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OPTIMAL CONTROL FOR $n \times n$ COUPLED SYSTEMS GOVERNED BY PETROWSKY TYPE EQUATIONS WITH CONTROL-CONSTRAINED AND INFINITE NUMBER OF VARIABLES

H. A. EL-SAIFY — G. M. BAHAA

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ABSTRACT. The main goal of this paper is to study a distributed control problem for $n \times n$ coupled systems of Petrowsky type with an infinite number of variables by using the theorems of J. L. Lions et al. ([LIONS, J. L.: *Optimal Control of Systems Governed by Partial Differential Equations*. Grundlehren Math. Wiss. 170, Springer-Verlag, Berlin-Heidelberg-New York, 1971], [LIONS, J. L.—MAGENES, E.: *Non-Homogeneous Boundary Value Problem and Applications, Vol. I*. Grundlehren Math. Wiss. 181, Springer-Verlag, Berlin-Heidelberg-New York, 1972]) Yet the problem considered there is more general than the one in [EL-SAIFY, H. A.: *Boundary control problem with an infinite number of variables*, Internat. J. Math. Math. Sci. **28** (2001), 57–62], [KOTARSKI, W.—EL-SAIFY, H. A.—BAHAA, G. M.: *Optimal control of parabolic equation with an infinite number of variables for non-standard functional and time delay*, IMA J. Math. Control Inform. **19** (2002), 461–476]. The controls are allowed to be in the space $\left(L^2(0,T;L^2(\mathbb{R}^\infty))\right)^n$. The necessary and sufficient conditions for optimality of the control are obtained and the set of inequalities defining the optimal control of these systems are also obtained. This study is carried out in two directions, the first one for the problem with mixed Dirichlet conditions, and the second one for the same problem with mixed Neumann conditions. Several mathematical examples and a real example for derived optimality conditions are presented.

2000 Mathematics Subject Classification: Primary 35R15, 49J20, 49K20; Secondary 93C20.
Keywords: optimal control problem, Dirichlet and Neumann conditions, existence and uniqueness of solution, $n \times n$ coupled systems of Petrowsky type, operator with an infinite number of variables.

Introduction

Infinite dimensional systems can be used to describe many physical phenomena in the real world. Well-known examples are heat conduction, vibration of elastic material, diffusion-reaction processes, population systems, and many others. Thus, the optimal control theory for infinite dimensional systems has a wide range of applications in engineering, economics, and some other fields. On the other hand, this theory has its own mathematical interests since it is regarded as a generalization for the classical calculus of variations and it generates many interesting mathematical questions ([14]).

The necessary and sufficient conditions of optimality for systems governed by different types of partial differential operators defined on spaces with a finite number of variables are discussed for example in [15]–[17].

The optimal control problem of systems governed by different types of operators defined on spaces with an infinite number of variables are initiated and proved in [3], [4], [8], [9], [11]. The interest in the study of this class of operators is stimulated by problems in quantum field theory ([2], [3]).

In [6], [7], we have obtained the set of inequalities that characterize the optimal control for $n \times n$ coupled systems governed by elliptic, parabolic and hyperbolic equations of infinite number of variables with different conditions.

In [12], the author have obtained the optimal control of a system governed by Petrowsky type equation with an infinite number of variables, where the system contained only one equation.

Here, we study the optimal control problem for $n \times n$ coupled systems of Petrowsky type equation involving 2ℓth order self-adjoint operator with an infinite number of variables. A set of inequalities that characterizes this optimal control is obtained and this set is studied in order to construct algorithms useful to numerical computations for the approximation of the control. We refer, for instance, to [10], [16] for the application of similar results in quantum field and as a physical example. In [10] the authors studied the local controllability problem for the Navier-Stokes equations that was described by an $n \times n$ coupled systems. In [16], they analyze the controllability of the motion of a fluid by means of the action of a vibrating shell coupled at the boundary of the fluid.

This paper is organized as follows. In Section 1, we introduce spaces of functions of infinitely many variables. Section 2 contains some facts and new results which enables us to formulate the $n \times n$ coupled Dirichlet problem for Petrowsky type of equations with an infinite number of variables. In Section 3, the control problem for this case is formulated, then we give the necessary and sufficient conditions for the control to be optimal. In Section 4, we study the optimal control for $n \times n$ coupled Neumann problem of Petrowsky type. In all our considered problems the control is of distributed type.

1. Some functional spaces

([1], [2], [12])

Below we consider functions of points $x \in \mathbb{R}^\infty = \mathbb{R}^1 \times \mathbb{R}^1 \times \dots$, the coordinate notation for such points is $x = (x_k)_{k=1}^\infty$, $x_k \in \mathbb{R}^1$. Let $\big(p_k(x_k)\big)_{k=1}^\infty$ be a fixed sequence of continuous positive probability weights, i.e., $\int_{\mathbb{R}^1} p_k(x)\, \mathrm{d}x = 1$. The measure on $\mathbb{R}^\infty$ given by

$$\begin{aligned} \mathrm{d}\varrho(x) &= \big(p_1(x_1)\,\mathrm{d}x_1\big) \otimes \big(p_2(x_2)\,\mathrm{d}x_2\big) \otimes \dots \\ &= \big(\mathrm{d}\varrho_1(x_1)\big) \otimes \big(\mathrm{d}\varrho_2(x_2)\big) \otimes \dots \end{aligned}$$

is called a *(weighted) product measure.*

The space $L^2\big(\mathbb{R}^\infty, \mathrm{d}\varrho(x)\big)$ can be understood as the infinite tensor product $\bigotimes_{k=1}^{\infty} \big(L^2(\mathbb{R}^1, \mathrm{d}\varrho_k(x_k))\big) = L^2\big(\mathbb{R}^1 \times \mathbb{R}^1 \times \dots, \mathrm{d}\varrho_1(x_1) \otimes \mathrm{d}\varrho_2(x_2) \otimes \dots\big)$, i.e., $L^2\big(\mathbb{R}^\infty, \mathrm{d}\varrho(x)\big)$ is the space of all square integrable functions on $\mathbb{R}^\infty$ with weight. We shall often set $L^2\big(\mathbb{R}^\infty, \mathrm{d}\varrho(x)\big) = L^2\big(\mathbb{R}^\infty\big)$.

It is a classical result that $L^2(\mathbb{R}^\infty)$ is a Hilbert space for the scalar product

$$(\phi, \psi)_{L^2(\mathbb{R}^\infty)} = \int_{\mathbb{R}^\infty} \phi(x)\psi(x)\, \mathrm{d}\varrho(x)\,.$$

The Sobolev spaces $W^\ell\big(\mathbb{R}^\infty, \mathrm{d}\varrho(x)\big)$ ($\ell = 1, 2, \dots$), which we shall denote by $W^\ell(\mathbb{R}^\infty)$, with a weight will be encountered much more frequently; they are defined as follows. $W^\ell(\mathbb{R}^\infty)$ is the completion of $C_0^\infty(\mathbb{R}^\infty)\big|_{\mathbb{R}^\infty}$ with respect to the inner product

$$(\phi, \psi)_{W^\ell(\mathbb{R}^\infty)} = \sum_{|\alpha| \le \ell} (\mathrm{D}^\alpha \phi, \mathrm{D}^\alpha \psi)_{L^2(\mathbb{R}^\infty)}\,, \qquad \psi, \phi \in C_0^\infty(\mathbb{R}^\infty)\big|_{\mathbb{R}^\infty}$$

where

$$\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} (\partial x_2)^{\alpha_2} \dots}\,, \qquad |\alpha| = \sum_{i=1}^{\infty} \alpha_i\,,$$

$C_0^\infty(\mathbb{R}^\infty)$ is the collection of infinitely differentiable compactly supported functions on $\mathbb{R}^\infty$, and Γ is the boundary of $\mathbb{R}^\infty$ (Γ is meant to be the boundary of the support of the measure $\mathrm{d}\varrho(x)$).

Obviously, $\|\phi\|_{L^2(\mathbb{R}^\infty)} \le \|\phi\|_{W^\ell(\mathbb{R}^\infty)}$ ($\phi \in W^\ell(\mathbb{R}^\infty)$) and $W^\ell(\mathbb{R}^\infty)$ is dense in $L^2(\mathbb{R}^\infty)$. Therefore $W^\ell(\mathbb{R}^\infty)$ and $L^2(\mathbb{R}^\infty)$ can be taken as the positive and zero spaces, and we construct the corresponding negative Sobolev spaces $W^{-\ell}(\mathbb{R}^\infty)$, the spaces of generalized functions, so we have the following chain

$$\begin{gathered} W^\ell(\mathbb{R}^\infty) \subseteq L^2(\mathbb{R}^\infty) \subseteq W^{-\ell}(\mathbb{R}^\infty)\,, \\ \|\phi\|_{W^\ell(\mathbb{R}^\infty)} \ge \|\phi\|_{L^2(\mathbb{R}^\infty)} \ge \|\phi\|_{W^{-\ell}(\mathbb{R}^\infty)}\,. \end{gathered}$$

The space $W_0^\ell(\mathbb{R}^\infty)$ is a proper subspace of $W^\ell(\mathbb{R}^\infty)$ consisting of all the functions $\phi \in W^\ell(\mathbb{R}^\infty)$ such that $\mathrm{D}^\beta\phi|_\Gamma = 0$, where $|\beta| \le \ell - 1$.

Clearly, $W_0^\ell(\mathbb{R}^\infty)$ is dense in $L^2(\mathbb{R}^\infty)$. Constructing the corresponding negative space $W_0^{-\ell}(\mathbb{R}^\infty)$, we get the chain

$$W_0^\ell(\mathbb{R}^\infty) \subseteq L^2(\mathbb{R}^\infty) \subseteq W_0^{-\ell}(\mathbb{R}^\infty),$$
$$\|\phi\|_{W_0^\ell(\mathbb{R}^\infty)} \ge \|\phi\|_{L^2(\mathbb{R}^\infty)} \ge \|\phi\|_{W_0^{-\ell}(\mathbb{R}^\infty)}.$$

$L^2(0,T;W^\ell(\mathbb{R}^\infty))$ denotes the space of measurable functions $t \mapsto \phi(t)$: $]0,T[\, \to W^\ell(\mathbb{R}^\infty)$ for the Lebesgue measure $\mathrm{d}t$ and such that

$$\left(\int_0^T \|\phi(t)\|_{W^\ell(\mathbb{R}^\infty)}\,\mathrm{d}t\right)^{\frac{1}{2}} = \|\phi\|_{L^2(0,T;W^\ell(\mathbb{R}^\infty))} < \infty,$$

where the variable t denotes the "time"; we assume that $t \in \,]0,T[$, $T < \infty$. This space is a Hilbert space with respect to the scalar product

$$(\phi,\psi)_{L^2(0,T;W^\ell(\mathbb{R}^\infty))} = \int_0^T \big(\phi(t),\psi(t)\big)_{W^\ell(\mathbb{R}^\infty)}\,\mathrm{d}t.$$

Since $W^\ell(\mathbb{R}^\infty)$ is Hilbert space, then the dual of $L^2(0,T;W^\ell(\mathbb{R}^\infty))$ is the space $L^2(0,T;W^{-\ell}(\mathbb{R}^\infty))$.

Analogously we can define the space $L^2(0,T;L^2(\mathbb{R}^\infty))$, which we shall denote by $L^2(Q)$, $Q = \mathbb{R}^\infty \times \,]0,T[$, $\Sigma = \Gamma \times \,]0,T[$, where Σ is the lateral boundary of Q.

It is easy to construct the following Sobolev spaces $(W^\ell(\mathbb{R}^\infty))^n$ by the n-times Cartesian product as follows:

$$(W^\ell(\mathbb{R}^\infty))^n = \underbrace{W^\ell(\mathbb{R}^\infty) \times W^\ell(\mathbb{R}^\infty) \times \cdots \times W^\ell(\mathbb{R}^\infty)}_{n\text{-times}}$$

with the norm defined by

$$\|\overline{\phi}\|_{(W^\ell(\mathbb{R}^\infty))^n} = \sum_{i=1}^n \|\phi_i\|_{W^\ell(\mathbb{R}^\infty)},$$

where $\overline{\phi} = (\phi_1,\phi_2,\ldots,\phi_n) = (\phi_i)_{i=1}^n$ is a vector function and $\phi_i \in W^\ell(\mathbb{R}^\infty)$, also we can construct the Cartesian product for the above Hilbert spaces. Finally we have the following chains:

$$\big(L^2(0,T;W^\ell(\mathbb{R}^\infty))\big)^n \subseteq \big(L^2(Q)\big)^n \subseteq \big(L^2(0,T;W^{-\ell}(\mathbb{R}^\infty))\big)^n,$$
$$\big(L^2(0,T;W_0^\ell(\mathbb{R}^\infty))\big)^n \subseteq \big(L^2(Q)\big)^n \subseteq \big(L^2(0,T;W_0^{-\ell}(\mathbb{R}^\infty))\big)^n,$$

where $\big(L^2(0,T;W^{-\ell}(\mathbb{R}^\infty))\big)^n$ and $\big(L^2(0,T;W_0^{-\ell}(\mathbb{R}^\infty))\big)^n$ are the dual spaces of $\big(L^2(0,T;W^\ell(\mathbb{R}^\infty))\big)^n$ and $\big(L^2(0,T;W_0^\ell(\mathbb{R}^\infty))\big)^n$ respectively.

2. Some facts and results

([5], [6], [7], [12])

We consider a family of operators $\mathcal{A}(t) \in \mathcal{L}\Big(\big(W_0^{\ell}(\mathbb{R}^\infty)\big)^n; \big(W_0^{-\ell}(\mathbb{R}^\infty)\big)^n\Big)$ such that

$$\begin{aligned}
&\mathcal{A}(t)\big(\overline{\phi} = (\phi_1, \phi_2, \dots, \phi_n)\big) \\
&= \Bigg(\sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha}\,\phi_1(x) + q(x,t)\phi_1(x), \sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha}\,\phi_2(x) \\
&\qquad + q(x,t)\phi_2(x), \dots, \sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha}\,\phi_n(x) + q(x,t)\phi_n(x)\Bigg) \\
&= \begin{pmatrix}
\sum\limits_{|\alpha|\le \ell}\sum\limits_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sum\limits_{|\alpha|\le \ell}\sum\limits_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q
\end{pmatrix}_{n\times n}
\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}_{n\times 1}
\end{aligned}$$

i.e.,

$$\mathcal{A}(t)\phi_i(x) = \sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha}\,\phi_i(x) + q(x,t)\phi_i(x)\,, \qquad i = 1, 2, \dots, n\,.$$

So $\mathcal{A}(t)$ is an $n \times n$ diagonal matrix of the following bounded self-adjoint elliptic partial differential operator of 2ℓth order with an infinite number of variables, which takes the form

$$\begin{aligned}
A(t)\phi_i(x) &= \sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\frac{1}{\sqrt{p_k(x_k,t)}}\frac{\partial^{2\alpha}}{\partial x_k^{2\alpha}}\sqrt{p_k(x_k,t)}\phi_i(x) + q(x,t)\phi_i(x)\,, \\
&= \sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha}\,\phi_i(x) + q(x,t)\phi_i(x)
\end{aligned}$$

where

$$\mathrm{D}_k\,\phi_i(x) = \frac{1}{\sqrt{p_k(x_k,t)}}\frac{\partial}{\partial x_k}\sqrt{p_k(x_k,t)}\phi_i(x)$$

and $q(x,t)$ is a real valued function in x which is bounded and measurable on $\mathbb{R}^\infty$, such that $q(x,t) \ge c_0 > 0$, c_0 is constant.

Let M be n-square matrix of coefficients such that

$$\begin{aligned} M(\phi_1,\phi_2,\dots,\phi_n) &= \left(\sum_{j=1}^{n} a_{1j}\phi_j, \sum_{j=1}^{n} a_{2j}\phi_j,\dots,\sum_{j=1}^{n} a_{nj}\phi_j\right) \\ &= \big(a_{11}\phi_1 + a_{12}\phi_2 + \cdots + a_{1n}\phi_n, a_{21}\phi_1 + a_{22}\phi_2 + \cdots + a_{2n}\phi_n, \\ &\qquad \dots, a_{n1}\phi_1 + a_{n2}\phi_2 + \cdots + a_{nn}\phi_n\big) \\ &= \begin{pmatrix} 1 & -1 & \dots & -1 \\ 1 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}_{n\times n} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}_{n\times 1} \end{aligned}$$

i.e.,

$$M\phi_i = \sum_{j=1}^{n} a_{ij}\phi_j\,, \qquad i = 1,2,\dots,n\,,$$

where $a_{ij} = \begin{cases} 1 & \text{if } i \ge j\,, \\ -1 & \text{if } i < j \end{cases}$ is the coupled term.

Let $S(t) = \mathcal{A}(t) + M$ (so it is $n \times n$ matrix) takes the form

$$S(t) = \begin{pmatrix} \sum\limits_{|\alpha|\le \ell}\sum\limits_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q + 1 & \dots & -1 \\ 1 & \dots & -1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & \sum\limits_{|\alpha|\le \ell}\sum\limits_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q + 1 \end{pmatrix}_{n\times n}.$$

That is

$$\begin{aligned} S(t)\phi_i(x) &= \sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha}\,\phi_i(x) + q(x,t)\phi_i(x) + \sum_{j=1}^{n} a_{ij}\phi_j(x) \\ &= \mathcal{A}(t)\phi_i(x) + M\phi_i(x)\,, \qquad i = 1,2,\dots,n\,. \end{aligned}$$

For each $t \in\,]0,T[$, we define a family of bilinear forms on $\big(W_0^{\ell}(\mathbb{R}^\infty)\big)^n$ by:

$$\pi\big(t;\overline{\phi},\overline{\psi}\big) = \big(S(t)\overline{\phi},\overline{\psi}\big)_{(L^2(\mathbb{R}^\infty))^n}\,, \qquad \overline{\phi} = (\phi_i)_{i=1}^n,\ \overline{\psi} = (\psi_i)_{i=1}^n \in \big(W_0^{\ell}(\mathbb{R}^\infty)\big)^n\,,$$

where $S(t)$ maps $\left(W_0^{\ell}(\mathbb{R}^\infty)\right)^n$ onto $\left(W_0^{-\ell}(\mathbb{R}^\infty)\right)^n$ and takes the above form. Then

$$\begin{aligned}
&\pi\left(t;\overline{\phi},\overline{\psi}\right) = \\
&= \sum_{i=1}^{n}\left(S(t)\phi_i,\psi_i\right)_{L^2(\mathbb{R}^\infty)} \\
&= \sum_{i=1}^{n}\left(A(t)\phi_i(x)+\sum_{j=1}^{n}a_{ij}\phi_j(x),\ \psi_i(x)\right)_{L^2(\mathbb{R}^\infty)} \\
&= \sum_{i=1}^{n}\left(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\mathrm{D}_k^{2\alpha}\,\phi_i(x)+q(x,t)\phi_i(x)+\sum_{j=1}^{n}a_{ij}\phi_j(x),\ \psi_i(x)\right)_{L^2(\mathbb{R}^\infty)} \\
&= \sum_{i=1}^{n}\int_{\mathbb{R}^\infty}\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}\mathrm{D}_k^{\alpha}\,\phi_i(x)\,\mathrm{D}_k^{\alpha}\,\psi_i(x)\ \mathrm{d}\varrho(x)+\sum_{i=1}^{n}\int_{\mathbb{R}^\infty}q(x,t)\phi_i(x)\psi_i(x)\ \mathrm{d}\varrho(x) \\
&\qquad +\sum_{i=1}^{n}\int_{\mathbb{R}^\infty}\sum_{j=1}^{n}a_{ij}\phi_j(x)\psi_i(x)\ \mathrm{d}\varrho(x)\,.
\end{aligned} \tag{1}$$

The bilinear form (1) is coercive on $\left(W_0^{\ell}(\mathbb{R}^\infty)\right)^n$ that is, there exists $\lambda\in\mathbb{R}$, such that:

$$\pi\left(t;\overline{\phi},\overline{\phi}\right)\ge\lambda\left\|\,\overline{\phi}\,\right\|^2_{(W_0^{\ell}(\mathbb{R}^\infty))^n}\,,\qquad \lambda>0\,. \tag{2}$$

It is well known that the ellipticity of $A(t)$ is sufficient for the coercitivness of $\pi\left(t;\overline{\phi},\overline{\psi}\right)$ on $\left(W_0^{\ell}(\mathbb{R}^\infty)\right)^n$. By taking into account the form of a_{ij}, we have

$$\begin{aligned}
&\pi\left(t;\overline{\phi},\overline{\psi}\right) = \\
&= \sum_{i=1}^{n}\int_{\mathbb{R}^\infty}\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}\mathrm{D}_k^{\alpha}\,\phi_i(x)\,\mathrm{D}_k^{\alpha}\,\psi_i(x)\ \mathrm{d}\rho+\sum_{i=1}^{n}\int_{\mathbb{R}^\infty}q(x,t)\phi_i(x)\psi_i(x)\ \mathrm{d}\rho \\
&\quad +\sum_{i=j=1}^{n}\int_{\mathbb{R}^\infty}\phi_j(x)\psi_i(x)\ \mathrm{d}\rho+\sum_{i>j}^{n}\int_{\mathbb{R}^\infty}\phi_j(x)\psi_i(x)\ \mathrm{d}\rho-\sum_{i<j}^{n}\int_{\mathbb{R}^\infty}\phi_j(x)\psi_i(x)\ \mathrm{d}\varrho\,.
\end{aligned}$$

Then

$$
\begin{aligned}
&\pi(t;\overline{\phi},\overline{\phi}) = \\
&= \sum_{i=1}^{n}\Big(\int_{\mathbb{R}^\infty} \sum_{|\alpha|\le \ell} \sum_{k=1}^{\infty} |\mathrm{D}_k^\alpha\, \phi_i(x)|^2 \,\mathrm{d}\rho + \int_{\mathbb{R}^\infty} q(x,t)|\phi_i(x)|^2 \,\mathrm{d}\rho + \int_{\mathbb{R}^\infty} |\phi_i(x)|^2 \,\mathrm{d}\varrho \Big) \\
&\ge \sum_{i=1}^{n}\Big(\sum_{|\alpha|\le \ell} \sum_{k=1}^{\infty} \|D_k^\alpha \phi_i(x)\|^2_{L^2(\mathbb{R}^\infty)} + c_0 \|\phi_i(x)\|^2_{L^2(\mathbb{R}^\infty)} + \|\phi_i(x)\|^2_{L^2(\mathbb{R}^\infty)} \Big) \\
&= \sum_{i=1}^{n} \|\phi_i(x)\|^2_{W_0^\ell(\mathbb{R}^\infty)} + c_0 \sum_{i=1}^{n} \|\phi_i(x)\|^2_{L^2(\mathbb{R}^\infty)} \\
&\ge \sum_{i=1}^{n} \|\phi_i(x)\|^2_{W_0^\ell(\mathbb{R}^\infty)} \\
&= \|\,\overline{\phi}\,\|^2_{(W_0^\ell(\mathbb{R}^\infty))^n}\,;
\end{aligned}
$$

$$
\begin{aligned}
&\big(\forall \overline{\phi},\overline{\psi} \in (W_0^\ell(\mathbb{R}^\infty))^n\big)\big(\text{the function } t \mapsto \pi(t;\overline{\phi},\overline{\psi}) \\
&\qquad\qquad \text{is continuously differentiable in }]0,T[\big)\,, \\
&\qquad \pi(t;\overline{\phi},\overline{\psi}) = \pi(t;\overline{\psi},\overline{\phi})\,.
\end{aligned} \tag{3}
$$

Under the above consideration, using [15; Chap. I, Theorem 1.4], we can formulate the following $n \times n$ mixed Dirichlet problem for Petrowsky type equation with an infinite number of variables ([5], [6], [7]).

Assume that (2) and (3) hold, $f_i = f_i(x,t)$, $y_{i,0}(x)$ and $y_{i,1}(x)$ are given in $L^2(Q)$, $W_0^\ell(\mathbb{R}^\infty)$ and $L^2(\mathbb{R}^\infty)$ respectively. Then there exists a unique $\overline{y} = (y_i)_{i=1}^n \in \big(L^2(Q)\big)^n$ satisfying: for all i, $1 \le i \le n$,

$$
\begin{aligned}
y_i,\ \frac{\partial y_i}{\partial x_k},\ \frac{\partial y_i}{\partial t} &\in L^2(Q)\,, \\
\frac{\partial^2 y_i}{\partial t^2} + S(t) y_i &= f_i \qquad \text{in } Q\,, \\
y_i &= 0 \qquad \text{on } \Sigma
\end{aligned}
$$

with the initial conditions

$$
\begin{aligned}
y_i(x,0) &= y_{i,0}(x) \qquad \text{in } \mathbb{R}^\infty\,, \\
\frac{\partial y_i(x,0)}{\partial t} &= y_{i,1}(x) \qquad \text{in } \mathbb{R}^\infty\,.
\end{aligned}
$$

Note. The operator $\frac{\partial^2}{\partial t^2} + S(t)$ is well posed in the sense of Petrowsky type with an infinite number of variables and maps $\big(L^2(0,T;W_0^\ell(\mathbb{R}^\infty))\big)^n$ onto $\big(L^2(0,T;W_0^{-\ell}(\mathbb{R}^\infty))\big)^n$.

3. Optimal control problem for Petrowsky type equation with mixed Dirichlet conditions

The space $\mathcal{U} = \big(L^2(Q)\big)^n$ is the space of controls. If $f_i \in L^2(Q)$, $y_{i,0} \in W_0^\ell(\mathbb{R}^\infty)$ and $y_{i,1} \in L^2(\mathbb{R}^\infty)$ and if (2) and (3) hold, then for a control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}$, the state of the system $\overline{y}(\overline{u}) = (y_i(\overline{u}))_{i=1}^n$, which is dependent on x, t and is denoted by $\overline{y}(x,t;\overline{u})$ and is given by the solution of:

$$\begin{aligned}
\frac{\partial^2 y_i(\overline{u})}{\partial t^2} + S(t) y_i(\overline{u}) &= f_i + u_i && \text{in } Q\,,\\
y_i(\overline{u}) &= 0 && \text{on } \Sigma\,,\\
y_i(x,0;\overline{u}) &= y_{i,0}(x) && \text{in } \mathbb{R}^\infty\,,\\
\frac{\partial y_i(x,0;\overline{u})}{\partial t} &= y_{i,1}(x) && \text{in } \mathbb{R}^\infty\,,\\
y_i(\overline{u}) \in L^2(Q)\,, \qquad & \frac{\partial y_i(\overline{u})}{\partial t} \in L^2(Q)\,.
\end{aligned} \tag{4}$$

The observation is given by:

$$\big(z_i(\overline{u})\big)_{i=1}^n = \overline{z}(\overline{u}) = \overline{y}(\overline{u}) = \big(y_i(\overline{u})\big)_{i=1}^n$$

i.e.,

$$z_i(\overline{u}) = y_i(\overline{u}) \qquad \text{for all} \quad 1 \le i \le n\,.$$

For some $\overline{z}_d = (z_{i,d})_{i=1}^n \in (L^2(Q))^n$, the cost function $J(\overline{u})$ is given by

$$J(\overline{u}) = \sum_{i=1}^n \|y_i(\overline{u}) - z_{i,d}\|_{L^2(Q)}^2 + \sum_{i=1}^n (N_i u_i, u_i)_{L^2(Q)}\,,$$

which is equivalent to

$$J(\overline{u}) = \sum_{i=1}^n \int_Q \big(y_i(\overline{u}) - z_{i,d}\big)^2 \,\mathrm{d}\varrho\,\mathrm{d}t + \sum_{i=1}^n (N_i u_i, u_i)_{L^2(Q)}\,, \tag{5}$$

where

$$N = (N_i)_{i=1}^n \in \mathcal{L}\Big(\big(L^2(Q)\big)^n, \big(L^2(Q)\big)^n\Big)$$

is a diagonal matrix of Hermitian positive definite operators: $N\overline{u} = (N_i u_i)_{i=1}^n$,

$$(N\overline{u}, \overline{u})_{(L^2(Q))^n} \ge \zeta \|\overline{u}\|_{(L^2(Q))^n}^2\,, \qquad \zeta > 0\,. \tag{6}$$

Our problem is to find

$$\inf_{\overline{u} \in \mathcal{U}_{\mathrm{ad}}} J(\overline{u})\,,$$

where the set of admissible controls $\mathcal{U}_{\mathrm{ad}}$ is closed convex subset of $(L^2(Q))^n$.

Under the given considerations, we apply [15; Chap. II, Theorem 1.4] to obtain the necessary and sufficient conditions of optimality:

THEOREM 1. *We assume that (2), (3) and (6) hold. The cost function is given by (5). Then the optimal control $\overline{u} = (u_i)_{i=1}^n$ exists and the necessary and sufficient conditions of optimality are given by (4) with the following system of partial differential equations and inequalities*

$$\begin{aligned} \frac{\partial^2 p_i(\overline{u})}{\partial t^2} + S^*(t)p_i(\overline{u}) &= y_i(\overline{u}) - z_{i,d}\,, && \textit{in } Q\,,\\ p_i(\overline{u}) &= 0\,, && \textit{on } \Sigma\,,\\ p_i(x,T;\overline{u}) &= 0\,, && \textit{in } \mathbb{R}^\infty\,,\\ \frac{\partial p_i(x,T;\overline{u})}{\partial t} &= 0\,, && \textit{in } \mathbb{R}^\infty \end{aligned} \tag{7}$$

and

$$\sum_{i=1}^n \big(p_i(\overline{u}) + N_i u_i,\, v_i - u_i\big)_{L^2(Q)} \ge 0 \qquad \textit{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}$$

with

$$p_i(\overline{u}),\ \frac{\partial p_i(\overline{u})}{\partial t} \in L^2(Q)\,,$$

where $S^*(t)$ is the adjoint of $S(t)$ defined by

$$S^*(t)\phi_i = A(t)\phi_i + \sum_{j=1}^n a_{ji}\phi_j\,,$$

a_{ji} is the transpose of a_{ij} and $p_i(\overline{u})$ is the adjoint state.

Proof. As in [5], [6], [7], the control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}$ is optimal if and only if

$$\sum_{i=1}^n J_i'(\overline{u})(v_i - u_i) \ge 0\,, \qquad \text{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}\,,$$

which is equivalent to:

$$\sum_{i=1}^n \big(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(Q)} + \sum_{i=1}^n (N_i u_i, v_i - u_i)_{L^2(Q)} \ge 0\,,$$

which may be written as

$$\sum_{i=1}^n \int_0^T \big(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(\mathbb{R}^\infty)}\, \mathrm{d}t + \sum_{i=1}^n (N_i u_i, v_i - u_i)_{L^2(Q)} \ge 0\,. \tag{8}$$

We shall now transform (8) as follows: we scalar multiply both sides of the first equation in (7) by $\left(y_i(\overline{v}) - y_i(\overline{u})\right)$, and integrating between 0, T gives us

$$\sum_{i=1}^{n} \int_0^T \left(\left(\frac{\partial^2}{\partial t^2} + S^*(t) \right) p_i(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^\infty)} \mathrm{d}t$$

$$= \sum_{i=1}^{n} \int_0^T \left(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^\infty)} \mathrm{d}t\,.$$

By taking into account conditions (4) and (7), after applying Green's formula to the left side of (7) with noting that if $\phi_i \in L^2(Q)$, $\phi_i' \in L^2(Q)$, $\phi_i'' \in L^2(0,T; W_0^{-\ell}(\mathbb{R}^\infty))$ and if ψ_i has the same properties, then

$$\int_0^T (\phi_i'', \psi_i)\, \mathrm{d}t = \left(\phi_i'(T), \psi_i(T)\right) - \left(\phi_i'(0), \psi_i(0)\right) - \left(\phi_i(T), \psi_i'(T)\right)$$

$$+ \left(\phi_i(0), \psi_i'(0)\right) + \int_0^T (\phi_i, \psi_i'')\, \mathrm{d}t\,.$$

Then we have

$$\int_0^T \left(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^\infty)} \mathrm{d}t$$

$$= \int_0^T \left(\left(\frac{\partial^2}{\partial t^2} + S^*(t) \right) p_i(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^\infty)} \mathrm{d}t$$

$$= \int_0^T \left(\left(\frac{\partial^2}{\partial t^2} + A(t) \right) p_i(\overline{u}) + \sum_{j=1}^{n} a_{ji} p_j(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^\infty)} \mathrm{d}t$$

$$= \int_0^T \left(p_i(\overline{u}),\, \left(\frac{\partial^2}{\partial t^2} + A(t) + \sum_{j=1}^{n} a_{ij} \right) \left(y_i(\overline{v}) - y_i(\overline{u}) \right) \right)_{L^2(\mathbb{R}^\infty)} \mathrm{d}t$$

$$= \int_0^T \left(p_i(\overline{u}), v_i - u_i \right)_{L^2(\mathbb{R}^\infty)} \mathrm{d}t\,.$$

Hence (8) becomes

$$\sum_{i=1}^{n} \int_{0}^{T} (p_i(\overline{u}), v_i - u_i)_{L^2(\mathbb{R}^\infty)} \, \mathrm{d}t + \sum_{i=1}^{n} (N_i u_i, v_i - u_i)_{L^2(Q)} \geq 0 ,$$
$$\text{for all} \quad \overline{v} = (v_i)_{i=1}^{n} \in \mathcal{U}_{\mathrm{ad}} .$$

i.e.,

$$\sum_{i=1}^{n} (p_i(\overline{u}) + N_i u_i, \, v_i - u_i)_{L^2(Q)} \geq 0 \qquad \text{for all} \quad \overline{v} = (v_i)_{i=1}^{n} \in \mathcal{U}_{\mathrm{ad}} .$$

The theorem is proved. □

EXAMPLE 1. If we take $n = 2$, then the optimality system is given by [5], [6], [7]:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y_1(\overline{u}) + \Big(\displaystyle\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} (-1)^{|\alpha|} \mathrm{D}_k^{2\alpha} + q(x,t) \Big) y_1(\overline{u}) + y_1(\overline{u}) - y_2(\overline{u}) = f_1 + u_1 \\ \hfill \text{in } Q , \\ \dfrac{\partial^2}{\partial t^2} y_2(\overline{u}) + \Big(\displaystyle\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} (-1)^{|\alpha|} \mathrm{D}_k^{2\alpha} + q(x,t) \Big) y_2(\overline{u}) + y_1(\overline{u}) + y_2(\overline{u}) = f_2 + u_2 \\ \hfill \text{in } Q , \\ y_1(\overline{u}) = 0 , \qquad y_2(\overline{u}) = 0 \qquad \text{on } \Sigma , \\ y_1(x,0;\overline{u}) = y_{1,0}(x;\overline{u}) , \qquad y_2(x,0;\overline{u}) = y_{2,0}(x;\overline{u}) \qquad \text{in } \mathbb{R}^\infty , \\ \dfrac{\partial y_1(x,0;\overline{u})}{\partial t} = y_{1,1}(x;\overline{u}) , \qquad \dfrac{\partial y_2(x,0;\overline{u})}{\partial t} = y_{2,1}(x;\overline{u}) \qquad \text{in } \mathbb{R}^\infty , \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} p_1(\overline{u}) + \Big(\displaystyle\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} (-1)^{|\alpha|} \mathrm{D}_k^{2\alpha} + q(x,t) \Big) p_1(\overline{u}) + p_1(\overline{u}) + p_2(\overline{u}) \\ \hfill = y_1(\overline{u}) - z_{1,d} \qquad \text{in } Q , \\ \dfrac{\partial^2}{\partial t^2} p_2(\overline{u}) + \Big(\displaystyle\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} (-1)^{|\alpha|} \mathrm{D}_k^{2\alpha} + q(x,t) \Big) p_2(\overline{u}) - p_1(\overline{u}) + p_2(\overline{u}) \\ \hfill = y_2(\overline{u}) - z_{2,d} \qquad \text{in } Q , \\ p_1(\overline{u}) = 0 , \qquad p_2(\overline{u}) = 0 \qquad \text{on } \Sigma , \\ p_1(x,t;\overline{u}) = 0 , \qquad p_2(x,t;\overline{u}) = 0 \qquad \text{in } \mathbb{R}^\infty , \\ \dfrac{\partial p_1(x,t;\overline{u})}{\partial t} = 0 , \qquad \dfrac{\partial p_2(x,t;\overline{u})}{\partial t} = 0 \qquad \text{in } \mathbb{R}^\infty , \end{cases} \tag{10}$$

$$\int_Q \big(p_1(\overline{u}) + N_1 u_1\big)(v_1 - u_1) + \big(p_2(\overline{u}) + N_2 u_2\big)(v_2 - u_2)\, \mathrm{d}\varrho\, \mathrm{d}t \geq 0 \qquad \text{for all} \quad \overline{v} = (v_1, v_2) \in \mathcal{U}_{\mathrm{ad}}, \tag{11}$$

where $\overline{u} = (u_1, u_2) \in \mathcal{U}_{\mathrm{ad}}$ and $\overline{P}(\overline{u}) = \big(p_1(\overline{u}), p_2(\overline{u})\big)$ is the adjoint state.

EXAMPLE 2. In the case of no constraints on the control ($\mathcal{U}_{\mathrm{ad}} = \mathcal{U}$), we obtain

$$u_1 = -N_1^{-1}\big(P_1(u)|_{\Sigma}\big), \qquad u_2 = -N_2^{-1}\big(P_2(u)|_{\Sigma}\big). \tag{12}$$

The optimal control is determined by simultaneously solving (9), (10) (where we eliminate u_1 with the aid of (12)) and then utilizing (12).

EXAMPLE 3. If we take

$$\mathcal{U}_{\mathrm{ad}} = \{v : \ v_i \geq 0 \ \text{on} \ \Sigma, \ i = 1, 2\}$$

and $N = \nu \times$ Identity, then the inequality (11) gives us

$$\begin{aligned} &u_1 \geq 0, \quad p_1(u) + \nu_1 u_1 \geq 0, \quad u_1(p_1(u) + \nu_1 u_1) = 0 \qquad \text{on } \Sigma, \\ &u_2 \geq 0, \quad p_2(u) + \nu_2 u_2 \geq 0, \quad u_2(p_2(u) + \nu_2 u_2) = 0 \qquad \text{on } \Sigma. \end{aligned} \tag{13}$$

EXAMPLE 4. If we take

$$\mathcal{U}_{\mathrm{ad}} = \{u : \ u_1 \ \text{arbitrary in} \ L^2(\mathbb{R}^\infty), \ u_2 \geq 0 \ \text{a.e. in} \ \mathbb{R}^\infty\}. \tag{14}$$

Thus there are no constraints on u_1, then the inequality (11) is equivalent to

$$\begin{aligned} p_1(u) + N_1 u_1 &= 0, \\ p_2(u) + N_2 u_2 &\geq 0, \quad u_2 \geq 0, \\ u_2\big(p_2(u) + N_2 u_2\big) &= 0. \end{aligned} \tag{15}$$

Thus under the hypotheses of (14), the optimal control is given by the solution of the system of equations and inequalities

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}y_1 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)y_1 + y_1 - y_2 + N_1^{-1}p_1 = f_1\,,\\
&\frac{\partial^2}{\partial t^2}y_2 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)y_2 + y_1 + y_2 - f_2 \ge 0\,,\\
&\frac{\partial^2}{\partial t^2}p_1 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)p_1 + p_1 + p_2 - y_1 = -z_{1,d}\,,\\
&\frac{\partial^2}{\partial t^2}p_2 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)p_2 - p_1 + p_2 - y_2 = -z_{2,d}\,,\\
&p_2 + N_2\Big(\frac{\partial^2}{\partial t^2}y_1 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)y_2 + y_1 + y_2 - f_2\Big) \ge 0\,,\\
&\Big(\frac{\partial^2}{\partial t^2}y_2 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)y_2 + y_1 + y_2 - f_2\Big)\cdot\\
&\quad\cdot\Big[p_2 + N_2\Big(\frac{\partial^2}{\partial t^2}y_2 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)y_2 + y_1 + y_2 - f_2\Big)\Big] = 0\,,\\
&\qquad\qquad y_1 = y_2 = p_1 = p_2 = 0 \qquad\qquad \text{on } \Sigma\,,\\
&\quad y_1(x,0;\overline{u}) = y_{1,0}(x;\overline{u})\,, \qquad y_2(x,0;\overline{u}) = y_{2,0}(x;\overline{u}) \qquad \text{in } \mathbb{R}^\infty\,,\\
&\quad \frac{\partial y_1(x,0;\overline{u})}{\partial t} = y_{1,1}(x;\overline{u})\,, \qquad \frac{\partial y_2(x,0;\overline{u})}{\partial t} = y_{2,1}(x;\overline{u}) \qquad \text{in } \mathbb{R}^\infty\,,\\
&\quad p_1(x,t;\overline{u}) = 0\,, \qquad p_2(x,t;\overline{u}) = 0 \qquad \text{in } \mathbb{R}^\infty\,,\\
&\quad \frac{\partial p_1(x,t;\overline{u})}{\partial t} = 0\,, \qquad \frac{\partial p_2(x,t;\overline{u})}{\partial t} = 0 \qquad \text{in } \mathbb{R}^\infty\,.
\end{aligned}
\tag{16}
$$

Further

$$
\begin{aligned}
u_1 &= -N_1^{-1}p_1\,,\\
u_2 &= \frac{\partial^2}{\partial t^2}y_2 + \Big(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha} + q(x,t)\Big)y_2 + y_1 + y_2 - f_2\,.
\end{aligned}
\tag{17}
$$

4. Optimal control for $n \times n$ coupled Neumann problem of Petrowsky type

([5], [6], [7], [12])

Since the bilinear form (1) is coercive on $\left(W_0^\ell(\mathbb{R}^\infty)\right)^n$ and since $\left(W_0^\ell(\mathbb{R}^\infty)\right)^n \subseteq \left(W^\ell(\mathbb{R}^\infty)\right)^n$, it is also coercive on $\left(W^\ell(\mathbb{R}^\infty)\right)^n$, i.e.

$$\pi\left(t;\overline{\phi},\overline{\phi}\right) \geq \lambda \|\overline{\phi}\|_{(W^\ell(\mathbb{R}^\infty))^n}, \qquad \lambda > 0. \tag{18}$$

For $\overline{\psi},\overline{\phi} \in \left(W^\ell(\mathbb{R}^\infty)\right)^n$ the function $t \mapsto \pi\left(t;\overline{\psi},\overline{\phi}\right)$ is continuously differentiable on $]0,T[$ and

$$\pi\left(t;\overline{\psi},\overline{\phi}\right) = \pi\left(t;\overline{\phi},\overline{\psi}\right). \tag{19}$$

We formulate the following lemma which defines the $n \times n$ mixed Neumann problem and enables us to state our control problem.

LEMMA 1. *Assuming that* (18) *and* (19) *hold, for given* $\overline{f} = \overline{f}(x,t) \in \left(L^2(Q)\right)^n$, $y_{i,1} \in L^2(\mathbb{R}^\infty)$, $y_{i,0} \in W^\ell(\mathbb{R}^\infty)$ *there exists a unique element* y_i *satisfying:*

$$y_i,\ \frac{\partial y_i}{\partial x_k},\ \frac{\partial y_i}{\partial t} \in L^2(Q),$$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y_i + S(t) y_i = f_i & \textit{in } Q, \\ \dfrac{\partial}{\partial \nu_A} y_i = 0 & \textit{on } \Sigma, \\ y_i(x,0) = y_{i,0}(x) & \textit{in } \mathbb{R}^\infty, \\ \dfrac{\partial}{\partial t} y_i(x,0) = y_{i,1}(x) & \textit{in } \mathbb{R}^\infty, \end{cases} \tag{20}$$

where $\frac{\partial}{\partial \nu_A}$ *is the so-called co-normal derivative with respect to* $A(t)$.

Proof. From (20), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y_i(t) + S(t) y_i = f_i \qquad \text{in } Q \quad \text{for all } 1 \leq i \leq n.$$

This equation is equivalent to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} y_i(t), \phi_i\right)_{L^2(\mathbb{R}^\infty)} + \left(S(t) y_i, \phi_i\right)_{L^2(\mathbb{R}^\infty)} = \left(f_i(t), \phi_i\right)_{L^2(\mathbb{R}^\infty)}. \tag{21}$$

Let us define

$$\left(f_i(t), \phi_i\right)_{L^2(\mathbb{R}^\infty)} = \int_{\mathbb{R}^\infty} f_i(x,t) \phi_i(x)\, \mathrm{d}\varrho(x), \qquad \phi_i \in W^\ell(\mathbb{R}^\infty).$$

In this way we obtain an element of $L^2(0,T;W^\ell(\mathbb{R}^\infty))$.

So equation (21) can be written as:

$$\int_0^T \Big(\frac{\mathrm{d}^2}{\mathrm{d}t^2}y_i(t),\phi_i(x)\Big)_{L^2(\mathbb{R}^\infty)}\,\mathrm{d}t+\int_0^T \big(S(t)y_i(t),\phi_i(x)\big)_{L^2(\mathbb{R}^\infty)}\,\mathrm{d}t$$

$$=\int_0^T \big(f_i(t),\phi_i(x)\big)_{L^2(\mathbb{R}^\infty)}\,\mathrm{d}t\,,$$

$$\int_0^T \Big(\frac{\mathrm{d}^2}{\mathrm{d}t^2}y_i(t),\phi_i(x)\Big)_{L^2(\mathbb{R}^\infty)}\,\mathrm{d}t+\int_0^T\Big\{\int_{\mathbb{R}^\infty}\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}\big((-1)^{|\alpha|}\,\mathrm{D}_k^{2\alpha}\,y_i(t)\big)\phi_i(x)\,\mathrm{d}\rho$$

$$+\int_{\mathbb{R}^\infty} q(x,t)y_i(t)\phi_i(x)\,\mathrm{d}\varrho\Big\}\,\mathrm{d}t+\int_0^T\int_{\mathbb{R}^\infty}\sum_{j=1}^{n}a_{ij}y_j(t)\phi_i(x)\,\mathrm{d}\varrho\,\mathrm{d}t$$

$$=\int_0^T \big(f_i(t),\phi_i(x)\big)_{L^2(\mathbb{R}^\infty)}\,\mathrm{d}t\,.$$

Applying Green's formula to the left side, we get

$$\int_Q \frac{\mathrm{d}^2}{\mathrm{d}t^2}y_i(t)\phi_i(x)\,\mathrm{d}\varrho\,\mathrm{d}t+\int_Q\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}\mathrm{D}_k^{\alpha}\,y_i(t)\,\mathrm{D}_k^{\alpha}\,\phi_i(x)\,\mathrm{d}\varrho\,\mathrm{d}t$$

$$-\int_\Sigma\frac{\partial y_i(t)}{\partial\nu_A}\phi_i(x)\,\mathrm{d}\Sigma+\int_Q\sum_{j=1}^{n}a_{ij}y_j(t)\phi_i(x)\,\mathrm{d}\varrho\,\mathrm{d}t+\int_Q q(x,t)y_i(t)\phi_i(x)\,d\varrho\,\mathrm{d}t$$

$$=\int_Q f_i(x,t)\phi_i(x)\,\mathrm{d}\varrho\,\mathrm{d}t\,.$$

Then we have

$$\int_\Sigma\frac{\partial y_i(t)}{\partial\nu_A}\phi_i(x)\,\mathrm{d}\Sigma=0\,,$$

i.e.,

$$\frac{\partial y_i}{\partial\nu_A}=0\qquad\text{on }\Sigma\quad\text{for all }i=1,2,\dots,n\,.$$

□

Note. In this case the operator $\frac{\partial^2}{\partial t^2}+S(t)$ maps $\big(L^2(0,T;W^\ell(\mathbb{R}^\infty))\big)^n$ onto $\big(L^2(0,T;W^{-\ell}(\mathbb{R}^\infty))\big)^n$.

Formulation of our control problem. The space $\left(L^2(Q)\right)^n$ being the space of controls, for a control $\overline{u} = (u_i)_{i=1}^n \in \left(L^2(Q)\right)^n$ the state $\overline{y}(\overline{u}) = \left(y_i(\overline{u})\right)_{i=1}^n$ of system is given by the solution of:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y_i(\overline{u}) + S(t) y_i(\overline{u}) = f_i + u_i & \text{in } Q\,, \\ \dfrac{\partial}{\partial \nu_A} y_i(\overline{u}) = 0 & \text{on } \Sigma\,, \\ y_i(x,0;\overline{u}) = y_{i,0}(x)\,, & \text{in } \mathbb{R}^\infty\,, \\ \dfrac{\partial y_i(x,0;\overline{u})}{\partial t} = y_{i,1}(x) & \text{in } \mathbb{R}^\infty\,, \end{cases} \tag{22}$$

$$y_i(\overline{u}),\ \frac{\partial y_i(\overline{u})}{\partial t} \in L^2(Q) \qquad \text{for all} \quad 1 \le i \le n\,.$$

The cost function is given by (5), hence by using the general theory of J. L. Lions ([15; Chap. I, Theorem 1.4, Chap. II, Theorem 1.4]), there exists a unique optimal control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}$ such that

$$J(\overline{u}) = \inf J(\overline{v}) \qquad \text{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}\,. \tag{23}$$

Moreover, we have the following theorem that gives the characterization of the optimal control.

THEOREM 2. *Problem* (23) *admits a unique solution* $\overline{u} = (u_i)_{i=1}^n \in \left(L^2(Q)\right)^n$. *Moreover, it is characterized by* (22) *with the following system*

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} p_i(\overline{u}) + S^*(t) p_i(\overline{u}) = y_i(\overline{u}) - z_{i,d} && \textit{in } Q\,, \\ &\frac{\partial}{\partial \nu_A} p_i(\overline{u}) = 0 && \textit{on } \Sigma\,, \\ &p_i(x,T;\overline{u}) = 0\,, \quad \frac{\partial p_i(x,T;\overline{u})}{\partial t} = 0 && \textit{in } \mathbb{R}^\infty\,, \\ &p_i(\overline{u}),\ \frac{\partial p_i(x,T;\overline{u})}{\partial t} \in L^2(Q)\,, \end{aligned} \tag{24}$$

$$\sum_{i=1}^n \int_Q \left(p_i(\overline{u}) + N_i u_i\right)(v_i - u_i)\, \mathrm{d}\varrho\, \mathrm{d}t \ge 0 \qquad \textit{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}\,,$$

where $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}$, $p_i(\overline{u})$ *is the adjoint state for* $1 \le i \le n$.

Proof. Since $J(\overline{u})$ is differentiable and $\mathcal{U}_{\mathrm{ad}}$ is bounded, the optimal control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}$ is characterized by

$$\sum_{i=1}^n J_i'(\overline{u})(v_i - u_i) \ge 0\,, \qquad \text{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}\,,$$

which is equivalent to

$$\sum_{i=1}^{n}\big(y_i(\overline{u}) - z_{i,d},\ y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(Q)} + \sum_{i=1}^{n}(N_i u_i, v_i - u_i)_{L^2(Q)} \geq 0\,. \tag{25}$$

Now, we are forming the scalar product of the first equation in (24) with $y_i(\overline{v}) - y_i(\overline{u})$ and integrating between 0, T; we have

$$\int_0^T \big(y_i(\overline{u}) - z_{i,d},\ y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(\mathbb{R}^\infty)}\,dt$$

$$= \int_0^T \Big(\frac{\partial^2}{\partial t^2}p_i(\overline{u}) + S^*(t)p_i(\overline{u}),\ y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^\infty)}\,dt$$

$$= \int_0^T \Big(\frac{\partial^2}{\partial t^2}p_i(\overline{u}) + A(t)p_i(\overline{u}) + \sum_{j=1}^{n} a_{ji}p_j(u),\ y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^\infty)}\,dt$$

$$= \int_0^T \Big(\frac{\partial^2}{\partial t^2}p_i(\overline{u}), y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^\infty)}\,dt + \int_0^T \big(A(t)p_i(\overline{u}), y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(\mathbb{R}^\infty)}\,dt$$

$$+ \int_0^T \Big(\sum_{j=1}^{n} a_{ji}p_j(\overline{u}), y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^\infty)}\,dt\,.$$

Applying Green's formula to the right hand side, we get

$$\int_0^T \Big(p_i(\overline{u}), \frac{\partial^2}{\partial t^2}\big(y_i(\overline{v}) - y_i(\overline{u})\big)\Big)_{L^2(\mathbb{R}^\infty)}\,dt + \int_0^T \big(p_i(\overline{u}), A(t)\big(y_i(\overline{v}) - y_i(\overline{u})\big)\big)_{L^2(\mathbb{R}^\infty)}\,dt$$

$$+ \int_0^T \Big(p_i(\overline{u}), \sum_{j=1}^{n} a_{ij}\big(y_j(\overline{v}) - y_j(\overline{u})\big)\Big)_{L^2(\mathbb{R}^\infty)}\,dt$$

$$- \int_0^T \Big(\frac{\partial}{\partial \nu_A}p_i(\overline{u}), \big(y_i(\overline{v}) - y_i(\overline{u})\big)\Big)_{L^2(\Sigma)}\,dt + \int_0^T \Big(p_i(\overline{u}), \frac{\partial}{\partial \nu_A}\big(y_i(\overline{v}) - y_i(\overline{u})\big)\Big)_{L^2(\Sigma)}\,dt\,.$$

Using the conditions in (22) and (24), we have

$$\int_0^T \big(y_i(\overline{u}) - z_{i,d},\ y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(\mathbb{R}^\infty)}\,dt = \int_0^T \big(p_i(\overline{u}), v_i - u_i\big)_{L^2(\mathbb{R}^\infty)}\,dt\,.$$

Therefore (25) becomes:

$$\sum_{i=1}^{n} \int_{Q} \big(p_i(\overline{u}) + N_i u_i\big)(v_i - u_i)\, \mathrm{d}\varrho\, \mathrm{d}t \geq 0 \qquad \text{for all} \quad \overline{v} = (v_i)_{i=1}^{n} \in \mathcal{U}_{\mathrm{ad}}\,,$$

which completes the proof. □

EXAMPLE 5. Let $n = 2$; then the optimality system is given as in Example 1 with the following boundary conditions ([6], [7])

$$\frac{\partial}{\partial \nu_A} y_1(\overline{u}) = \frac{\partial}{\partial \nu_A} y_2(\overline{u}) = \frac{\partial p_1(\overline{u})}{\partial \nu_A} = \frac{\partial p_2(\overline{u})}{\partial \nu_A} = 0 \qquad \text{on} \quad \Sigma\,.$$

Also we can construct analogues of previous examples.

REAL EXAMPLE. (see [27]) We shall use the following notation:

$Q = Q_T = \Omega \times]0,T[$, Ω is an open subset of $\mathbb{R}^N$,
$\Sigma = \Sigma_T = \Gamma \times]0,T[$,
$\Gamma =$ boundary of Ω,
$\Sigma =$ lateral boundary of Q.

Let us take the second order evolution problem. We consider a function $a(x,t)$ such that

$$a(x,t) \in C^1\big(]0,T]; L^\infty(\Omega)\big)\,.$$

We introduce

$$V = \big\{\overline{\psi}: \ \overline{\psi}, \Delta\overline{\psi} \in \big(L^2(\Omega)\big)^n\big\}\,, \qquad H = \big(L^2(\Omega)\big)^n\,,$$

and

$$a(t; \overline{\phi}, \overline{\psi}) = \sum_{i=1}^{n} \int_{\Omega} a(x,t) \Delta\phi_i \Delta\psi_i \ \mathrm{d}x \qquad \text{for all} \quad \overline{\phi}, \overline{\psi} \in V\,,$$

where

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \qquad \text{is the Laplacian operator.}$$

We now consider a system whose state is given by

$$\begin{gathered}
\frac{\partial^2 y_i(\overline{v})}{\partial t^2} + \Delta\big(a \Delta y_i(\overline{v})\big) = f_i + v_i\,, \qquad \overline{v} \in \big(L^2(Q)\big)^n\,, \\
\Delta y_i(\overline{v}) = 0\,, \quad \frac{\partial \Delta y_i(\overline{v})}{\partial n} = 0 \qquad \text{on } \Sigma\,, \\
\Delta y_i(x,0;\overline{v}) = y_{i,0}(x)\,, \quad \frac{\partial y_i}{\partial t}(x,0;\overline{v}) = y_{1,i}(x)\,, \qquad x \in \Omega\,, \\
y_{i,0} \in V\,, \quad y_{i,1} \in L^2(\Omega)\,.
\end{gathered} \tag{26}$$

If we assume that the cost function is given by

$$J(\overline{v}) = \sum_{i=1}^{n} \int_{Q} \big(y_i(x,t;\overline{v}) - z_{i,d}\big)^2 \,\mathrm{d}x\,\mathrm{d}t + (N_i v_i, v_i)_{\mathcal{U}}\,, \qquad \mathcal{U} = L^2(Q)\,,$$

which means that the observation $\overline{y}(\overline{v})$ belongs to $(L^2(Q))^n$, then from [15; Chap. IV, Theorem 2.1], we see that the optimal control is determined by

$$\begin{aligned} \frac{\partial^2 p_i(\overline{u})}{\partial t^2} + \Delta\big(a\Delta p_i(\overline{u})\big) &= y_i(\overline{u}) - z_{i,d} && \text{in } Q\,,\\ \Delta p_i(\overline{u}) = 0\,, \qquad \frac{\partial \Delta p_i(\overline{u})}{\partial n} &= 0 && \text{on } \Sigma\,,\\ \Delta p_i(x,T;\overline{u}) = 0\,, \quad \frac{\partial p_i}{\partial t}(x,T;\overline{u}) &= 0 && \text{on } \Omega\,, \end{aligned}$$

to which we add (26) (where $\overline{v} = \overline{u}$) and

$$\sum_{i=1}^{n} \int_{Q} \big(p_i(\overline{u}) + N_i u_i\big)(v_i - u_i)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \qquad \text{for all} \quad \overline{v} \in \mathcal{U}_{\mathrm{ad}}\,.$$

Comments. It is evident that by modifying

- the boundary conditions,
- the nature of the control (distributed, boundary),
- the nature of the observation,
- the initial differential system

an infinity of variations on the above problem are possible to study.

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