 7 to the theory of sequential instruments and to the problem of coexistence of quantum observables.

2. Positive operator measures and their Naimark dilations

2.1. Dilations.

Let $\mathcal{H}$ be a complex Hilbert space, and let $L(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. Let Ω be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of Ω. Let $E: \mathcal{A} \to L(\mathcal{H})$ be an operator measure, i.e., a set function which is σ-additive with respect to the strong operator topology of $L(\mathcal{H})$. The Orlicz-Pettis lemma [14; p. 318, Theorem 1] implies that it is equivalent to require E to be σ-additive with respect to the weak operator topology. (We mostly consider only operator measures whose values are positive operators, and then this equivalence is more elementary, see for instance [2; p. 6, Proposition 1].) We say that E is *positive* if $E(X) \geq 0$ for all $X \in \mathcal{A}$, and *normalized* if $E(\Omega) = I$. Positive normalized operator measures are also called *semispectral measures*, and when they are projection valued we call them *spectral measures*.

Consider a semispectral measure $E: \mathcal{A} \to L(\mathcal{H})$ and let $(\mathcal{K}, P, V)$ be a (*Naimark*) *dilation* of E into a spectral measure P, that is, $P: \mathcal{A} \to L(\mathcal{K})$ is a spectral measure acting on a Hilbert space $\mathcal{K}$ and $V: \mathcal{H} \to \mathcal{K}$ an isometric linear map such that

$$E(X) = V^*P(X)V \tag{1}$$

for all $X \in \mathcal{A}$. The dilation $(\mathcal{K}, P, V)$ of E is *minimal* if $\mathcal{K}$ is the closed linear span of $\{P(X)V\varphi : \ \varphi \in \mathcal{H}, \ X \in \mathcal{A}\}$. A minimal dilation of E always exists, and it is unique up to an isometric isomorphism; a proof may be conveniently found e.g. in [34]. Naimark's original work on dilation theory is in [31], and a noncommutative generalization is given in [39]. An essentially parallel development is in [37; pp. 481–483].

2.2. Dilations of spectral measures.

There are important cases of equation (1) where the semispectral measure E is actually a spectral measure; see subsection 2.3. To characterize those situations we need the following probably well-known result. (The proof below contains a slight simplification due to the referee.)

LEMMA 2.2.1. *For any two projection operators P and R on a Hilbert space $\mathcal{H}$, the product PRP is a projection if and only if $PR = RP$.*

Proof. If $PR = RP$, then PRP is a projection. On the other hand, it PRP is a projection, then $PRPRP = PRP$ and thus $((I-P)RP)^*((I-P)RP) = PR(I-P)RP = 0$. It follows that $(I-P)RP = 0$, and so $RP = PRP$, implying $RP = PR$. □

COROLLARY 2.2.2. *Let $(\mathcal{K}, P, V)$ be a Naimark dilation of E. For any $X \in \mathcal{A}$, $E(X) = V^*P(X)V$ is a projection if and only if the projection VV^* commutes with the projection $P(X)$.*

Proof. If the projection operators VV^* and $P(X)$ commute with each other, then $E(X)^2 = V^*P(X)VV^*P(X)V = V^*VV^*P(X)V = V^*P(X)V = E(X)$, since $V^*V = I_{\mathcal{H}}$. Assume next that $E(X)^2 = E(X)$ so that $V^*P(X)VV^*P(X)V = V^*P(X)V$. But then also $VV^*P(X)VV^*P(X)VV^* = VV^*P(X)VV^*$, and so $VV^*P(X)VV^*$ is a projection. By Lemma 2.2.1 this means that the projections VV^* and $P(X)$ commute with each other. □

COROLLARY 2.2.3. *A dilation $(\mathcal{K}, P, V)$ of a spectral measure E is minimal if and only if $\mathcal{K} = V(\mathcal{H})$.*

Proof. For any $X \in \mathcal{A}$ and $\varphi \in \mathcal{H}$, $VV^*(P(X)V\varphi) = P(X)VV^*V\varphi = P(X)V\varphi$, showing that $P(X)V\varphi \in V(\mathcal{H})$. Therefore, $(\mathcal{K}, P, V)$ can be minimal only if $\mathcal{K} = V(\mathcal{H})$. If $\mathcal{K} = V(\mathcal{H})$, then clearly $(\mathcal{K}, P, V)$ is minimal. □

2.3. Dilations and measurements.

In the dilation $(\mathcal{K}, P, V)$ of a semispectral measure E the original Hilbert space $\mathcal{H}$ may be identified with the closed subspace $V(\mathcal{H})$ of the dilation space $\mathcal{K}$. If E represents an observable of a quantum system described by the Hilbert space $\mathcal{H}$, then, in general, neither P nor $\mathcal{K}$ has any direct physical meaning for the system under study. This is typically so with the minimal dilations. It is, however, possible to dilate each semispectral measure E in terms of the tensor product structure identifying $\mathcal{K}$ as a tensor product $\mathcal{H} \otimes \mathcal{H}_0$ of $\mathcal{H}$ with some auxiliary Hilbert space $\mathcal{H}_0$, which may represent for instance an environment or a (part of a) measuring apparatus. This involves constructing a so-called completely positive instrument for E and using a Naimark-Stinespring type two-variable dilation theorem in conjunction with expressing a normal representation of $L(\mathcal{H})$ as a multiple of the identity representation [18], [20], [32]. In that formulation, for each semispectral measure E acting in $\mathcal{H}$ there is a Hilbert space $\mathcal{H}_0$, a unit vector $\phi \in \mathcal{H}_0$, a unitary operator U on $\mathcal{K} = \mathcal{H} \otimes \mathcal{H}_0$, and a spectral measure P acting on $\mathcal{H}_0$, that is, $I \otimes P$ acting on $\mathcal{K}$, such that

$$E(X) = V_\phi^* U^* \big(I \otimes P(X)\big) U V_\phi$$

for all $X \in \mathcal{A}$, where the isometric map $V_\phi : \mathcal{H} \to \mathcal{K}$ is defined by $V_\phi(\varphi) = \varphi \otimes \phi$. Letting $P[\phi]$ denote the projection onto the subspace generated by ϕ, we now have $VV^* = I \otimes P[\phi]$ so that in view of Corollary 2.2.2, $E(X)$ is a projection if and only if $I \otimes P[\phi]$ commutes with $P_U(X) := U^*\big(I \otimes P(X)\big)U$. Since $V(\mathcal{H}) = \mathcal{H} \otimes [\phi]$, the measurement dilation $\big(\mathcal{H} \otimes \mathcal{H}_0, P_U, V_\phi\big)$ is never minimal for a projection measure E.

We note that the above type of measurement dilation of E, composed of $\mathcal{H}_0$, U, P, and V_ϕ can be interpreted as a dilation of E in two different ways, with $X \mapsto U^* I \otimes P(X) U$ as the spectral measure and V_ϕ as the isometric map, or $X \mapsto I \otimes P(X)$ as the spectral measure and UV_ϕ as the isometric map. The two dilations are, of course, unitarily equivalent.

3. Integration with respect to a positive operator measure

3.1. General results.

Let $E\colon \mathcal{A} \to L(\mathcal{H})$ be an operator measure. For any $\varphi, \psi \in \mathcal{H}$ we denote by $E_{\psi,\varphi}$ the complex measure defined by $E_{\psi,\varphi}(X) = \langle \psi \mid E(X)\varphi \rangle$. Let $f\colon \Omega \to \mathbb{C}$ be a measurable function and let $\mathcal{D}\big(L^E(f)\big)$ denote the set of those vectors $\varphi \in \mathcal{H}$ for which f is $E_{\psi,\varphi}$-integrable for each $\psi \in \mathcal{H}$. The set $\mathcal{D}\big(L^E(f)\big)$ is a vector subspace of $\mathcal{H}$ and there is a unique linear operator $L^E(f)$, with the domain $\mathcal{D}(L^E(f))$, such that

$$\langle \psi \mid L^E(f)\varphi \rangle = \int_\Omega f \, \mathrm{d}E_{\psi,\varphi}$$

for all $\psi \in \mathcal{H}$. We call $L^E(f)$ the *integral* of f with respect to E and we also write $L^E(f) = \int f \, \mathrm{d}E$. Further details can be found in [21].

Let $\mathcal{D}_f^E$ denote the set of vectors $\varphi \in \mathcal{H}$ for which f is square integrable with respect to the complex measure $E_{\varphi,\varphi}$. The following result, taken from [21], clarifies the role of the square integrability domain $\mathcal{D}_f^E$ for the operator $L^E(f)$.

THEOREM 3.1.1.

(a) *If the operator measure E is positive, then $\mathcal{D}_f^E$ is a subspace of $\mathcal{H}$ and $\mathcal{D}_f^E \subseteq \mathcal{D}(L^E(f))$.*

(b) *If the operator measure E is projection valued, then $\mathcal{D}_f^E = \mathcal{D}(L^E(f))$.*

We emphasize that for semispectral measures the natural domain of the operator $L^E(f)$ is the set $\mathcal{D}(L^E(f))$ and not $\mathcal{D}_f^E$, which, in fact, may be much smaller than $\mathcal{D}(L^E(f))$.

Consider now a semispectral measure $E\colon \mathcal{A} \to L(\mathcal{H})$ and let $(\mathcal{K}, P, V)$ be a Naimark dilation of it. For any measurable function $f\colon \Omega \to \mathbb{C}$ the operator $L^P(f) = \int f \, \mathrm{d}P$ is densely defined with the domain $\mathcal{D}(L^P(f)) = \mathcal{D}_f^P$, see e.g. [38]. It is natural to ask under what conditions the 'projected' operator $V^*L^P(f)V$ equals the operator $L^E(f)$. This is answered in the following result proved in [22].

THEOREM 3.1.2. *With the above notations, the operator $V^*L^P(f)V$ is the restriction of the operator $L^E(f)$ to $\mathcal{D}_f^E$. In particular, $V^*L^P(f)V = L^E(f)$ if and only if $\mathcal{D}_f^E = \mathcal{D}(L^E(f))$.*

Remark 3.1.3. Let now $E\colon \mathcal{A} \to L(\mathcal{H})$ be an operator measure which is not necessarily positive. For any fixed $\xi \in \mathcal{H}$, the map $E_\xi : \mathcal{A} \to \mathcal{H}$ defined by $E_\xi(X) = E(X)\xi$ is (norm) σ-additive, i.e. a vector measure. A measurable function $f\colon \Omega \to \mathbb{C}$ is integrable with respect to the vector measure E_ξ (in the sense of [14]) if and only if f is integrable with respect to each complex measure $E_{\eta,\xi}$, $\eta \in \mathcal{H}$. This follows from the proof of [21; Lemma A1] combined with [27; Theorem 2.4] or [42; Corollary 3.6]. Thus the integration theory of measurable functions with respect to operator measures could also be based on the theory of vector measures.

3.2. Moment operators.

The multi-indexed moment operators $L^E(\mathbf{x}^{\mathbf{k}}) = \int_{\mathbb{R}^n} \mathbf{x}^{\mathbf{k}}\, \mathrm{d}E$, $\mathbf{k} = (k_1, \dots, k_n)$, $\mathbf{k} \in \mathbb{N}^n$, $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_n^{k_n}$, of a semispectral measure $E\colon \mathcal{B}(\mathbb{R}^n) \to L(\mathcal{H})$ are an important case of the above type of operators. For short, we denote $E[\mathbf{k}] = L^E(\mathbf{x}^{\mathbf{k}})$. If $(\mathcal{K}, P, V)$ is a spectral dilation of E, then, by the multiplicativity of P, one gets $P[\mathbf{k}] = L^P(\mathbf{x}^{\mathbf{k}}) = \big(L^P(\mathbf{x})\big)^{\mathbf{k}} = P[\mathbf{1}]^{\mathbf{k}}$, where $P[\mathbf{1}] = A_1 \cdots A_n$, with $A_i = \int_{\mathbb{R}^n} x_i\, \mathrm{d}P$, $i = 1, \dots, n$, being the selfadjoint operator defined by the ith coordinate projection of E. However, the operator $V^* A_1^{k_1} \cdots A_n^{k_n} V$ is equal to the corresponding moment operator $E[\mathbf{k}]$ of E only when the square integrability domain $\mathcal{D}^E_{\mathbf{x}^{\mathbf{k}}}$ equals the domain $\mathcal{D}\big(E[\mathbf{k}]\big)$. Examples demonstrating that the operator $V^* A_1^{k_1} \cdots A_n^{k_n} V$ can be a proper restriction of the operator $E[\mathbf{k}]$ are easy to construct, see e.g. [22].

By the spectral theorem, the operators $P[\mathbf{k}]$, $\mathbf{k} \in \mathbb{N}^n$, (even $P[\mathbf{1}]$ alone) determine the spectral measure $P\colon \mathcal{B}(\mathbb{R}^n) \to L(\mathcal{K})$. However, neither the operators $V^*P[\mathbf{k}]V$, $\mathbf{k} \in \mathbb{N}^n$, nor the operators $E[\mathbf{k}]$, $\mathbf{k} \in \mathbb{N}^n$, need determine the semispectral measure $E\colon \mathcal{B}(\mathbb{R}^n) \to L(\mathcal{H})$. A well-known example, presented also in [15], going back to the classic works on the Stieltjes moment problem, is the following. Let E_1 and E_2 be the semispectral measures concentrated on $[0,\infty)$ and defined by $E_1(X) = \frac{1}{24} I \int_X \mathrm{e}^{-x^{1/4}}\, \mathrm{d}x$ and $E_2(X) = \frac{1}{24} I \int_X \mathrm{e}^{-x^{1/4}} [1 - \sin(x^{1/4})]\, \mathrm{d}x$. Then $E_1[k] = \frac{1}{4}(4k+3)! = E_2[k]$ for all $k \in \mathbb{N}$ though clearly $E_1 \neq E_2$.

4. The canonical phase as the Toeplitz measure

4.1. Phase observables.

Covariant phase observables are an important class of physical quantities which cannot be represented by spectral measures. For a recent overview, see e.g. [35]. Such observables can be characterized in various equivalent ways, for instance as follows. Let $\big(|n\rangle\big)_{n\in\mathbb{N}} \subset \mathcal{H}$ be an orthonormal basis (number basis) of $\mathcal{H}$. Then any sequence of unit vectors $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ defines a (phase shift covariant) semispectral measure $E\colon \mathcal{B}\big([0,2\pi)\big) \to L(\mathcal{H})$ through

$$E(X) = \sum_{n,m\in\mathbb{N}} \langle \xi_n \mid \xi_m \rangle \frac{1}{2\pi} \int_X \mathrm{e}^{\mathrm{i}(n-m)x}\, \mathrm{d}x\, |n\rangle\langle m|\,, \qquad X \in \mathcal{B}\big([0,2\pi)\big)\,,$$

with the (covariance) property

$$\mathrm{e}^{\mathrm{i}\,xN} E(X)\, \mathrm{e}^{-\mathrm{i}\,xN} = E(X \dot{+} x)\,,$$

where $N = \sum_{n\in\mathbb{N}} n|n\rangle\langle n|$ and $\dot{+}$ denotes addition modulo 2π. Conversely, any covariant phase observable is of that form for some sequence of unit vectors $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}$. We note that the structure of the phase observables can be determined both by direct methods ([35]) and by either using a semispectral measure generalization of Mackey's imprimitivity theorem ([6]) or combining Mackey's original theorem with a Naimark dilation ([8]).

4.2. The canonical phase.

The canonical phase observable $E_{\mathrm{can}} : \mathcal{B}([0,2\pi)) \to L(\mathcal{H})$ is defined by a constant sequence $\xi_n = \xi$ for all n. We consider some of its properties next. The moment operators $E_{\mathrm{can}}[n] = \int_0^{2\pi} x^n \,\mathrm{d}E_{\mathrm{can}}(x)$, $n \in \mathbb{N}$, are bounded selfadjoint operators. The same is true of the Susskind-Glogower sine and cosine operators $L^{E_{\mathrm{can}}}(\sin)$ and $L^{E_{\mathrm{can}}}(\cos)$, respectively. If $(\mathcal{K}, P, V)$ is any Naimark dilation of E_{can} into a projection measure P, then, for all $n \in \mathbb{N}$, $E_{\mathrm{can}}[n] = V^*A^nV = (V^*AV)^n$, as well as $L^{E_{\mathrm{can}}}(\sin) = V^*\sin(A)V$ and $L^{E_{\mathrm{can}}}(\cos) = V^*\cos(A)V$, where $A = \int_0^{2\pi} x \,\mathrm{d}P$.

4.3. Minimal dilation of E_{can}.

The minimal Naimark dilation of E_{can} is easily constructed. Indeed, let $(e_k)_{k\in\mathbb{Z}}$, with $e_k(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}kx}$, $x \in [0,2\pi)$, be the Fourier basis of $L^2([0,2\pi))$. As is evident from the construction of E_{can}, the canonical spectral measure $M\colon X \mapsto M_{\chi_X}$, $M_{\chi_X} f = \chi_X f$, on $L^2([0,2\pi))$ and the mapping $V\colon \mathcal{H} \to L^2([0,2\pi))$, for which $V(|n\rangle) = e_n$ for all $n \in \mathbb{N}$, constitute a Naimark dilation for it. We now prove the minimality of this dilation.

THEOREM 4.3.1. *The closed linear span D of the vectors $M_{\chi_X}\psi$, $\psi \in V(\mathcal{H})$, $X \in \mathcal{B}([0,2\pi))$, is the whole of $L^2([0,2\pi))$.*

Proof. If $n \in \mathbb{N}$, then $e_n = \chi_{[0,2\pi)} V(|n\rangle) \in D$. By approximating e_{-n} uniformly on $[0,2\pi)$ by linear combinations of characteristic functions of sets in $\mathcal{B}([0,2\pi))$, we see in the limit that for any $n \in \mathbb{N}$ the function $e_{-n} = \sqrt{2\pi} e_{-n} e_0$ is in D. Since D thus contains the Hilbert basis $(e_k)_{n\in\mathbb{Z}}$ of $L^2([0,2\pi))$, we have $D = L^2([0,2\pi))$. □

4.4. Toeplitz measure.

The canonical phase E_{can} is unitarily equivalent to the Toeplitz measure on the Hardy subspace H^2 of $L^2([0,2\pi))$. Indeed, recalling that $H^2 = V(\mathcal{H})$ and denoting $\Pi = VV^*$ the Naimark projection of $L^2([0,2\pi))$ onto H^2, we may write

$$E_{\mathrm{can}}(X) = V^* M_{\chi_X} V = V^* \Pi M_{\chi_X} V\,,$$

where $X \mapsto \Pi M_{\chi_X}$ is the Toeplitz measure. For any bounded Borel function $f\colon [0,2\pi) \to \mathbb{R}$, the selfadjoint operators $L^{E_{\mathrm{can}}}(f)$ and $\Pi(L^M(f)) = \Pi(M_f)$ are thus unitarily equivalent, too. In particular, this means that their spectra $\sigma(L^{E_{\mathrm{can}}}(f))$ and $\sigma(\Pi(M_f))$ are the same.

THEOREM 4.4.1. *Let $f\colon [0,2\pi) \to \mathbb{R}$ be a bounded Borel function. The spectrum of the operator $L^{E_{\mathrm{can}}}(f)$ is the closed interval* $[\operatorname{ess\,inf} f, \operatorname{ess\,sup} f]$.

Proof. The operator $L^{E_{\mathrm{can}}}(f)$ is unitarily equivalent to the Toeplitz operator $\Pi(M_f)$. By the Hartman-Wintner theorem ([12; p. 179]), the spectrum of the Toeplitz operator $\Pi(M_f)$ is $[\operatorname{ess\,inf} f, \operatorname{ess\,sup} f]$. □

As an immediate application of the above theorem one may determine, for instance, the spectra of the effect operators $E_{\mathrm{can}}(X)$, the moment operators of E_{can}, as well as those of the Glogower-Susskind sine and cosine operators. In particular, for any $X \in \mathcal{B}([0,2\pi))$ for which both X and X' have nonzero Lebesgue measure, $\sigma(E_{\mathrm{can}}(X)) = [0,1]$. Similarly, for any $n \in \mathbb{N}$, $\sigma(E_{\mathrm{can}}[n]) = [0,(2\pi)^n]$, and also $\sigma(L^{E_{\mathrm{can}}}(\sin)) = [-1,1]$ and $\sigma(L^{E_{\mathrm{can}}}(\cos)) = [-1,1]$. For an alternative study of these questions see, e.g. [16], [26]; see also [29] for a Toeplitz operator reformulation of the phase operator of [16].

Remark 4.4.2. We note in passing the interesting approach to the study of the canonical phase observable E_{can} used by Ozawa in [33]. By means of Robinson's nonstandard analysis Ozawa constructs a spectral dilation of E_{can} acting on a big Hilbert space. In addition to the numbers states, Ozawa distinguishes what he calls macroscopic states. Explaining the notions of nonstandard analysis would take us too far afield here. However, the theory of the so-called Loeb measures, central in modern nonstandard analysis, seems to deserve a mention in this context as a technique with potential applications to quantum mechanics. As far as we know, the first use of Loeb measures in the context of positive operator measures is announced in [44].

5. Schrödinger pairs dilated

5.1. Phase space observables.

Let again $(|n\rangle)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and let $N = \sum_{n\geq 0} n\,|n\rangle\langle n|$, $a = \sum_{n\geq 0} \sqrt{n+1}\,|n\rangle\langle n+1|$, and $a^* = \sum_{n\geq 0} \sqrt{n+1}\,|n+1\rangle\langle n|$ be the associated number, lowering, and raising operators with their natural domains, respectively.

Then $a^*a = N$ and $aa^* = N + I$. Moreover, the closures Q and P of the operators $\frac{1}{\sqrt{2}}(a^* + a)$ and $\frac{\mathrm{i}}{\sqrt{2}}(a^* - a)$ are unitarily equivalent to the Schrödinger position and momentum operators, respectively, see e.g. [36; Chapter IV].

Let $\lambda\colon \mathcal{B}(\mathbb{C}) \to [0, \infty]$ be the two-dimensional Lebesgue measure, let $D_z = \mathrm{e}^{za^* - \bar{z}a}$, $z \in \mathbb{C}$, be the (unitary) shift operator, and let φ be a fixed unit vector. The map $\mathbb{C} \ni z \mapsto D_z\varphi \in \mathcal{H}$ is norm-continuous and $\frac{1}{\pi}\int_{\mathbb{C}} D_z|\varphi\rangle\langle\varphi|D_z^* \,\mathrm{d}\lambda(z) = I$ in the weak sense. The formula

$$E^\varphi(Z) = \frac{1}{\pi}\int_Z D_z\,|\varphi\rangle\langle\varphi|D_z^*\,\mathrm{d}\lambda(z)\,, \qquad Z \in \mathcal{B}(\mathbb{C})\,,$$

defines a semispectral measure, the so-called phase space observable E^φ associated with the vector state φ.

5.2. Minimal dilation.

As is well known, all the phase space observables E^φ, $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, have the same minimal dilation to a spectral measure, see for instance [7] or [40]. We next recall this construction.

Let $\mathcal{K}$ denote the Hilbert space $L^2\big(\mathbb{C}, \frac{1}{\pi}\,\mathrm{d}\lambda\big)$ and let $M\colon \mathcal{B}(\mathbb{C}) \to L(\mathcal{K})$ be the canonical spectral measure $Z \mapsto M_{\chi_Z}$, $M_{\chi_Z}\psi = \chi_Z\psi$. Consider the mapping $V_\varphi\colon \mathcal{H} \to \mathcal{K}$ defined by

$$\big(V_\varphi\psi\big)(z) := \langle\varphi \mid D_z^*\psi\rangle_{\mathcal{H}}\,, \qquad z \in \mathbb{C}.$$

The mapping V_φ is linear and isometric. The projection onto the closed subspace $V_\varphi(\mathcal{H})$ is easily seen to be $\Pi_\varphi = \sum_{n=0}^{\infty} P[\phi_n^\varphi]$, where $\phi_n^\varphi = V_\varphi\,|\,n\rangle$, $n \geq 0$.

For any $\psi, \xi \in \mathcal{H}$, and for each $Z \in \mathcal{B}(\mathbb{C})$,

$$\langle\xi \mid E^\varphi(Z)\psi\rangle = \langle V_\varphi\xi \mid M(Z)V_\varphi\psi\rangle\,,$$

which shows that $\big(\mathcal{K}, M, V_\varphi\big)$ is a dilation of E^φ. The minimality of the dilation follows from the irreducibility of the representation $z \mapsto D_z$.

5.3. Schrödinger couple.

Let $L^M(f) = \int_{\mathbb{C}} f(z)\,\mathrm{d}M(z)$ be the normal operator defined by the canonical spectral measure M and the Borel function $f\colon \mathbb{C} \to \mathbb{C}$. Then $L^M(f)$ is a multiplication operator, $\big(L^M(f)\phi\big)(z) = f(z)\phi(z)$, $z \in \mathbb{C}$, $\phi \in \mathcal{D}\big(L^M(f)\big)$. Now $V_\varphi^* L^M(f) V_\varphi \subseteq L^{E^\varphi}(f)$ and the domain of $V_\varphi^* L^M(f) V_\varphi$ is $D_f^{E_\varphi}$.

Let $f = f_1 + \mathrm{i}\, f_2$ be the decomposition of f into real and imaginary parts. By definition, f is $E_{\psi,\varphi}$-integrable if and only if f_1 and f_2 are $E_{\psi,\varphi}$-integrable, in which case $\int f \,\mathrm{d}E_{\psi,\varphi} = \int f_1 \,\mathrm{d}E_{\psi,\varphi} + \mathrm{i} \int f_2 \,\mathrm{d}E_{\psi,\varphi}$. This shows that

$$L^E(f) = L^E(f_1) + \mathrm{i}\, L^E(f_2)\,,$$
$$L^E(\overline{f}) = L^E(f_1) - \mathrm{i}\, L^E(f_2)\,.$$

Assume that $L^E(f)$ is densely defined. Then $L^E(\overline{f}) \subseteq L^E(f)^*$ and thus the real and imaginary parts of the operator $L^E(f)$ are simply

$$\operatorname{Re} L^E(f) = \frac{1}{2}\left[L^E(f) + L^E(f)^*\right] = \frac{1}{2}\left[L^E(f) + L^E(\overline{f})\right],$$
$$\operatorname{Im} L^E(f) = \frac{1}{2\mathrm{i}}\left[L^E(f) - L^E(f)^*\right] = \frac{1}{2\mathrm{i}}\left[L^E(f) - L^E(\overline{f})\right].$$

By a direct computation one also observes that the operator $\operatorname{Re} L^E(f)$, resp. $\operatorname{Im} L^E(f)$, is a restriction of the operator $L^E(\operatorname{Re} f)$, resp. $L^E(\operatorname{Im} f)$.

Consider now a phase space observable defined by a number state $|\,s\rangle$, that is, a fixed element of the orthonormal basis $\{|\,n\rangle\}_{n\in\mathbb{N}}$, and denote it by $E^{|s\rangle}$. Consider the identity function $z \mapsto z$. The operators $L^{E^{|s\rangle}}(z)$ and $L^{E^{|s\rangle}}(\overline{z})$ can then be determined and one gets the operator equations $L^{E^{|s\rangle}}(z) = a$ and $L^{E^{|s\rangle}}(\overline{z}) = a^*$ ([21]). Therefore, $\operatorname{Re} L^{E^{|s\rangle}}(z) = \frac{1}{2}(a + a^*)$ and $\operatorname{Im} L^{E^{|s\rangle}}(z) = \frac{1}{2\mathrm{i}}(a - a^*)$. Also, by a direct computation one verifies that $L^{E^{|s\rangle}}(\operatorname{Re} z) = L^{E^{|s\rangle}}(x) \subseteq \frac{1}{\sqrt{2}}Q$ and $L^{E^{|s\rangle}}(\operatorname{Im} z) = L^{E^{|s\rangle}}(y) \subseteq \frac{1}{\sqrt{2}}P$, where Q and P are the Schrödinger couple associated with (a, a^*). Therefore, integrating the functions $z \mapsto z$ and $z \mapsto \overline{z}$ with respect to the phase space operator measure $E^{|s\rangle}$ we obtain the well-known operator relations

$$\frac{1}{2}(a + a^*) = \operatorname{Re} L^{E^{|s\rangle}}(z) \subseteq L^{E^{|s\rangle}}(\operatorname{Re} z) = L^{E^{|s\rangle}}(x) \subseteq \frac{1}{\sqrt{2}}Q\,,$$
$$\frac{1}{2\mathrm{i}}(a - a^*) = \operatorname{Im} L^{E^{|s\rangle}}(z) \subseteq L^{E^{|s\rangle}}(\operatorname{Im} z) = L^{E^{|s\rangle}}(y) \subseteq \frac{1}{\sqrt{2}}P\,.$$

On the other hand, it is well known that the closure of $\frac{1}{\sqrt{2}}(a + a^*)$ is Q and that of $\frac{1}{\sqrt{2}\mathrm{i}}(a - a^*)$ is P, see, e.g. [36].

Consider now the minimal dilation $\left(L^2(\mathbb{C}, \frac{1}{\pi}\,\mathrm{d}\lambda), M, V_{|s\rangle}\right)$ of $E^{|s\rangle}$. Integrating the functions $z \mapsto z$ and $z \mapsto \overline{z}$ with respect to M we get the mutually commuting multiplicative operators $L^M(z) = M_z$ and $L^M(\overline{z}) = M_{\overline{z}}$. Clearly, now $\operatorname{Re} L^M(z) = \frac{1}{2}(M_z + M_{\overline{z}}) = M_x = L^M(\operatorname{Re} z)$ and $\operatorname{Im} L^M(z) = M_y = L^M(\operatorname{Im} z)$.

Therefore,

$$\operatorname{Re}(V_{|s\rangle}^* L^M(z) V_{|s\rangle}) = V_{|s\rangle}^* M_x V_{|s\rangle} \subseteq \frac{1}{2}(a+a^*) \subseteq \frac{1}{\sqrt{2}} Q,$$
$$\operatorname{Im}(V_{|s\rangle}^* L^M(z) V_{|s\rangle}) = V_{|s\rangle}^* M_y V_{|s\rangle} \subseteq \frac{1}{2\mathrm{i}}(a-a^*) \subseteq \frac{1}{\sqrt{2}} P.$$

This shows that the Schrödinger pair (Q, P), which is totally noncommutative in the sense that there is no nonzero vector mapped to zero by all commutators of the spectral projections of Q and P, is obtained (as an extension of) the Naimark projections of the two mutually commuting selfadjoint coordinate operators M_x and M_y acting on the Hilbert space $L^2(\mathbb{C}, \frac{1}{\pi}\,\mathrm{d}\lambda)$.

6. Operator bimeasures and their dilations

6.1. Operator bimeasures and dilations.

In the following we let $\mathcal{A}_i$ be a σ-algebra of subsets of Ω_i, $i = 1, 2$. Let $\mathcal{A}_1 \times \mathcal{A}_2$ denote the Cartesian product of $\mathcal{A}_1$ and $\mathcal{A}_2$, and $\mathcal{A}_1 \otimes \mathcal{A}_2$ the σ-algebra generated in the power set of $\Omega_1 \times \Omega_2$ by the products $X \times Y$, $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$.

DEFINITION 6.1.1. A mapping $B\colon \mathcal{A}_1 \times \mathcal{A}_2 \to L(\mathcal{H})$ is a called a *positive operator bimeasure* if, for all $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$, the functions $\mathcal{A}_2 \ni Y \mapsto B(X, Y) \in L(\mathcal{H})$ and $\mathcal{A}_1 \ni X \mapsto B(X, Y) \in L(\mathcal{H})$ are positive operator measures. A positive (scalar) bimeasure $\beta\colon \mathcal{A}_1 \times \mathcal{A}_2 \to [0, \infty)$ is defined similarly.

For positive operator bimeasures there is the following analogue of the Naimark dilation theorem, see [43; Theorem 4.2]:

THEOREM 6.1.2. *Let $B\colon \mathcal{A}_1 \times \mathcal{A}_2 \to L(\mathcal{H})$ be a positive operator bimeasure. There exists a Hilbert space $\mathcal{K}$ with a bounded linear map $V\colon \mathcal{H} \to \mathcal{K}$ and two projection measures $P_1\colon \mathcal{A}_1 \to L(\mathcal{K})$, $P_2\colon \mathcal{A}_2 \to L(\mathcal{K})$ such that $P_1(X)P_2(Y) = P_2(Y)P_1(X)$ and $B(X, Y) = V^* P_1(X) P_2(Y) V$ for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_1$. If $B(\Omega_1 \times \Omega_2) = I$, then P_1 and P_2 can be taken to be spectral measures.*

It is natural to ask if a positive operator bimeasure on $\mathcal{A}_1 \times \mathcal{A}_2$ extends to an operator measure defined on $\mathcal{A}_1 \otimes \mathcal{A}_2$. In [43; Theorem 4.3] (in the setting of σ-rings) the problem was given equivalent reformulations as follows:

THEOREM 6.1.3. *The following conditions are equivalent:*

(i) *for every bimeasure $\beta\colon \mathcal{A}_1 \times \mathcal{A}_2 \to [0, \infty)$ there is a measure $\mu\colon \mathcal{A}_1 \otimes \mathcal{A}_2 \to [0, \infty)$ such that $\mu(X \times Y) = \beta(X, Y)$ for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$;*

(ii) *for every Hilbert space $\mathcal{H}$ and any two commuting projection valued measures $P_1: \mathcal{A}_1 \to L(\mathcal{H})$, $P_2: \mathcal{A}_2 \to L(\mathcal{H})$, there is a projection valued measure $P: \mathcal{A}_1 \otimes \mathcal{A}_2 \to L(\mathcal{H})$ such that $P(X \times Y) = P_1(X)P_2(Y)$ for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$;*

(iii) *for every Hilbert space $\mathcal{H}$ and every positive operator bimeasure B: $\mathcal{A}_1 \times \mathcal{A}_2 \to L(\mathcal{H})$ there is a positive operator measure $E: \mathcal{A}_1 \otimes \mathcal{A}_2 \to L(\mathcal{H})$ such that $E(X \times Y) = B(X,Y)$ for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$;*

(iv) *for every Hilbert space $\mathcal{H}$ and any two commuting positive operator measures $E_1: \mathcal{A}_1 \to L(\mathcal{H})$ and $E_2: \mathcal{A}_2 \to L(\mathcal{H})$, there is a positive operator measure $E: \mathcal{A}_1 \otimes \mathcal{A}_2 \to L(\mathcal{H})$ such that $E(X \times Y) = E_1(X)E_2(Y)$ for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$.*

It is known that these equivalent conditions do not always hold. The generic counterexample based on a nonmeasurable decomposition of the unit interval has surfaced in the literature in various guises at least since half a century ago. A fairly explicit appearance is in [30] whose authors attribute the idea to [1] and also refer to [17; p. 214]. In the context of commuting spectral measures essentially the same method is used in [5], [13], and in the context of positive scalar bimeasures in [3; p. 33, Exercise], [19] (with a reference to [3]), and [9].

Various sufficient conditions have been noted. In [10] and [11] the problem of sequentially composing instruments (see below) leads to the need to extend positive scalar bimeasures on standard Borel spaces or compact metrizable spaces. A topological regularity condition is utilized for positive bimeasures in [3] and in a more general situation in [19]. A sufficient condition for commuting spectral measures based on Rohlin's notion of Lebesgue space is given in [5]. In [4] it is observed that without any extra condition on the underlying measure spaces the extension problem has a positive solution provided that attention is confined to tensor products of spectral measures. An instructive treatment of this circle of ideas is contained in W. Ricker's review of [4] in Math. Reviews 96m:47038.

In [43] it was shown that for any (separately) regular positive operator bimeasure E on the Cartesian product $\mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2)$ of the Borel σ-algebras of two locally compact Hausdorff spaces Ω_1 and Ω_2, there is a unique regular positive operator measure $E: \mathcal{B}(\Omega_1 \times \Omega_2) \to L(\mathcal{H})$ such that $E(X \times Y) = B(X,Y)$ for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$. This was obtained as a corollary of a general Stinespring type dilation theorem in conjunction with a result [43; Proposition 3.2] proved there using the theory of tensor products of C^*-algebras. We now give an elementary proof of that proposition.

LEMMA 6.1.4. *Let Ω be a locally compact Hausdorff space and let $E\colon \mathcal{B}(\Omega) \to L(\mathcal{H})$ be a projection valued measure. Let $\mathcal{F}$ denote the linear span of the functions $\chi_X T$ where $X \in \mathcal{B}(\Omega)$ and $T \in E(\mathcal{B}(\Omega))'$, the commutant of the range of E. There is a unique linear map $\Psi\colon \mathcal{F} \to L(\mathcal{H})$ such that $\Psi(\chi_X T) = E(X)T$ for all $X \in \mathcal{B}(\Omega)$ and $T \in E(\mathcal{B}(\Omega))'$. Moreover, $\|\Psi(f)\| \le \sup_{\omega\in\Omega}\|f(\omega)\|$ for all $f \in \mathcal{F}$.*

P r o o f . The existence and uniqueness of Ψ are seen by standard arguments. Now write $f \in \mathcal{F}$ in the form $f = \sum_{j=1}^{n} \chi_{X_j} T_j$ with $X_j \in \mathcal{B}(\Omega)$, pairwisely disjoint, $T_j \in E(\mathcal{B}(\Omega))'$, and denote $M = \sup_{\omega\in\Omega}\|f(\omega)\|$. For $\xi \in \mathcal{H}$ we have

$$
\begin{aligned}
\|\Psi(f)\xi\|^2 &= \left\|\left(\sum_{j=1}^{n} E(X_j)T_j\right)\xi\right\|^2 \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \langle E(X_j)T_j\xi \mid E(X_k)T_k\xi\rangle \\
&= \sum_{j=1}^{n} \|E(X_j)T_j\xi\|^2 = \sum_{j=1}^{n} \|T_jE(X_j)\xi\|^2 \\
&\le \sum_{j=1}^{n} M^2\|E(X_j)\xi\|^2 \le M^2\|\xi\|^2 .
\end{aligned}
$$

□

In the setting of this lemma we denote by $\tilde{\mathcal{F}}$ the closure of $\mathcal{F}$ in the space of bounded $L(\mathcal{H})$-valued functions on Ω equipped with the norm $f \mapsto \sup_{\omega\in\Omega}\|f(\omega)\|$. The lemma shows that there is a unique bounded linear extension $\tilde{\Psi}\colon \tilde{\mathcal{F}} \to L(\mathcal{H})$ of Ψ. We denote $\tilde{\Psi}(f) = \int_\Omega f \,\mathrm{d}E$ for any $f \in \tilde{\mathcal{F}}$. For a locally compact Hausdorff space Ω, we let $C_0(\Omega)$ denote the space of continuous complex functions on Ω vanishing at infinity. We recall that there is a natural bijective correspondence between the sets of the positive linear maps $\Phi\colon C_0(\Omega) \to L(\mathcal{H})$ and the positive operator measures $E\colon \mathcal{B}(\Omega) \to L(\mathcal{H})$ which are regular in the sense that the complex measures $E_{\varphi,\psi}$ are regular for all $\varphi, \psi \in \mathcal{H}$, see for instance [34; pp. 49–50].

THEOREM 6.1.5. *Let Ω_1 and Ω_2 be locally compact Hausdorff spaces. Let $B\colon C_0(\Omega_1) \times C_0(\Omega_2) \to L(\mathcal{H})$ be a positive bilinear map. There is a unique positive linear map $\Phi\colon C_0(\Omega_1 \times \Omega_2) \to L(\mathcal{H})$ such that $\Phi(f \otimes g) = B(f,g)$ for all $f \in C_0(\Omega_1)$, $g \in C_0(\Omega_2)$.*

Proof. By [43; Theorems 2.2, 3.1] there is a Hilbert space $\mathcal{K}$ with a bounded linear map $V: \mathcal{H} \to \mathcal{K}$ and two representations $\pi: C_0(\Omega_1) \to L(\mathcal{K})$ and $\rho: C_0(\Omega_2) \to L(\mathcal{K})$, with commuting ranges, such that $B(f,g) = V^*\pi(f)\rho(g)V$ for all $f \in C_0(\Omega_1)$, $g \in C_0(\Omega_2)$. Let $P: \mathcal{B}(\Omega_1) \to L(\mathcal{K})$ be the regular projection valued measure corresponding to π and R the one corresponding to ρ. If $f_1, \ldots, f_n \in C_0(\Omega_1)$ and $g_1, \ldots, g_n \in C_0(\Omega_2)$, then by Lemma 6.1.4 the integral

$$\int_{\Omega_2} \left(\int_{\Omega_1} \left(\sum_{i=1}^{n} f_i \otimes g_i \right)(x,y)\, \mathrm{d}P(x) \right) \mathrm{d}R(y)$$

is well defined, and

$$\begin{aligned} \left\| \int_{\Omega_2} \left(\int_{\Omega_1} \left(\sum_{i=1}^{n} f_i \otimes g_i \right)(x,y)\, \mathrm{d}P(x) \right) \mathrm{d}R(y) \right\| &\le \sup_{y \in \Omega_2} \left\| \sum_{i=1}^{n} g_i(y) \pi(f_i) \right\| \\ &\le \sup_{y \in \Omega_2} \sup_{x \in \Omega_1} \left| \sum_{i=1}^{n} f_i(x) g_i(y) \right| . \end{aligned}$$

Since the space $\left\{ \sum_{i=1}^{n} f_i \otimes g_i : n \in \mathbb{N},\ f_1, \ldots, f_n \in C_0(\Omega_1),\ g_1, \ldots, g_n \in C_0(\Omega_2) \right\}$ is by the Stone-Weierstrass theorem norm dense in $C_0(\Omega_1 \times \Omega_2)$, there is thus a unique bounded linear map $F: C_0(\Omega_1 \times \Omega_2) \to L(\mathcal{K})$ such that $F(f \otimes g) = \pi(f)\rho(g)$. Now define $\Phi(h) = V^*F(h)V$ for all $h \in C_0(\Omega_1 \times \Omega_2)$. Then

$$\Phi(f \otimes g) = V^*F(f \otimes g)V = V^*\pi(f)\rho(g)V = B(f,g)$$

for all $f \in C_0(\Omega_1)$, $g \in C_0(\Omega_2)$. Note that F is a representation of $C_0(\Omega_1 \times \Omega_2)$ in $L(\mathcal{K})$ since $F\left(\left(\sum_{i=1}^{n} f_i \otimes g_i \right) \left(\sum_{j=1}^{p} h_j \otimes k_j \right) \right) = \sum_{i=1}^{n} \sum_{j=1}^{p} \pi(f_i)\pi(h_j)\rho(g_i)\rho(k_j) = F\left(\sum_{i=1}^{n} f_i \otimes g_i \right) F\left(\sum_{j=1}^{p} h_j \otimes k_j \right)$ (by density). Thus Φ is positive. □

7. Sequential instruments and the problem of coexistence

Two immediate applications of the preceeding considerations suggest themselves. The first gives the existence of sequential instruments and it is due to [11]. The second application refers to the problem of coexistence of quantum observables as studied, e.g. in [23], [24], [25]. To close this paper we shall briefly comment on these applications.

7.1. Sequential instruments.

The notion of an instrument, i.e. an operation valued measure, has been introduced primarily to describe the conditional state changes of a quantum system under a measurement, conditioned with respect to a measurement outcome. The sequential composing of instruments reflects the possibility of performing measurements on a quantum system one after the other.

Let $\mathcal{T}(\mathcal{H})$ denote the Banach space of the trace class operators on $\mathcal{H}$, equipped with the trace norm, and let $L(\mathcal{T}(\mathcal{H}))$ denote the set of bounded linear transformations on $\mathcal{T}(\mathcal{H})$. Let $\mathcal{A}$ be a σ-algebra of a set Ω. A map $\mathcal{I}\colon \mathcal{A} \to L(\mathcal{T}(\mathcal{H}))$ is called an *instrument* if for each vector state (unit vector) $\varphi \in \mathcal{H}$ the map

$$\mathcal{A} \ni X \mapsto \operatorname{tr}[\mathcal{I}(X)(P[\varphi])] \in [0,1]$$

is a probability measure. (For alternative definitions, see e.g. [10].) The maps $\mathcal{I}(X)$ are bounded positive contractive transformations on $\mathcal{T}(\mathcal{H})$, and they are called *operations*. An instrument $\mathcal{I}\colon \mathcal{A} \to L(\mathcal{T}(\mathcal{H}))$ defines an observable $E\colon \mathcal{A} \to L(\mathcal{H})$ through the formula

$$\langle \varphi \mid E(X)\varphi \rangle = \operatorname{tr}[\mathcal{I}(X)(P[\varphi])], \qquad X \in \mathcal{A}, \ \varphi \in \mathcal{H},$$

and E is called the associate observable of $\mathcal{I}$.

Consider now two instruments $\mathcal{I}_i\colon \mathcal{A}_i \to L(\mathcal{T}(\mathcal{H}))$, $i = 1,2$. Then for any $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$ one may compose the operations $\mathcal{I}_1(X)$ and $\mathcal{I}_2(Y)$ in either order, for instance as $\mathcal{I}_2(Y) \circ \mathcal{I}_1(X)$, first performing the operation $\mathcal{I}_1(X)$ and then $\mathcal{I}_2(Y)$. This raises the question of whether the operation valued map

$$(X,Y) \mapsto \mathcal{I}_2(Y) \circ \mathcal{I}_1(X) \tag{2}$$

can be extended to an instrument $\mathcal{I}_{12}\colon \mathcal{A}_1 \otimes \mathcal{A}_2 \to L(\mathcal{T}(\mathcal{H}))$. This question was first studied in [11] under some topological assumptions on $(\Omega_i, \mathcal{A}_i)$, $i = 1,2$. Here we shall demonstrate the possibility of extending (2) without any topological assumptions but assuming, instead, that the instruments are completely positive. Before doing that we still consider the map (2).

For each vector state $\varphi \in \mathcal{H}$ the function

$$\mathcal{A}_1 \times \mathcal{A}_2 \ni (X,Y) \mapsto \mu_\varphi(X,Y) := \operatorname{tr}\Big[\mathcal{I}_2(Y)\Big(\mathcal{I}_1(X)\big(P[\varphi]\big)\Big)\Big] \in [0,1] \tag{3}$$

is a probability bimeasure. By the duality $\mathcal{T}(\mathcal{H})^* \cong L(\mathcal{H})$, the bimeasures μ_φ, $\varphi \in \mathcal{H}$, define a positive operator bimeasure, a biobservable, $B_{12}\colon \mathcal{A}_1 \times \mathcal{A}_2 \to L(\mathcal{H})$ satisfying

$$\langle \varphi \mid B_{12}(X,Y)\varphi \rangle = \mu_\varphi(X,Y)$$

for all $\varphi \in \mathcal{H}$, $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$. The marginal observables, i.e. positive operator measures $X \mapsto B_{12}(X,\Omega_2)$ and $Y \mapsto B_{12}(\Omega_1,Y)$, are easily determined. In terms of the dual instrument $\mathcal{I}^*$ of $\mathcal{I}$, defined by $\operatorname{tr}[T\mathcal{I}^*(X)(A)] =$

$\operatorname{tr}[\mathcal{I}(X)(T)A]$ for all $T \in \mathcal{T}(\mathcal{H})$, $A \in L(\mathcal{H})$, $X \in \mathcal{A}$, these marginal observables are simply

$$X \mapsto \mathcal{I}_1(X)^*(I) = E_1(X), \tag{4}$$

$$Y \mapsto \mathcal{I}_1(\Omega_1)^*(\mathcal{I}_2^*(Y)(I)) = \mathcal{I}_1(\Omega_1)^*(E_2(Y)) \tag{5}$$

with E_i being the associate observable of the instrument $\mathcal{I}_i$, $i = 1,2$. It is worth pointing out that the (dual) operation $\mathcal{I}_1(\Omega_1)^*$ corresponds to the fact that the relevant measurement has been performed, but the result has not yet been registered and it is (apart from a trivial case) never an identity transformation. The second marginal observable is therefore a kind of 'noisy' version of E_2, where E_2 is affected by the first measurement, an E_1-measurement.

We now come to consider the possibility of extending the bimeasures (2) and (3) to the σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ using the complete positivity of the instruments without any topological assumptions.

An instrument $\mathcal{I}\colon \mathcal{A} \to L(\mathcal{T}(\mathcal{H}))$ is said to be *completely positive* if all its dual operations $\mathcal{I}^*(X)\colon L(\mathcal{H}) \to L(\mathcal{H})$, $X \in \mathcal{A}$, are completely positive. According to a fundamental result of Ozawa [32; Theorem 5.1], an instrument $\mathcal{I}$ is completely positive if and only if it has a Stinespring type dilation in the sense that there is a Hilbert space $\mathcal{H}_0$, a spectral measure $P\colon \mathcal{A} \to L(\mathcal{H}_0)$, a unit vector $\phi \in \mathcal{H}_0$, and a unitary map $U\colon \mathcal{H} \otimes \mathcal{H}_0 \to \mathcal{H} \otimes \mathcal{H}_0$ such that for each $X \in \mathcal{A}$ and for each vector state $\varphi \in \mathcal{H}$,

$$\mathcal{I}(X)(P[\varphi]) = \operatorname{tr}_0[P[U(\varphi \otimes \phi)]I \otimes P(X)], \tag{6}$$

where $\operatorname{tr}_0\colon \mathcal{T}(\mathcal{H}\otimes\mathcal{H}_0) \to \mathcal{T}(\mathcal{H})$ denotes the partial trace over $\mathcal{H}_0$. The associate observable E of $\mathcal{I}$ then satisfies

$$E(X) = V_\phi^* U^*(I \otimes P(X))UV_\phi$$

for all $X \in \mathcal{A}$ (compare Section 2.3).

Consider now two completely positive instruments $\mathcal{I}_i\colon \mathcal{A}_i \to L(\mathcal{T}(\mathcal{H}))$, and let $\mathcal{M}_i = (\mathcal{H}_i, P_i, \phi_i, U_i)$ constitute their Stinespring type dilations. To compose these instruments sequentially, for instance "first $\mathcal{M}_1$ and then $\mathcal{M}_2$", let I_{12} denote the isometric isomorphism $\mathcal{H} \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H} \otimes \mathcal{H}_2 \otimes \mathcal{H}_1$ which switches the positions of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the tensor product $\mathcal{H} \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $I_{12}^{-1} = I_{21}$ and put

$$\widetilde{U}_1 = U_1 \otimes I_2, \qquad \widetilde{U}_2 = I_{21} U_2 \otimes I_1 I_{12},$$

where I_i is the identity operator on $\mathcal{H}_i$, $i = 1,2$. Consider now the composite quadruple $\mathcal{M}_{12} = (\mathcal{H}_1 \otimes \mathcal{H}_2, P_1 \otimes P_2, \phi_1 \otimes \phi_2, \widetilde{U}_2\widetilde{U}_1)$. The completely positive instrument $\mathcal{I}_{12}: \mathcal{A}_1 \otimes \mathcal{A}_2 \to L(\mathcal{T}(\mathcal{H}))$ is now easily determined, and one gets

$$\begin{aligned}
&\mathcal{I}_{12}(X \times Y)(P[\varphi]) \\
&= \mathrm{tr}_{12}\Big[P\big[\tilde{U}_2\tilde{U}_1(\varphi \otimes \phi_1 \otimes \phi_2)\big]I \otimes P_1(X) \otimes P_2(Y)\Big] \\
&= \mathrm{tr}_{12}\Big[\tilde{U}_2\Big(\big(P[U_1(\varphi \otimes \phi_1)]I \otimes P_1(X)\big) \otimes P[\phi_2]\Big)\tilde{U}_2^* I \otimes I_1 \otimes P_2(Y)\Big] \\
&= \mathrm{tr}_2\Big[\tilde{U}_2\Big(\mathcal{I}_1(X)(P[\varphi]) \otimes P[\phi_2]\Big)\tilde{U}_2^* I \otimes P_2(Y)\Big] \\
&= \mathcal{I}_2(Y)\Big(\mathcal{I}_1(X)(P[\varphi])\Big) = \mathcal{I}_2(Y) \circ \mathcal{I}_1(X)(P[\varphi])
\end{aligned} \tag{7}$$

for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$, and for any vector state $\varphi \in \mathcal{H}$. The associate observable E_{12} is then

$$\begin{aligned}
E_{12}(X \times Y) &= \mathcal{I}_{12}(X \times Y)^*(I) = \big(\mathcal{I}_2(Y) \circ \mathcal{I}_1(X)\big)^*(I) \\
&= \mathcal{I}_1(X)^*\big(\mathcal{I}_2^*(Y)(I)\big) = \mathcal{I}_1(X)^*\big(E_2(Y)\big).
\end{aligned} \tag{8}$$

Its marginal observables are

$$E_{12}(X \times \Omega_2) = E_1(X), \qquad E_{12}(\Omega_1 \times Y) = \mathcal{I}_1(\Omega_1)^*\big(E_2(Y)\big), \tag{9}$$

which are the same as those of equations (4) and (5).

The instrument $\mathcal{I}_{12}$ of the 'sequential measurement' $\mathcal{M}_{12}$ is the composition of the instruments $\mathcal{I}_1$ and $\mathcal{I}_2$ of the 'measurements' $\mathcal{M}_1$ and $\mathcal{M}_2$, that is, $\mathcal{I}_{12} = \mathcal{I}_2 \circ \mathcal{I}_1$ in the sense of equations (2). Similarly, one gets the sequential instrument $\mathcal{I}_{21}: \mathcal{A}_2 \otimes \mathcal{A}_1 \to L(\mathcal{T}(\mathcal{H}))$ with $\mathcal{I}_{21} = \mathcal{I}_1 \circ \mathcal{I}_2$ corresponding to performing first the 'measurement' $\mathcal{M}_2$ and then $\mathcal{M}_1$.

7.2. The problem of the coexistence of observables.

Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ and $\mathcal{A}_i \subset \mathcal{P}(\Omega_i)$ be σ-algebras, and $E: \mathcal{A} \to L(\mathcal{H})$ and $E_i: \mathcal{A}_i \to L(\mathcal{H})$, $i = 1, 2$, be quantum observables. We say that E_1 and E_2 are *functions* of E if there are measurable functions $f_1: \Omega \to \Omega_1$ and $f_2: \Omega \to \Omega_2$ such that for each $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$, $E_1(X) = E\big(f_1^{-1}(X)\big)$, and $E_2(Y) = E\big(f_2^{-1}(Y)\big)$. This corresponds exactly to the case where the observables E_1 and E_2 have a common measurement (dilation) $(\mathcal{H}_0, P_U, V_\phi)$ so that

$$\begin{aligned}
E_1(X) &= V_\phi^* U^*\big(I \otimes P(f_1^{-1}(X))\big)UV_\phi, \qquad X \in \mathcal{A}_1, \\
E_2(Y) &= V_\phi^* U^*\big(I \otimes P(f_2^{-1}(Y))\big)UV_\phi, \qquad Y \in \mathcal{A}_2,
\end{aligned}$$

for some measurable (pointer) functions $f_i: \Omega \to \Omega_i$, $i = 1, 2$. In this sense it is justified to say that the observables E_1 and E_2 are functions of a third observable E exactly when they can be *measured jointly*. There is another related physical notion which aims to formulate the idea that two observables can be measured together: the observables E_1 and E_2 are *coexistent* if there is a third observable E such that the ranges of E_1 and E_2 are contained in that of E, that is,

$$\{E_1(X): X \in \mathcal{A}_1\} \cup \{E_2(Y): Y \in \mathcal{A}_2\} \subset \{E(Z): Z \in \mathcal{A}\}.$$

This notion goes back to G. Ludwig [28].

It is well known that any two spectral measures E_1 and E_2 are functions of a third spectral measure E exactly when the ranges of E_1 and E_2 are contained in that of E, see e.g. [41]. In the case of semispectral measures the situation is different. Clearly, if two semispectral measures E_1 and E_2 are functions of a third semispectral measure E, then the ranges of E_1 and E_2 are contained in the range of E. It is an open question whether the converse statement holds true also for semispectral measures.

On the basis of Theorem 6.1.2 one can give a characterization of the case where the observables E_1 and E_2 are functions of a third observable. To formulate this result, we say that the observables E_1 and E_2 have a *biobservable* if there is a positive operator bimeasure $B\colon \mathcal{A}_1 \times \mathcal{A}_2 \to L(\mathcal{H})$ such that for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$, $E_1(X) = B(X, \Omega_2)$ and $E_2(Y) = B(\Omega_1, Y)$, and they have a *joint observable* if there is a positive operator measure $E\colon \mathcal{A}_1 \otimes \mathcal{A}_2 \to L(\mathcal{H})$ such that for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$, $E_1(X) = E(X \times \Omega_2)$ and $E_2(Y) = E(\Omega_1 \times Y)$. Moreover, we say that an observable is regular if it is regular as a semispectral measure.

THEOREM 7.2.1. *Let Ω_1 and Ω_2 be locally compact Hausdorff spaces and $E_i\colon \mathcal{B}(\Omega_i) \to L(\mathcal{H})$ regular observables, $i = 1,2$. The following statements are equivalent:*

(i) *E_1 and E_2 have a biobservable;*
(ii) *E_1 and E_2 have a joint observable;*
(iii) *E_1 and E_2 are functions of a third observable.*

Remark 7.2.2.

(a) If the observables E_1 and E_2 have completely positive instruments $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_1 \circ \mathcal{I}_2 = \mathcal{I}_2 \circ \mathcal{I}_1$, then $E_{12} = E_{21}$ is a joint observable of E_1 and E_2.

(b) Let $E_i\colon \mathcal{A}_i \to L(\mathcal{H}_i)$, $i = 1,2$, be operator measures. By defining $\widetilde{E}_1(X) = E_1(X) \otimes I_2$ and $\widetilde{E}_2(Y) = I_1 \otimes E_2(Y)$ we get two mutually commuting operator measures $\widetilde{E}_i\colon \mathcal{A}_i \to L(\mathcal{H}_1 \otimes \mathcal{H}_2)$. For any $\xi_1 \in \mathcal{H}_1$ and $\xi_2 \in \mathcal{H}_2$ the mapping

$$\begin{aligned} X \times Y \mapsto \langle \xi_1 \otimes \xi_2 \mid \widetilde{E}_1(X)\widetilde{E}_2(Y)\xi_1 \otimes \xi_2 \rangle &= \langle \xi_1 \otimes \xi_2 \mid E_1(X) \otimes E_2(Y)\, \xi_1 \otimes \xi_2 \rangle \\ &= \langle \xi_1 \mid E_1(X)\xi_1 \rangle \langle \xi_2 \mid E_2(Y)\xi_2 \rangle \end{aligned}$$

extends by basic measure theory to a σ-additive function on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Any operator measure is norm bounded in view of the uniform boundedness principle and the fact that any vector measure is bounded ([14]). It follows by a standard density argument that the additive extension of the mapping $X \times Y \mapsto E_1(X) \otimes E_2(Y)$ to the algebra generated by the sets $X \times Y$, $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$, is σ-additive with respect to the weak (and by the Orlicz-Pettis lemma also to the strong) operator topology. By a standard argument involving the Fréchet-Riesz representation theorem this mapping extends uniquely to an operator measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$. In view of the well-known connection between the Hilbert-Schmidt operators and the tensor product of Hilbert spaces, this observation generalizes the main result of [4].

REFERENCES

[1] ANDERSEN, E. S.—JESSEN, B.: *On the introduction of measures in infinite product sets*, Danske Vid. Selsk. Mat.-Fys. Medd. **25** (1948).

[2] BERBERIAN, S. K.: *Notes on Spectral Theory*, D. van Nostrand Co., Inc., Princeton, 1966.

[3] BERG, C.—CHRISTENSEN, J. P. R.—RESSEL, P.: *Harmonic Analysis on Semigroups*. Grad. Texts in Math. 100, Springer, New York, 1984.

[4] BIRMAN, M.—SOLOMYAK, M.: *Tensor products of a finite number of spectral measures is always a spectral measure*, Integral Equations Operator Theory **24** (1996), 179–187.

[5] BIRMAN, M. SH.—VERSHIK, A. M.—SOLOMYAK, M. Z.: *Product of commuting spectral measures need not be countably additive*, Funktsional Anal. i Prilozhen. **13(1)** (1978), 61–66 [English translation: Functional Anal. Appl. **13** (1979), 48–49].

[6] CASSINELLI, G.—DE VITO, E.: *Square-integrability modulo a subgroup*, Trans. Amer. Math. Soc. **355** (2003), 1443–1465.

[7] CASSINELLI, G.—DE VITO, E.—LAHTI, P.—LEVRERO, A.: *Phase space observables and isotypic spaces*, J. Math. Phys. **41** (2000), 5883–5896.

[8] CASSINELLI, G.—DE VITO, E.—LAHTI, P.—PELLONPÄÄ, J.-P.: *Covariant localizations in the torus and the phase observables*, J. Math. Phys. **43** (2002), 693–704.

[9] CHEN, P. D.—LI, J. F.: *On the existence of product stochastic measures*, Acta Math. Appl. Sinica **7** (1991), 120–134.

[10] DAVIES, E. B.: *The Quantum Theory of Open Systems*, Academic Press, London-New York-San Francisco, 1976.

[11] DAVIES, E. B.—LEWIS, J. T.: *An operational approach to quantum probability*, Comm. Math. Phys. **17** (1970), 239–260.

[12] DOUGLAS, R. G.: *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[13] DUDLEY, R. M.: *A note on products of spectral measures*. In: Vector and Operator Valued Measures and Applications (Proc. Sympos., Alta, Utah, 1972), Academic Press, New York, 1973, pp. 125–126.

[14] DUNFORD, N.—SCHWARTZ, J. T.: *Linear Operators, Part I: General Theory*, Interscience Publishers, New York, 1958.

[15] DVUREČENSKIJ, A.—LAHTI, P.—YLINEN, K.: *Positive operator measures determined by their moment operators*, Rep. Math. Phys. **45** (2000), 139–146.

[16] GARRISON, J. C.—WONG, J.: *Canonically conjugate pairs, uncertainty relations, and phase operators*, J. Math. Phys. **11** (1970), 2242–2249.

[17] HALMOS, P. R.: *Measure Theory*, Van Nostrand, Toronto, 1950.

[18] HOLEVO, A. S.: *Probabilistic and Statistical Aspects of Quantum Theory*, North Holland, Amsterdam, *1982*.

[19] KARNI, S.—MERZBACH, E.: *On the extension of bimeasures*, J. Anal. Math. **55** (1990), 1–16.

[20] KRAUS, K.: *States, Effects, and Operations*, Springer-Verlag, Berlin, 1983.

[21] LAHTI, P.—MĄCZYŃSKI, M.—YLINEN, K.: *The moment operators of phase space observables and their number margins*, Rep. Math. Phys. **41** (1998), 319–331.

[22] LAHTI, P.—PELLONPÄÄ, J.-P.—YLINEN, K.: *Operator integrals and phase space observables*, J. Math. Phys. **40** (1999), 2181–2189.

[23] LAHTI, P.—PULMANNOVÁ, S.: *Coexistent observables and effects in quantum mechanics*, Rep. Math. Phys. **39** (1997), 339–351.

[24] LAHTI, P.—PULMANNOVÁ, S.: *Coexistence vs. functional coexistence of quantum observables*, Rep. Math. Phys. **47** (2001), 199–212.

[25] LAHTI, P.—PULMANNOVÁ, S.—YLINEN, K.: *Coexistent observables and effects in a convexity approach*, J. Math. Phys. **39** (1998), 6364-6371.

[26] LERNER, E. C.—HUANG, H. W.—WALTERS, G. E.: *Some mathematical properties of oscillator phase operators*, J. Math. Phys. **11** (1970), 1679–1684.

[27] LEWIS, D. R.: *Integration with respect to vector measures*, Pacific J. Math. **33** (1970), 157-165.

[28] LUDWIG, G.: *Foundations of Quantum Mechanics, Vol. I*, Springer-Verlag, Berlin, 1983.

[29] MLAK, W.: *Notes on quantum circular operators*. Preprint 303, Institute of Mathematics, Polish Academy of Sciences, 1984.

[30] MARCZEWSKI, E.—RYLL-NARDZEWSKI, C.: *Remarks on the compactness and non direct products of measures*, Fund. Math. **40** (1953), 165–170.

[31] NAIMARK, M. A.: *On a representation of additive operator set functions*, Comptes Rendus (Doklady) Acad. Sci. URSS **41** (1943), 359–361.

[32] OZAWA, M.: *Quantum measuring processes of continuous observables*, J. Math. Phys. **25** (1984), 79–87.

[33] OZAWA, M.: *Phase operator problem and macroscopic extension of quantum mechanics*, Ann. Physics **257** (1997), 65–83.

[34] PAULSEN, V. I.: *Completely Bounded Maps and Dilations*. Pitman Res. Notes Math. Ser. 146, Longman, London, 1986.

[35] PELLONPÄÄ, J.-P.: *Covariant phase observables in quantum mechanics*, Ann. Univ. Turku. Ser. A I no. 288 (2002).

[36] PUTNAM, C. P.: *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer-Verlag, Berlin, 1967.

[37] RIESZ, F.—SZ.-NAGY, B.: *Functional Analysis*, Dover Publications, Inc., New York, 1990.

[38] RUDIN, W.: *Functional Analysis*, McGraw Hill, New York, 1973.

[39] STINESPRING, W. F.: *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211-216.

[40] STULPE, W.: *Classical Representations of Quantum Mechanics Related to Statistically Complete Observables*, Wissenschaft und Technik Verlag, Berlin, 1997.

[41] VARADARAJAN, V. S.: *Geometry of Quantum Theory*, Springer, Berlin, 1985.
[42] YLINEN, K.: *On vector bimeasures*, Ann. Mat. Pura Appl. **117** (1978), 115–138.
[43] YLINEN, K.: *Positive operator bimeasures and a noncommutative generalization*, Studia Math. **118** (1996), 157–168.
[44] YLINEN, K.: *Aspects of operator measures and bimeasures susceptible to a nonstandard treatment.* In: Mathematisches Forschungsinstitut Oberwolfach, Tagungsbericht 08/1999, Nonstandard Analysis and Related Methods, and their Applications 21. 2.–27. 2. 1999. http://www.mfo.de/Meetings/Documents/1999/08/Report_08_99.ps.

Received July 7, 2003
Revised October 23, 2003

* *Department of Physics*
University of Turku
FIN-20014 Turku
FINLAND
E-mail: pekka.lahti@utu.fi

** *Department of Mathematics*
University of Turku
FIN-20014 Turku
FINLAND
E-mail: ylinen@utu.fi