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NOTES ABOUT THE HYPERCYCLICITY CRITERION

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ABSTRACT. A Banach space operator T is said to be hypercyclic if there exists a vector x such that the orbit $\{T^n x\}$ is dense in the space. The most used tool to discover hypercyclic operators is known as the Hypercyclicity Criterion. Given an operator T satisfying the Hypercyclicity Criterion, we characterize the subsequences of natural numbers $\{n_k\}$ for which we can assert that the sequence of powers $\{T^{n_k}\}$ also satisfies it. In order to show that, some equivalent conditions of the Hypercyclicity Criterion are studied. Finally as a consequence of this analysis we show that hypercyclic operators with a dense subset of almost periodic vectors satisfy the Hypercyclicity Criterion.

A bounded linear operator T on a Banach space $\mathcal{B}$ is said to be *hypercyclic* if there exists a vector $x \in \mathcal{B}$ (which is called later *hypercyclic vector*) such that the orbit $\{T^n x\}$ is dense in $\mathcal{B}$. If an operator has hypercyclic vectors, then it has a dense subset of them: every vector in the orbit $T^n x$ is hypercyclic for T. Moreover, it is well known that the set of hypercyclic vectors is a dense G_δ subset.

In [Ro], S. Rolewicz discovered the first example of hypercyclic operator defined on Banach space. Many hypercyclic operators have been discovered ever since then. The fundamental tool in this development is known as the Hypercyclicity Criterion. The Hypercyclicity Criterion is the best known sufficient condition to ensure that an operator is hypercyclic. We will use the interesting weaker version introduced in [BP].

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HYPERCYCLICITY CRITERION. *Let T be a bounded linear operator defined on a separable Banach space $\mathcal{B}$. Assume that $\{n_k\}$ is an increasing sequence of positive integers for which there exist*

a) *a dense subset $X \subset \mathcal{B}$ such that $T^{n_k}x \to 0$ for every $x \in X$;*
b) *a dense subset $Y \subset \mathcal{B}$ and a sequence of mappings $S_{n_k}: Y \to Y$ such that $T^{n_k}S_{n_k}y \to y$ and $S_{n_k}y \to 0$ for every $y \in Y$.*

Then there exists a vector $z \in \mathcal{B}$ such that $\{T^{n_k}z\}$ is dense in $\mathcal{B}$.

Observe that if an operator satisfies the Hypercyclicity Criterion for a sequence $\{n_k\}$, then it satisfies the Hypercyclicity Criterion for any subsequence $\{r_k\} \subset \{n_k\}$. As a consequence, operators satisfying the Hypercyclicity Criterion are Hereditarily Hypercyclic. That is, there exists a sequence $\{n_k\}$ such that for any subsequence $\{r_k\}$ there exists a vector $x \in \mathcal{B}$ so that $\{T^{r_k}x\}$ is dense. Actually Hypercyclicity Criterion is equivalent to the Hereditarily Hypercyclicity notion (see [BP]).

In [GoS; Theorem 3.2] the following sufficient condition for Hypercyclicity appears:

THE THREE OPEN SET'S CONDITION. *Let T be a bounded operator defined on a Banach space $\mathcal{B}$. Suppose that for any two nonvoid open sets U, V and for any neighbourhood of the origin W, there exists a positive integer n such that $T^n(U) \cap W \neq \emptyset$ and $T^n(W) \cap V \neq \emptyset$. Then T is hypercyclic.*

Moreover the Three open set's condition is equivalent to the Hypercyclicity Criterion. This result will play a central role in the proof of Theorem 5. Let us denote by B_k the balls with center at the origin and radius $1/k$.

LEMMA 1. *Suppose that T satisfies the Three open set's condition. Then there exist a hypercyclic vector z and a sequence of natural numbers $\{n_k\}$ such that $T^{n_k}z \to 0$ and $T^{n_k}(B_k) \cap (z + B_k) \neq \emptyset$.*

P r o o f. We will show that the set of vectors w satisfying $T^{r_k}w \to 0$ and $T^{r_k}(B_k) \cap (w + B_k) \neq \emptyset$ for some subsequence $\{r_k\}$ is a dense G_δ subset. Since the set of hypercyclic vectors is a dense G_δ subset, it is possible to find a hypercyclic vector z and a sequence $\{n_k\}$ satisfying the above conditions.

The set of vectors w satisfying $T^{r_k}w \to 0$ and $T^{r_k}(B_k) \cap (w + B_k) \neq \emptyset$ for some subsequence $\{r_k\}$ is

$$\bigcap_{k=1}^{\infty} \left(\bigcup_{n=1}^{\infty} T^{-n}(B_k) \cap \{z :\ T^n(B_k) \cap (z + B_k) \neq \emptyset\} \right),$$

which is clearly a dense G_δ subset. Let us denote by G_k the open set in the big parenthesis. The proof will be completed if we show that each G_k is dense. Given k, choose any z and $\varepsilon > 0$. Our task is to show that the ball of radius $\varepsilon > 0$

about z contains a point of G_k. We can suppose without loss of generality that $2\varepsilon < 1/k$. Write $B = B(z,\varepsilon)$. By the three open set condition there exists n such that $T^n(B_k) \cap B \neq \emptyset$ and $T^n(B) \cap B_k \neq \emptyset$. Pick $x, y \in B$ such that $T^n x \in B_k$ and $y \in T^n(B_k)$. To see that x lies in G_k, note that since $x, y \in B(z,\varepsilon)$ and $2\varepsilon < 1/k$, we have that $y \in (x + B_k)$, therefore $y \in T^n(B_k) \cap (x + B_k)$. Thus x belongs to G_k and lies within ε of z, as desired. □

THEOREM 2. *Let T be a bounded linear operator. Then T satisfies the Hypercyclicity Criterion if and only if T satisfies three open set's condition.*

Proof. The first implication is trivial. For the converse, let z and $\{n_k\}$ be the hypercyclic vector and the subsequence as in the previous lemma. The dense subsets required by the Hypercyclicity Criterion will be $X = Y = \{T^n z\}$. We see at once that $T^{-n_k}(z + B_k) \cap B_k \neq \emptyset$. Pick $w_k \in T^{-n_k}(z + B_k) \cap B_k$ and let us define $S_{n_k} T^n z = T^n w_k$ on the dense subset $X = \{T^n z\}$. Since w_k converges to zero, S_{n_k} converges pointwise to zero on each element of the orbit $T^n z$. On the other hand, since $T^{n_k} z \to 0$, we have that T^{n_k} converge pointwise to zero on $X = \{T^n z\}$.

Finally observe that $T^{n_k} S_{n_k} z = T^{n_k} w_k \in (z + B_k)$ for any k, therefore $T^{n_k} S_{n_k} z$ converges to z as k increases to infinity. Therefore $T^{n_k} S_{n_k} T^n z = T^n T^{n_k} w_k$ converges to $T^n z$ which yields the hypothesis of the Hypercyclicity Criterion. □

Remark 3.

1) From the proof of the above theorem we can deduce that an operator T satisfies the Hypercyclicity Criterion if and only if there exist a hypercyclic vector z, two neighbourhood basis $\{U_k\}$ and $\{V_k\}$ of z and 0 respectively and a subsequence $\{n_k\}$ such that $T^{n_k}(U_k) \cap V_k \neq \emptyset$ and $T^{n_k}(V_k) \cap U_k \neq \emptyset$ for each k.

2) A bounded linear operator T defined on a Banach space $\mathcal{B}$ is called *supercyclic* if there exists a vector x such that the set $\{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbb{N}\}$ is dense in $\mathcal{B}$. Derived from the Hypercyclicity Criterion, in [MS] the authors proved one sufficient condition for supercyclicity called Supercyclicity Criterion (see [MS]). It is possible to show that an operator T satisfies the Supercyclicity Criterion if and only if for any nonvoid open sets U, V and for any open neighbourhood of the origin W, there exist a positive integer n and a complex number λ such that $\lambda T^n(U) \cap W \neq \emptyset$ and $\lambda T^n(W) \cap V \neq \emptyset$.

In [BP], there are some results in the following line:

If T is hypercyclic and we have some control over the orbits of T on a dense subset, then T satisfies the Hypercyclicity Criterion.

For instance, in [BP] it is shown that a hypercyclic operator with a dense subset of periodic vectors satisfies the Hypercyclicity Criterion (recall that a vector x is called *periodic* for T if there exists a natural number n such that $T^n x = x$, and it is *almost periodic* if the orbit $\{T^n x\}$ is precompact). With the help of Theorem 2 we generalize the above result.

PROPOSITION 4. *Let T be a hypercyclic operator defined on a Banach space $\mathcal{B}$. Suppose that there exists a dense subset of almost periodic vectors. Then T satisfies the Hypercyclicity Criterion.*

Proof. We will show that T satisfies the Hypercyclicity Criterion using Remark 3.1). Pick z any hypercyclic vector for T. Let us denote $U_k := B(0, 1/k)$ and $V_k := z + U_k$. Since the set of hypercyclic vectors is dense, there exist a sequence of hypercyclic vectors u_k converging to zero and a subsequence $\{n_k\}$ such that $T^{n_k} u_k \in V_k$ for any k. We will show that there exists a subsequence $\{m_k\} \subset \{n_k\}$ such that $T^{n_k}(V_k) \cap U_k \neq \emptyset$ for any k and the proof will be completed.

Given k, let us denote $U := U_k$ and $V := V_k$. Pick $w \in z + B(0, 1/2k)$ an almost periodic vector for T. The orbit $\{T^n w\}$ is precompact, therefore there exist a subsequence $\{r_j\}$ of $\{n_k\}$, and $w_0 \in \mathcal{B}$ such that $\|T^{r_j} w - w_0\| \le 1/2k$ for all $j \ge k_0$. Since z is hypercyclic, there exists m_0 such that $\|T^{m_0} z + w_0\| < 1/2k$. By continuity, it is possible to find $r_l \in \{r_j\}$ large enough (and also $l \ge k_0$) such that $\|u_{r_l}\| \le \frac{1}{2k\|T^{m_0}\|}$ and $\|T^{m_0} T^{r_l} u_{r_l} + w_0\| \le 1/2k$. Let us consider the vector $h = w + T^{m_0} u_{r_l}$, which is clearly in V_k. Hence,

$$\|T^{r_l} h\| = \|T^{r_k} w + T^{r_k} T^{m_0} u_{r_l}\| \le \|T^{r_l} w - w_0\| + \|T^{r_l} T^{m_0} u_{r_l} + w_0\| \le 1/k .$$

If we take $m_k = r_l$, then $T^{m_k} h \in U$ and the proof is finished. □

In [An] it is shown the following deep result:

If T is hypercyclic, then T and T^n have the same set of hypercyclic vectors.

In [BP], using the deep result of Ansari, the authors proved that:

If T satisfies the Hypercyclicity Criterion, then T^n also satisfies it.

Theorem 5 provides an elementary proof of this result. Moreover, we show more than that.

THEOREM 5. *Let T satisfy the Hypercyclicity Criterion. Let $\{n_k\}$ be an increasing sequence of natural numbers so that the sequence $n_{k+1} - n_k$ is bounded. Then the sequence $\{T^{n_k}\}$ satisfies the Hypercyclicity Criterion, namely, there exists a subsequence $\{r_k\} \subset \{n_k\}$ such that $\{T^{r_k}\}$ is hereditarily hypercyclic.*

P r o o f. Since $\{n_{k+1} - n_k\}$ is bounded, there exists n such that $n_{k+1} - n_k < n$ for all k, this implies that for any k there exists a positive integer $n'_k \in \{n_k\}$, such that $kn \le n'_k < (k+1)n$. Since T satisfies the Hypercyclicity Criterion, for any nonempty set U, open neighbourhood of the origin B, $V = T^{-2n}(U)$ and $W = \bigcap_{j=1}^{2n-1} T^{-j}(B)$, there exists a positive integer p such that

$$T^p(U) \cap W \ne \emptyset \tag{1}$$

and

$$T^p(W) \cap T^{-2n}(V) \ne \emptyset\,. \tag{2}$$

The integer p satisfies $k_0 n \le p < (k_0+1)n$ for some k_0. On the other hand in the interval $\big[k_0 n,\, (k_0+1)n\big)$ there exists an element of the sequence $\{n_k\}$, which we denote by n'_{k_0}.

Suppose that $p < n'_{k_0}$. Set $j(p) = n'_{k_0+1} - p$ and $j'(p) = 2n - \big(n'_{k_0+1} - p\big)$. Note that $1 \le j(p), j'(p) \le 2n-1$, therefore $W \subset T^{-j(p)}(B)$ and $W \subset T^{-j'(p)}(B)$. Therefore from (1) and (2) we obtain:

$$T^p(U) \cap T^{-j(p)}(B) \ne \emptyset \tag{3}$$

and

$$T^p\big(T^{-j'(p)}(B)\big) \cap T^{-2n}(V) \ne \emptyset\,. \tag{4}$$

Conditions (3) and (4) are satisfied if and only if

$$T^{p+j(p)}(U) \cap B = T^{n'_{k_0}}(U) \cap B \ne \emptyset$$

and

$$T^{p-j'(p)+2n}(B) \cap V = T^{n'_{k_0}}(B) \cap V \ne \emptyset\,,$$

and the proof is complete for the case $n'_{k_0} > p$. Now suppose that $n'_{k_0} \le p$. In this case the same argument holds for $j(p) = p - n'_{k_0}$ and $j'(p) = 2n - (p - n'_{k_0})$. □

As a consequence we have:

COROLLARY 6. *If T satisfies the Hypercyclicity Criterion, then T^n also satisfies it.*

We finish with several open questions and comments.

1. Theorem 5 can be proved also following the Bès-Peris proof. However, we need to solve before it the following open question, which is a generalization of A n s a r i 's result (see [BP])

> *Let T be a hypercyclic operator and $\{n_k\}$ an increasing sequence of natural numbers with "bounded gaps" (that is $\{n_{k+1} - n_k\}$ bounded). Then the sequence $\{T^{n_k}\}$ is hypercyclic, that is, there exists a vector x so that the orbit $\{T^{n_k}x\}$ is dense.*

2. Using the techniques of Herrero in [He] and the DeLeeuw-Lyubich-Gicksberg decomposition (see [LG], [Ly] and [DQ]), it is not too difficult to generalize [He; Theorem 1]:

Let T be an operator on a Hilbert space with a dense subset of almost periodic vectors. Suppose that $T - \lambda$ is semi-Fredholm for any $\lambda \in \partial\mathbb{D}$ and $\sigma_p(T^\star) = \emptyset$. Then there exists a dense subset X such that $T^n x \to 0$ for any $x \in X$.

3. Let T and S be two hypercyclic operators satisfying the Hypercyclicity Criterion. Suppose that the direct sum $T \oplus S$ is hypercyclic on $H \oplus H$. Does $T \oplus S$ satisfy the Hypercyclicity Criterion?

4. Let T be an operator satisfying the Hypercyclicity Criterion and suppose that its adjoint $T^\star$ is hypercyclic. Does $T^\star$ satisfy the Hypercyclicity Criterion?

In relation with the last question, in [BP], it is proved that if an operator T and its adjoint $T^\star$ satisfy the Hypercyclicity Criterion, then they cannot satisfy the Hypercyclicity Criterion with respect to a common sequence $\{n_k\}$.

When we finished this paper, we were kindly informed by L. Bernal-González and K. Große-Erdman ([BG]) that they have obtained independently Theorem 2 and Proposition 4 in more generality.

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