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Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

RING-LIKE STRUCTURES CORRESPONDING TO GENERALIZED ORTHOMODULAR LATTICES

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ABSTRACT. Ring-like structures, so-called Boolean pseudorings, that are in a natural bijective correspondence with generalized orthomodular lattices are defined. In contrast to generalized orthomodular lattices, Boolean pseudorings are universal algebras and form a variety which turns out to be arithmetical and congruence regular. Distributive Boolean pseudorings resp. Boolean pseudorings with an associative addition operation are Boolean rings.

1. Introduction

There is a well-known duality between Boolean algebras and Boolean rings by means of which the lattice structure of a Boolean algebra $(L, \vee, \wedge, ', 0, 1)$ can be equivalently replaced by the structure of the corresponding Boolean ring $(L, +, \cdot, 0, 1)$. In this context the addition operation $+$ is defined as the symmetric difference $a + b := (a \wedge b') \vee (a' \wedge b) = (a \vee b) \wedge (a' \vee b')$. With respect to this operation $(L, +)$ forms a group. This fact implies that the equation $a + x = b$ has a unique solution, namely $x = a + b$, in contradistinction to the fact that the equation $a \vee x = b$ does not have a unique solution in general. Hence a ring-theoretic approach may be used in the theory of quantum logic since the

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operation $+$ can be interpreted more easily than the operation $\vee$. The possibility of solving the equation $a+x=b$ by $x=a+b$ may also be used in cryptology, where a is interpreted as a key since one has $(x+a)+a=x$ for all x. This is the so-called XOR coding applied to bits.

The duality between Boolean algebras and Boolean rings has been extended to more general structures and extensively investigated by many authors (cf. e.g. [2], [3], [4], [7], [8], [9], [10], [11], [12] and [15]). The aim of the present paper is to extend the above mentioned duality to generalized orthomodular lattices which were introduced in [13] and which generalize orthomodular lattices (cf. the monographs [14] and [1] on orthomodular lattices and the monograph [16] on orthomodular posets). We call the ring-like structures corresponding to generalized orthomodular lattices in a natural way, Boolean pseudorings. In Boolean pseudorings the equations $x+x=0$ and $xx=x$ holding in Boolean rings are preserved. It is proved that Boolean pseudorings are arithmetical and congruence regular and that ring distributivity in Boolean pseudorings is equivalent to lattice distributivity in the corresponding generalized orthomodular lattice. Also some other characterizations of the classicality of the physical system described by the corresponding algebraic structure are given. They partly deal with facts concerning the solvability of certain linear equations.

We hope that the results of this paper may be helpful for obtaining a new axiomatic approach to quantum logic and to the foundations of quantum mechanics, allowing a better physical interpretation.

2. Generalized orthomodular lattices

Definition 2.1. A *generalized orthomodular lattice* is a partial algebra $(L,\vee,\wedge,\ominus,0)$ of type $(2,2,2,0)$ where $(L,\vee,\wedge,0)$ is a lattice with 0, $x\ominus y$ is defined if and only if $y\leq x$, for each $a\in L$, $\big([0,a],\vee,\wedge,x\mapsto a\ominus x,0,a\big)$ is an ortholattice and $b\ominus a=(c\ominus a)\wedge b$ for all $a,b,c\in L$ with $a\leq b\leq c$.

Remark 2.1. If $(L,\vee,\wedge,\ominus,0)$ is a generalized orthomodular lattice and $b\in L$, then $\big([0,b],\vee,\wedge,x\mapsto b\ominus x,0,b\big)$ is an orthomodular lattice since $c\leq d\leq b$ implies $c\vee\big(d\wedge(b\ominus c)\big)=c\vee(d\ominus c)=d$.

Remark 2.2. A generalized orthomodular lattice is an orthomodular lattice if and only if it contains a greatest element. A generalized orthomodular lattice $(L,\vee,\wedge,\ominus,0)$ is distributive if and only if for every $a\in L$ the orthomodular lattice $\big([0,a],\vee,\wedge,x\mapsto a\ominus x,0,a\big)$ is a Boolean algebra.

EXAMPLE 2.1. If M is a set, κ an infinite cardinal, $L := \{A \subseteq M : |A| < \kappa\}$ or $L := \{A \subseteq M : |A| \leq \kappa\}$ and $A \ominus B := A \setminus B$ for all $A \in L$ and all $B \subseteq A$, then $(L, \cup, \cap, \ominus, \emptyset)$ is a generalized orthomodular lattice which is an orthomodular lattice if and only if $M \in L$. In this case $(L, \cup, \cap, x \mapsto M \ominus x, \emptyset, M)$ is the Boolean algebra of all subsets of M.

EXAMPLE 2.2. If H is a Hilbert space, L denotes the set of all finite-dimensional linear subspaces of H and $A \ominus M$ denotes the orthogonal complement of the linear subspace M of $A \in L$, then $(L, \vee, \cap, \ominus, \{0\})$ is a generalized orthomodular lattice which is an orthomodular lattice if and only if H is finite-dimensional.

3. Boolean pseudorings

DEFINITION 3.1. A *Boolean pseudoring* is an algebra $(R, +, \cdot, 0)$ of type $(2, 2, 0)$ satisfying the following identities:

$$\begin{aligned} x + y &= y + x\,, & xy &= yx\,, \\ & & (xy)z &= x(yz)\,, \\ x + x &= 0\,, & xx &= x\,, \\ x + 0 &= x\,, & x0 &= 0\,, \end{aligned}$$

$$\begin{aligned} (xy + x) + x &= xy\,, & \big((x + y) + xy\big) + xy &= x + y\,, \\ (x + y)x &= x + xy\,, & \big((x + y) + xy\big)x &= x\,, \end{aligned}$$

$$\begin{aligned} (xyz + x)y &= xyz + xy\,, \\ \big((xy + xz) + xyz\big)x &= (xy + xz) + xyz\,, \\ (xyz + x)(xy + x) &= xy + x\,. \end{aligned}$$

Now we are able to formulate and prove the natural bijective correspondence between generalized orthomodular lattices and Boolean pseudorings:

THEOREM 3.1. *For fixed L the formulas*

$$\begin{aligned} x + y &:= (x \vee y) \ominus (x \wedge y)\,, \\ xy &:= x \wedge y \end{aligned}$$

and

$$\begin{aligned} x \vee y &:= (x + y) + xy\,, \\ x \wedge y &:= xy\,, \\ a \ominus x &:= x + a \end{aligned}$$

induce mutually inverse bijections between the set of all generalized orthomodular lattices on L and the set of all Boolean pseudorings on L.

Proof. If $(L,\vee,\wedge,\ominus,0)$ is a generalized orthomodular lattice and $x+y := (x\vee y)\ominus(x\wedge y)$ and $xy := x\wedge y$ for all $x,y\in L$, then

$$\begin{aligned}(x+y)+xy &= \big(\big((x\vee y)\ominus(x\wedge y)\big)\vee(x\wedge y)\big)\ominus\big(\big((x\vee y)\ominus(x\wedge y)\big)\wedge x\wedge y\big)\\ &= x\vee y\,,\end{aligned}$$

$$\begin{aligned}&x+x = x\ominus x = 0\,,\\ &x+0 = x\ominus 0 = x\,,\\ &(xy+x)+x = x\ominus\big(x\ominus(x\wedge y)\big) = x\wedge y = xy\,,\\ &\big((x+y)+xy\big)+xy = (x\vee y)+(x\wedge y) = (x\vee y)\ominus(x\wedge y) = x+y\,,\\ &(x+y)x = \big((x\vee y)\ominus(x\wedge y)\big)\wedge x = x\ominus(x\wedge y) = x+xy\,,\\ &\big((x+y)+xy\big)x = (x\vee y)\wedge x = x\,,\end{aligned}$$

$$\begin{aligned}(xyz+x)y &= \big(x\ominus(x\wedge y\wedge z)\big)\wedge y = \big(x\ominus(x\wedge y\wedge z)\big)\wedge x\wedge y\\ &= (x\wedge y)\ominus(x\wedge y\wedge z) = xyz+xy\,,\end{aligned}$$

$$\begin{aligned}\big((xy+xz)+xyz\big)x &= \big((xy+xz)+(xy)(xz)\big)x = \big((x\wedge y)\vee(x\wedge z)\big)\wedge x\\ &= (x\wedge y)\vee(x\wedge z) = (xy+xz)+(xy)(xz)\\ &= (xy+xz)+xyz\,,\end{aligned}$$

$$(xyz+x)(xy+x) = \big(x\ominus(x\wedge y\wedge z)\big)\wedge\big(x\ominus(x\wedge y)\big) = x\ominus(x\wedge y) = xy+x\,,$$

$$\begin{aligned}&xy = x\wedge y \qquad \text{and}\\ &x+a = a\ominus x \qquad \text{if} \quad x\le a\end{aligned}$$

for all $x,y,z,a\in L$.

Conversely, let $(L,+,\cdot,0)$ be a Boolean pseudoring. Then $(L,\cdot,0)$ is a meet-semilattice with 0. Let $(L,\le)$ denote the corresponding poset. Now define $x\vee y := (x+y)+xy$, $x\wedge y := xy$ and $a\ominus x := x+a$ for all $x,y,a\in L$ with $x\le a$. Then

$$\begin{aligned}x(x\vee y) &= x\big((x+y)+xy\big) = \big((x+y)+xy\big)x = x\,,\\ y(x\vee y) &= y\big((x+y)+xy\big) = \big((y+x)+yx\big)y = y\end{aligned}$$

and hence $x,y\le x\vee y$. If $x,y\le z$, then

$$\begin{aligned}(x\vee y)z &= \big((x+y)+xy\big)z = \big((zx+zy)+zxy\big)z = (zx+zy)+zxy\\ &= (x+y)+xy = x\vee y\end{aligned}$$

and hence $x \vee y \leq z$. This shows that $x \vee y = \sup_{\leq}(x, y)$. Hence $(L, \vee, \wedge, 0)$ is a lattice with 0. Now assume $x, y, a \in L$ and $x \leq y \leq a$. Then

$$(a \ominus x)a = (x + a)a = (a + x)a = a + xa = a + x = x + a$$
$$= a \ominus x\,, \qquad \text{i.e.} \quad a \ominus x \leq a\,,$$

$$a \ominus (a \ominus x) = (x + a) + a = (ax + a) + a = ax = x\,,$$

$$(a \ominus y)(a \ominus x) = (y + a)(x + a) = (x + a)(y + a) = (ayx + a)(ay + a) = ay + a$$
$$= y + a = a \ominus y\,, \qquad \text{i.e.} \quad a \ominus y \leq a \ominus x\,,$$

$$(a \ominus x) \wedge x = (x + a)x = x + xa = x + x = 0\,,$$

$$(a \ominus x) \wedge y = (x + a)y = (ayx + a)y = ayx + ay = x + y = y \ominus x\,.$$

Hence $(L, \vee, \wedge, \ominus, 0)$ is a generalized orthomodular lattice and

$$(x \vee y) \ominus (x \wedge y) = xy + \big((x + y) + xy\big) = \big((x + y) + xy\big) + xy = x + y\,,$$
$$x \wedge y = xy$$

for all $x, y \in L$. □

The following corollary contains some easy observations:

COROLLARY 3.1. *Let $\mathcal{R} = (R, +, \cdot, 0)$ be a Boolean pseudoring, $b \in R$ and $(R, \leq)$ denote the poset corresponding to the meet-semilattice $(R, \cdot)$. Then*
$\big([0, b], +, \cdot, 0\big)$ is a subalgebra of $\mathcal{R}$.
$\mathcal{R}$ is distributive if and only if for each $a \in R$, $\big([0, a], +, \cdot, 0\big)$ is distributive.
$(R, +)$ is a semigroup if and only if for all $a \in R$, $\big([0, a], +\big)$ is a semigroup.

In the following let $\operatorname{Con} \mathcal{A}$ denote the set of all congruences of an algebra $\mathcal{A}$.

It is well known that orthomodular lattices are arithmetical, congruence regular and congruence uniform (cf. [5], [14] and [6]). This partly carries over to Boolean pseudorings.

THEOREM 3.2. *Boolean pseudorings are arithmetical and congruence regular.*

Proof. Let $\mathcal{R} = (R, +, \cdot, 0)$ be a Boolean pseudoring, $(R, \vee, \wedge, \ominus, 0)$ the corresponding generalized orthomodular lattice and $\Theta, \Phi \in \operatorname{Con} \mathcal{R}$.

First assume $(a, b) \in \Theta \circ \Phi$. Then there exists an element $c \in R$ with $a \Theta c \Phi b$. Since

$$\Theta \cap [0, a \vee b \vee c]^2,\ \Phi \cap [0, a \vee b \vee c]^2 \in \operatorname{Con}\big([0, a \vee b \vee c], \vee, \wedge, x \mapsto (a \vee b \vee c) \ominus x, 0, a \vee b \vee c\big)$$

and orthomodular lattices are congruence permutable (cf. [14]), we obtain

$$(a, b) \in \big(\Theta \cap [0, a \vee b \vee c]^2\big) \circ \big(\Phi \cap [0, a \vee b \vee c]^2\big)$$
$$= \big(\Phi \cap [0, a \vee b \vee c]^2\big) \circ \big(\Theta \cap [0, a \vee b \vee c]^2\big) \subseteq \Phi \circ \Theta$$

proving congruence permutability of $\mathcal{R}$. Moreover, $t(x,y,z) := (xy+yz)+zx$ is a majority term since

$$\begin{aligned} t(y,x,x) &= (yx+xx)+xy = (x+xy)+xy \\ &= \big((x\ominus(x\wedge y))\vee(x\wedge y)\big)\ominus\big((x\ominus(x\wedge y))\wedge x\wedge y\big) = x\,, \end{aligned}$$

$$t(x,y,x) = (xy+yx)+xx = (xy+xy)+x = 0+x = x\,,$$

$$t(x,x,y) = (xx+xy)+yx = (x+xy)+xy = t(y,x,x) = x\,.$$

This shows that $\mathcal{R}$ is congruence distributive and hence arithmetical.

Finally, assume $a \in R$ and $[a]\Theta = [a]\Phi$. Since for all $b \in R$ with $b \geq a$,

$$\Theta \cap [0,b]^2,\ \Phi \cap [0,b]^2 \in \operatorname{Con}\big([0,b],\vee,\wedge,x\mapsto b\ominus x,0,b\big)$$

and

$$[a]\big(\Theta\cap[0,b]^2\big) = [a]\Theta\cap[0,b] = [a]\Phi\cap[0,b] = [a]\big(\Phi\cap[0,b]^2\big)$$

and orthomodular lattices are congruence regular (cf. [5]), we obtain $\Theta\cap[0,b]^2 = \Phi\cap[0,b]^2$ for all $b \in R$ with $b \geq a$ and hence

$$\begin{aligned} \Theta &= \Theta\cap R^2 = \Theta\cap\bigcup_{b\geq a}[0,b]^2 = \bigcup_{b\geq a}\big(\Theta\cap[0,b]^2\big) = \bigcup_{b\geq a}\big(\Phi\cap[0,b]^2\big) \\ &= \Phi\cap\bigcup_{b\geq a}[0,b]^2 = \Phi\cap R^2 = \Phi \end{aligned}$$

proving congruence regularity of $\mathcal{R}$. □

Finally, we characterize the distributivity of a generalized orthomodular lattice by means of certain properties of the corresponding Boolean pseudoring. Surprisingly, lattice distributivity and ring distributivity turn out to be equivalent in this case thus allowing nice characterizations of the classicality of the physical system described by these algebraic structures.

THEOREM 3.3. *Let $\mathcal{L} = (L,\vee,\wedge,\ominus,0)$ be a generalized orthomodular lattice and $\mathcal{R} = (L,+,\cdot,0)$ the corresponding Boolean pseudoring. Then the following are equivalent:*

(i) *$\mathcal{L}$ is distributive.*
(ii) *$\mathcal{R}$ is distributive.*
(iii) *$+$ is associative.*
(iv) *$(x+y)+y = x$ for all $x,y \in L$.*
(v) *For all $a,b \in L$ there exists at most one $c \in L$ with $a+c=b$.*
(vi) *For all $a,b \in L$ there exists at least one $c \in L$ with $c \leq a\vee b$ and $a+c=b$.*
(vii) *For all $a,b \in L$ there exists exactly one $c \in L$ with $c \leq a\vee b$ and $a+c=b$.*
(viii) *$\mathcal{R}$ is a Boolean ring.*

Proof. If $x, y, a \in L$ and $x, y \le a$, then

$$x + y = (x \vee y) \ominus (x \wedge y) = \bigl(a \ominus (x \wedge y)\bigr) \wedge (x \vee y) = (x \vee y) \wedge \bigl((a \ominus x) \vee (a \ominus y)\bigr).$$

In [15] the symmetric difference $(x \vee y) \wedge (x' \vee y')$ in an orthomodular lattice $(L, \vee, \wedge, ', 0, 1)$ was denoted by $+_2$ and it was proved there that for an arbitrary orthomodular lattice $(L, \vee, \wedge, ', 0, 1)$ the following are equivalent:

(1) $(L, \vee, \wedge, ', 0, 1)$ is a Boolean algebra.
(2) $(L, +_2, \cdot)$ is distributive.
(3) $+_2$ is associative.
(4) For all $a, b \in L$ there exists at most one $c \in L$ with $a +_2 c = b$.
(5) For all $a, b \in L$ there exists at least one $c \in L$ with $a +_2 c = b$.

Using Remark 2.2 and Corollary 3.1 the equivalence of (i), (ii), (iii), (v), (vi) and (vii) follows. Obviously, (iii) implies (iv). Now assume that (iv) holds. Further assume $x, y, a \in L$ and $x, y \le a$. Then

$$\begin{aligned} x &= (x + y) + y \\ &= \Bigl(\bigl((x \vee y) \wedge \bigl((a \ominus x) \vee (a \ominus y)\bigr)\bigr) \vee y\Bigr) \\ &\quad \wedge \Bigl(\bigl((a \ominus x) \wedge (a \ominus y)\bigr) \vee (x \wedge y) \vee (a \ominus y)\Bigr) \\ &\le (x \wedge y) \vee (a \ominus y) \end{aligned}$$

whence $x \vee (a \ominus y) \le (x \wedge y) \vee (a \ominus y) \le x \vee (a \ominus y)$. Therefore $(x \wedge y) \vee (a \ominus y) = x \vee (a \ominus y)$. It is well known (see [14]) that this is equivalent to the fact that the elements x and y of the orthomodular lattice $\bigl([0, a], \vee, \wedge, x \mapsto a \ominus x, 0, a\bigr)$ commute. Since x, y were arbitrary elements of $[0, a]$, this means that $\bigl([0, a], \vee, \wedge, x \mapsto a \ominus x, 0, a\bigr)$ is a Boolean algebra, and since a was an arbitrary element of L, this proves (i) according to Remark 2.2. Finally, (viii) is equivalent to the conjunction of (ii) and (iii). □

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