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PRIME IDEALS AND POLARS IN $DR\ell$-MONOIDS AND PSEUDO BL-ALGEBRAS

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ABSTRACT. Dually residuated lattice ordered monoids ($DR\ell$-monoids) are a generalization of lattice ordered groups embracing also algebras closely related to logic like pseudo MV-algebras (GMV-algebras) or pseudo BL-algebras. In the paper, the concepts of a prime ideal and a polar in a $DR\ell$-monoid are established and their basic properties are shown. Since pseudo BL-algebras are in fact the duals of certain bounded $DR\ell$-monoids, the analogous properties of pseudo BL-algebras are immediately obtained.

1. Preliminaries

An algebra $\mathcal{A} = \langle A; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown\rangle$ of type $(2,0,2,2,2,2)$ is a *dually residuated lattice ordered monoid*, simply a *$DR\ell$-monoid*, if

(1) $\langle A; +, 0, \vee, \wedge\rangle$ is an ℓ-monoid, i.e., $\langle A; +, 0\rangle$ is a monoid, $\langle A; \vee, \wedge\rangle$ is a lattice and $+$ distributes over $\vee$ and $\wedge$;

(2) for any $x, y \in A$, $x \rightharpoonup y$ is the least $s \in A$ such that $s + y \geqslant x$, and $x \leftharpoondown y$ is the least $t \in A$ such that $y + t \geqslant x$;

(3) $\mathcal{A}$ fulfils the identities

$$\big((x \rightharpoonup y) \vee 0\big) + y \leqslant x \vee y\,, \qquad y + \big((x \leftharpoondown y) \vee 0\big) \leqslant x \vee y\,.$$

In the original definition the validity of the inequalities $x \rightharpoonup x \geqslant 0$ and $x \leftharpoondown x \geqslant 0$ was also desired, but analogously as in [11], one can prove that we always have $x \rightharpoonup x = x \leftharpoondown x = 0$. Notice next that the condition (2) is equivalent to the following system of identities (see [18]):

$$\begin{aligned} (x \rightharpoonup y) + y \geqslant x\,, &\qquad y + (x \leftharpoondown y) \geqslant x\,,\\ x \rightharpoonup y \leqslant (x \vee z) \rightharpoonup y\,, &\qquad x \leftharpoondown y \leqslant (x \vee z) \leftharpoondown y\,,\\ (x + y) \rightharpoonup y \leqslant x\,, &\qquad (y + x) \leftharpoondown y \leqslant x\,.\end{aligned}$$

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The class of (noncommutative) $DR\ell$-monoids includes lattice ordered groups and also algebras being in close connection to fuzzy logic. For instance, pseudo BL-algebras and pseudo MV-algebras can be viewed as special cases of bounded $DR\ell$-monoids (see [13] and [18]). Recall that commutative $DR\ell$-monoids (called $DR\ell$-semigroups) were introduced by K. L. N. Swamy in [20] to be a common extension of commutative ℓ-groups and Brouwerian algebras. For basic properties of noncommutative $DR\ell$-monoids, see [10] or [12].

Let us recall some concepts from [12]. For any x of a $DR\ell$-monoid $\mathcal{A}$, the *absolute value* of x is defined by $|x| = x \vee (0 \rightharpoonup x) = x \vee (0 \leftharpoondown x)$, and $x^+ = x \vee 0$ is the *positive part* of x. For each $X \subseteq A$, X^+ will mean the set of all positive elements of X.

A subset I of a $DR\ell$-monoid $\mathcal{A}$ is said to be an *ideal of* $\mathcal{A}$ if it satisfies the following conditions:

(I1) $0 \in I$;
(I2) if $x, y \in I$, then $x + y \in I$;
(I3) for all $x \in I$, $y \in A$, $|y| \leqslant |x|$ implies $y \in I$.

This definition is a natural generalization of the concept of an ideal in commutative $DR\ell$-semigroups. Of course, if $\mathcal{A}$ is a pseudo MV-algebra, then ideals in both algebras coincide. In the case that $\mathcal{A}$ is an ℓ-group, the ideals are just the convex ℓ-subgroups.

Under the ordering by set inclusion, the set of all ideals becomes an algebraic Brouwerian lattice $\mathcal{I}(\mathcal{A})$ in which the relative pseudocomplement of I with respect to J is given by

$$I * J = \{a \in A : \ (\forall x \in I)(|a| \wedge |x| \in J)\}.$$

Further, for any ideal I one can assign the binary relations $\Theta_1(I)$ and $\Theta_2(I)$, respectively, defined by

$$\langle x, y\rangle \in \Theta_1(I) \iff ((x \rightharpoonup y) \vee (y \rightharpoonup x) \in I),$$

and

$$\langle x, y\rangle \in \Theta_2(I) \iff ((x \leftharpoondown y) \vee (y \leftharpoondown x) \in I),$$

respectively. In general, both $\Theta_1(I)$ and $\Theta_2(I)$ are congruence relations on the distributive lattice $\ell(\mathcal{A}) = \langle A; \vee, \wedge\rangle$, and in the quotient lattices $\ell(\mathcal{A})/\Theta_1(I)$ and $\ell(\mathcal{A})/\Theta_2(I)$ we have

$$[x]\Theta_1(I) \leqslant [y]\Theta_1(I) \iff (x \rightharpoonup y)^+ \in I, \tag{1.1}$$

and

$$[x]\Theta_2(I) \leqslant [y]\Theta_2(I) \iff (x \leftharpoondown y)^+ \in I, \tag{1.2}$$

respectively.

An ideal I of a $DR\ell$-monoid $\mathcal{A}$ is *normal* if either of the following equivalent conditions is satisfied:

(1) $(\forall x, y \in A)\big((x \rightharpoonup y)^+ \in I \iff (x \leftharpoondown y)^+ \in I\big)$;

(2) $(\forall x \in A)(x + I^+ = I^+ + x)$.

The normal ideals of any $DR\ell$-monoid correspond one-to-one to its congruence relations. Indeed, if I is a normal ideal, then $\Theta_1(I)$ and $\Theta_2(I)$ coincide and this binary relation $\Theta(I)$ is a congruence on $\mathcal{A}$ such that $[0]\Theta(I) = I$. Conversely, for any congruence relation Θ on $\mathcal{A}$, $[0]\Theta$ is a normal ideal, and in addition, $\Theta([0]\Theta) = \Theta$. Thus the mapping $I \mapsto \Theta(I)$ gives the isomorphism between the lattice $\mathcal{N}(\mathcal{A})$ of normal ideals and $\mathrm{Con}(\mathcal{A})$.

2. Prime ideals

An ideal I of a $DR\ell$-monoid $\mathcal{A}$ is said to be *prime* if it is a finitely meet-irreducible element in the ideal lattice $\mathcal{I}(\mathcal{A})$, that is,

$$(\forall J, K \in \mathcal{I}(\mathcal{A}))\big(I = J \cap K \implies (I = J \quad \text{or} \quad I = K)\big).$$

THEOREM 2.1. *For any ideal I of $\mathcal{A}$, the following conditions are equivalent:*

(1) *I is a prime ideal;*

(2) $(\forall J, K \in \mathcal{I}(\mathcal{A}))\big(J \cap K \subseteq I \implies (J \subseteq I \text{ or } K \subseteq I)\big)$;

(3) $(\forall x, y \in A)\big(|x| \wedge |y| \in I \implies (x \in I \text{ or } y \in I)\big)$;

(4) $(\forall x, y \in A)\big(0 \leqslant x \wedge y \in I \implies (x \in I \text{ or } y \in I)\big)$.

Proof.

(1) $\implies$ (2): If $J \cap K \subseteq I$, then $I = I \vee (J \cap K) = (I \vee J) \cap (I \vee K)$, as $\mathcal{I}(\mathcal{A})$ is a distributive lattice. Hence $I = I \vee J$ or $I = I \vee K$ and, consequently, $J \subseteq I$ or $K \subseteq I$.

(2) $\implies$ (3): Obviously, $|x| \wedge |y| \in I$ implies $I\big(|x| \wedge |y|\big) = I(x) \cap I(y) \subseteq I$, whence $I(x) \subseteq I$ or $I(y) \subseteq I$ and therefore $x \in I$ or $y \in I$.

(3) $\implies$ (4): It follows from $0 \leqslant x \wedge y = |x| \wedge |y|$.

(4) $\implies$ (1): Let $I = J \cap K$. If neither $I = J$ nor $I = K$, then there are $x \in J \setminus I$ and $y \in K \setminus I$. Moreover, we can assume $x, y \geqslant 0$. Then $0 \leqslant x \wedge y \in J \cap K = I$, whence $x \in I$ or $y \in I$, which is a contradiction. □

THEOREM 2.2. *For any proper ideal I of a $DR\ell$-monoid $\mathcal{A}$ and for each $a \notin I$, there exists a prime ideal P of $\mathcal{A}$ such that $I \subseteq P$ and $a \notin P$.*

Proof. By Zorn's Lemma there is an ideal P which is maximal with respect to the required property. Let $P = J \cap K$ for some $J, K \in \mathcal{I}(\mathcal{A}) \setminus \{P\}$. Then obviously $a \in J$ and $a \in K$, whence $a \in J \cap K = P$, which is a contradiction. This shows that P is prime. □

COROLLARY 2.3. *Let $\mathcal{A}$ be a DRℓ-monoid.*

(1) *Every ideal I of $\mathcal{A}$ is the intersection of all primes containing I.*
(2) *Every maximal ideal of $\mathcal{A}$ is prime.*

PROPOSITION 2.4. *Let $\{P_i\}_{i\in I}$ be a chain of prime ideals of a DRℓ-monoid $\mathcal{A}$. Then $P = \bigcap_{i\in I} P_i$ is a prime ideal of $\mathcal{A}$. Consequently, every prime ideal contains a minimal prime ideal.*

Proof. Suppose $0 \leqslant x \wedge y \in P$, and $x \notin P$, i.e., $x \notin P_j$ for some $j \in I$. Then $x \notin P_k$ for all $k \in I$ with $P_k \subseteq P_j$. Hence $y \in P_k$ for any such k, and so $y \in P_i$ for all $i \in I$, proving $y \in P$. □

PROPOSITION 2.5. *Let $\mathcal{B}$ be a DRℓ-submonoid of a DRℓ-monoid $\mathcal{A}$. Then any prime ideal Q of $\mathcal{B}$ is obtained in the form $Q = B \cap P$ for some prime ideal P of $\mathcal{A}$.*

Proof. If P is a prime ideal of $\mathcal{A}$, then certainly $Q = B \cap P$ is a prime ideal of $\mathcal{B}$.

Conversely, suppose that Q is a prime ideal of $\mathcal{B}$ and let $I(Q)$ be the ideal of $\mathcal{A}$ generated by Q, i.e.,

$$\begin{aligned} I(Q) &= \{a \in A:\ (\exists\, w_1,\dots,w_n \in Q)(|a| \leqslant |w_1| + \dots + |w_n|)\} \\ &= \{a \in A:\ (\exists\, w \in Q^+)(|a| \leqslant w)\} \end{aligned}$$

since $Q \in \mathcal{I}(\mathcal{B})$.

If $x \in B \setminus Q$, then $x \notin I(Q)$, because $x \in I(Q)$ if and only if $|x| \leqslant w$ for some $w \in Q^+$, which would mean $x \in Q$. Thus $I(Q) \cap (B \setminus Q) = \emptyset$. Therefore by Zorn's Lemma, there exists $P \in \mathcal{I}(\mathcal{A})$ that is maximal with the property $I(Q) \subseteq P$ and $P \cap (B \setminus Q) = \emptyset$. It is easy to see that $Q = B \cap P$ as $Q \subseteq B \cap P$ and $(P \cap B) \setminus Q = P \cap (B \setminus Q) = \emptyset$.

It remains to prove that P is prime. Suppose $P = J \cap K$ for some $J, K \in \mathcal{I}(\mathcal{A}) \setminus \{P\}$. Obviously, $J \cap (B \setminus Q) \neq \emptyset$ and $K \cap (B \setminus Q) \neq \emptyset$, i.e., there are $0 \leqslant a, b \in A$ such that $a \in J \cap (B \setminus Q)$ and $b \in K \cap (B \setminus Q)$. Hence $a \wedge b \in J \cap K \cap (B \setminus Q) = P \cap (B \setminus Q) = \emptyset$, which is a contradiction. We conclude that P is a prime ideal of $\mathcal{A}$ with the property $B \cap P = Q$ as required. □

Remark 2.6. If I is an ideal of $\mathcal{A}$, then any ideal J of I is also an ideal in $\mathcal{A}$ since I is a convex *DRℓ*-submonoid of $\mathcal{A}$, and hence we can consider the relative pseudocomplement $I * J$.

PROPOSITION 2.7. *Let I be an ideal of a DRℓ-monoid $\mathcal{A}$. Then the mappings $\varphi\colon P \mapsto I \cap P$ and $\psi\colon Q \mapsto I * Q$ are mutually inverse order preserving bijections between the prime ideals of $\mathcal{A}$ not exceeding I and the proper prime ideals of I.*

Proof. Obviously, if P is a prime ideal of $\mathcal{A}$ not containing I, then $\varphi(P) = I \cap P$ is a proper prime ideal of I.

Let now Q be a proper prime ideal in I; then

$$\psi(Q) = \bigvee\{H \in \mathcal{I}(\mathcal{A}) : \ I \cap H \subseteq Q\}$$

as $\psi(Q) = I * Q$. In order to show that $\psi(Q)$ is prime in $\mathcal{A}$, assume $\psi(Q) = J \cap K$ for some $J, K \in \mathcal{I}(\mathcal{A}) \setminus \{\psi(Q)\}$. Since the lattice $\mathcal{I}(\mathcal{A})$ is algebraic and distributive, it is clear that

$$\begin{aligned} I \cap \psi(Q) &= I \cap \bigvee\{H \in \mathcal{I}(\mathcal{A}) : \ I \cap H \subseteq Q\} \\ &= \bigvee\{I \cap H : \ H \in \mathcal{I}(\mathcal{A}) \ \& \ I \cap H \subseteq Q\} \\ &\subseteq Q \subseteq I \cap \psi(Q) \end{aligned}$$

since $Q \subseteq I$ and $Q \subseteq \psi(Q)$. Thus $I \cap \psi(Q) = Q$.

Further, $Q = I \cap \psi(Q) \subset I \cap J$ and similarly $Q \subset I \cap K$. (If, for instance, $J \cap I = Q$, then $J \subseteq \psi(Q)$.) Therefore we can find $a, b \in A$ such that $a \in (J \cap I) \setminus Q = J \cap (I \setminus Q)$ and $b \in (K \cap I) \setminus Q = K \cap (I \setminus Q)$. Hence $|a| \wedge |b| \in J \cap K \cap (I \setminus Q) = \psi(Q) \cap (I \setminus Q) = (\psi(Q) \cap I) \setminus Q = \emptyset$. Thus $\psi(Q)$ is a prime ideal of $\mathcal{A}$ and $I \nsubseteq \psi(Q)$.

Moreover, we have seen that $\varphi(\psi(Q)) = I \cap \psi(Q) = Q$. It remains to prove that conversely $\psi(\varphi(P)) = P$ for each prime ideal P of $\mathcal{A}$ such that $I \nsubseteq P$.

Obviously, $\psi(\varphi(P)) = \psi(I \cap P) = \bigvee\{H \in \mathcal{I}(\mathcal{A}) : \ I \cap H \subseteq I \cap P\}$ and hence $P \subseteq \psi(\varphi(P))$. Conversely, if $a \in \psi(\varphi(P))$ and $b \in I \setminus P$, then

$$\begin{aligned} |a| \wedge |b| \in I \cap \psi(\varphi(P)) &= I \cap \bigvee\{H \in \mathcal{I}(\mathcal{A}) : \ I \cap H \subseteq I \cap P\} \\ &= \bigvee\{I \cap H : \ H \in \mathcal{I}(\mathcal{A}) \ \& \ I \cap H \subseteq I \cap P\} \\ &\subseteq I \cap P \subseteq P. \end{aligned}$$

Since $b \notin P$ and P is prime, it follows $a \in P$ proving $\psi(\varphi(P)) = P$. □

Let I be an ideal of a $DR\ell$-monoid $\mathcal{A}$. In view of (1.1), $\ell(\mathcal{A})/\Theta_1(I)$ is totally ordered if and only if

$$(\forall x, y \in A)\big((x \rightharpoonup y)^+ \in I \ \text{ or } \ (y \rightharpoonup x)^+ \in I\big). \tag{2.1}$$

Similarly, $\ell(\mathcal{A})/\Theta_2(I)$ is totally ordered if and only if

$$(\forall x, y \in A)\big((x \leftharpoondown y)^+ \in I \ \text{ or } \ (y \leftharpoondown x)^+ \in I\big), \tag{2.2}$$

by (1.2).

PROPOSITION 2.8. *Let I be an ideal of $\mathcal{A}$. If $\ell(\mathcal{A})/\Theta_1(I)$ or $\ell(\mathcal{A})/\Theta_2(I)$ is totally ordered, then the set of all ideals exceeding I is totally ordered under set inclusion.*

Proof. Suppose that $I \subseteq J, K$ for some ideals such that $J \nsubseteq K$ and $K \nsubseteq J$. Then there exist $0 \leqslant a, b \in A$ such that $a \in J \setminus K$ and $b \in K \setminus J$. By (2.1), $(a \rightharpoonup b)^+ \in I$ or $(b \rightharpoonup a)^+ \in I$, say $(a \rightharpoonup b)^+ \in I$. Hence $0 \leqslant a \leqslant a \vee b = (a \rightharpoonup b)^+ + b \in K$, so that $a \in K$, which is a contradiction. □

PROPOSITION 2.9. *For any ideal I of $\mathcal{A}$, if $\{J \in \mathcal{I}(\mathcal{A}) : I \subseteq J\}$ is linearly ordered under inclusion, then I is prime.*

Proof. If $I = J \cap K$, then $I = J$ or $I = K$, because $J \subseteq K$ or $K \subseteq J$. □

COROLLARY 2.10. *If $\ell(\mathcal{A})/\Theta_1(I)$ or $\ell(\mathcal{A})/\Theta_2(I)$ is linearly ordered, then I is a prime ideal.*

PROPOSITION 2.11. *If $\mathcal{A}$ is a totally ordered DRℓ-monoid, then $\mathcal{I}(\mathcal{A})$ is totally ordered under set inclusion. Consequently, any ideal in $\mathcal{A}$ is prime.*

Proof. If $I \nsubseteq J$ and $J \nsubseteq I$, then there are $a, b \geqslant 0$ such that $a \in I \setminus J$ and $b \in J \setminus I$. Moreover, $a \leqslant b$ or $b \leqslant a$, say $a \leqslant b$. Therefore we have $0 \leqslant a \leqslant b \in J$, which yields $a \in J$, which is a contradiction.

Since $\mathcal{I}(\mathcal{A})$ is totally ordered under inclusion, it is easily seen that $\{J \in \mathcal{I}(\mathcal{A}) : I \subseteq J\}$ is totally ordered for each ideal I, and so I is prime. □

In the sequel, we shall characterize the prime ideals of $DR\ell$-monoids satisfying the identities

$$\begin{aligned} (x \rightharpoonup y)^+ \wedge (y \rightharpoonup x)^+ &= 0, \\ (x \leftharpoondown y)^+ \wedge (y \leftharpoondown x)^+ &= 0. \end{aligned} \tag{$*$}$$

For instance, $(*)$ is satisfied by any ℓ-group, by any linearly ordered $DR\ell$-monoid and also by any bounded $DR\ell$-monoid which is induced by a GMV-algebra (pseudo MV-algebra) or by a pseudo BL-algebra, respectively (see [18] and [13]). Note that the above identities are equivalent to the inequalities

$$\begin{aligned} (x \rightharpoonup y) \wedge (y \rightharpoonup x) &\leqslant 0, \\ (x \leftharpoondown y) \wedge (y \leftharpoondown x) &\leqslant 0. \end{aligned}$$

Any completely meet-irreducible ideal of $\mathcal{A}$ is called *regular*. Using a well-known property of algebraic lattices, we can easily see that any ideal is the intersection of a family of regular ideals.

THEOREM 2.12. *Let $\mathcal{A}$ be a $DR\ell$-monoid with $(*)$. For any ideal I, the following conditions are equivalent:*

(1) *I is prime.*
(2) $(\forall J, K \in \mathcal{I}(\mathcal{A}))(J \cap K \subseteq I \implies (J \subseteq I$ *or* $K \subseteq I))$.
(3) $(\forall x, y \in A)(|x| \wedge |y| \in I \implies (x \in I$ *or* $y \in I))$.
(4) $(\forall x, y \in A)(0 \leqslant x \wedge y \in I \implies (x \in I$ *or* $y \in I))$.
(5) $(\forall x, y \in A)(x \wedge y \in I \implies (x \in I$ *or* $y \in I))$.
(6) $(\forall x, y \in A)(x \wedge y = 0 \implies (x \in I$ *or* $y \in I))$.
(7) $(\forall x, y \in A)((x \rightharpoonup y)^+ \in I$ *or* $(y \rightharpoonup x)^+ \in I)$.
(8) $(\forall x, y \in A)((x \leftharpoondown y)^+ \in I$ *or* $(y \leftharpoondown x)^+ \in I)$.
(9) *$\ell(\mathcal{A})/\Theta_1(I)$ is linearly ordered.*
(10) *$\ell(\mathcal{A})/\Theta_2(I)$ is linearly ordered.*
(11) *$\{J \in \mathcal{I}(\mathcal{A}) : I \subseteq J\}$ is linearly ordered by set inclusion.*
(12) *I is equal to the intersection of a chain of regular ideals.*

Proof. The conditions (1)–(4) are equivalent by Theorem 2.1.

(4) $\implies$ (5): Since

$$(x \rightharpoonup (x \wedge y)) \wedge (y \rightharpoonup (x \wedge y)) = ((x \rightharpoonup x) \vee (x \rightharpoonup y)) \wedge ((y \rightharpoonup x) \vee (y \rightharpoonup y)) = (0 \vee (x \rightharpoonup y)) \wedge ((y \rightharpoonup x) \vee 0) = 0,$$

by $(*)$, it follows that $x \rightharpoonup (x \wedge y) \in I$ or $y \rightharpoonup (x \wedge y) \in I$, say $x \rightharpoonup (x \wedge y) \in I$. Then

$$(x \rightharpoonup (x \wedge y)) + (x \wedge y) = x \vee (x \wedge y) = x$$

belongs to I whenever $x \wedge y \in I$, proving (5).

(5) $\implies$ (6): Obvious.

(6) $\implies$ (7) and (6) $\implies$ (8): It follows from $(*)$.

(7) $\implies$ (9): By (2.1).

(8) $\implies$ (10): By (2.2).

(9) $\implies$ (11) and (10) $\implies$ (11): By Proposition 2.8.

(11) $\implies$ (12): By the previous remarks, I is equal to the intersection of some set of regular ideals which is a chain by (11).

(12) $\implies$ (1): It is a consequence of Proposition 2.4 since any regular ideal is prime. □

A poset $\langle P; \leqslant\rangle$ is a *root-system* if for each $p \in P$, $\{x \in P : x \geqslant p\}$ is totally ordered.

For instance, the prime ℓ-subgroups of any ℓ-group form a root-system (see e.g. [1]). Theorem 2.12 provides the following generalization of this fact:

COROLLARY 2.13. *If $\mathcal{A}$ fulfils $(*)$, then any ideal including a prime ideal is prime and the set of all prime ideals (and hence also the set of all regular ideals) is a root-system.*

3. Polars and minimal prime ideals

Let $\mathcal{A}$ be a *DRℓ*-monoid and $X \subseteq A$. The set

$$X^\perp = \{a \in A : (\forall x \in X)(|a| \wedge |x| = 0)\}$$

is called the *polar of* X. For any $a \in A$, we write briefly $a^\perp$ instead of $\{a\}^\perp$. A subset X of A is a *polar in* $\mathcal{A}$ if $X = Y^\perp$ for some $Y \subseteq A$.

PROPOSITION 3.1. *Let $\mathcal{A}$ be a DRℓ-monoid and $X, Y \subseteq A$. Then*

(1) $X \subseteq X^{\perp\perp}$;
(2) $X \subseteq Y \implies Y^\perp \subseteq X^\perp$;
(3) $X^\perp = X^{\perp\perp\perp}$;
(4) $X^\perp = I(X)^\perp$.

Proof. The properties (1)–(3) are straightforward. To prove (4), it is sufficient to check $X^\perp \subseteq I(X)^\perp$ since the other inclusion follows from (2). Let $x \in X^\perp$ and $y \in I(X)$, that is, $|y| \leqslant |x_1| + \cdots + |x_n|$ for some $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$. Then

$$\begin{aligned} |x| \wedge |y| &\leqslant |x| \wedge (|x_1| + \cdots + |x_n|) \\ &\leqslant (|x| \wedge |x_1|) + \cdots + (|x| \wedge |x_n|) \\ &= 0 + \cdots + 0 = 0 . \end{aligned}$$

Thus $x \in I(X)^\perp$ showing $X^\perp \subseteq I(X)^\perp$. □

COROLLARY 3.2. *A subset X of a DRℓ-monoid $\mathcal{A}$ is a polar in $\mathcal{A}$ if and only if $X = X^{\perp\perp}$.*

Proof. If $X = Y^\perp$ for some $Y \subseteq A$, then $X^{\perp\perp} = Y^{\perp\perp\perp} = Y^\perp = X$. □

Recall that a prime ideal I is called *minimal* if there exists no prime ideal J properly contained in I.

PROPOSITION 3.3. *Let $\mathcal{A}$ be a DRℓ-monoid and $X \subseteq A$. Then $X^\perp$ is equal to the intersection of all minimal prime ideals M of $\mathcal{A}$ such that $X \nsubseteq M$.*

Proof. Let M be a minimal prime ideal with $X \nsubseteq M$. Let $a \in X^\perp$ and $b \in X \setminus M$. Obviously, $|a| \wedge |b| = 0$, whence $a \in M$, because M is prime and $b \notin M$. Thus $X^\perp \subseteq M$.

If $a \notin X^\perp$, then $|a| \wedge |b| > 0$ for some $b \in X$. By Theorem 2.2 there exists a prime ideal not containing $|a| \wedge |b|$, and since any prime ideal includes a minimal prime ideal, there is a minimal prime ideal M such that $|a| \wedge |b| \notin M$. Therefore neither $|a|$ nor $|b|$ belongs to M, and hence $X \nsubseteq M$ and $a \notin M$. □

COROLLARY 3.4. *Let $\mathcal{A}$ be a DRℓ-monoid, $X \subseteq A$.*

(1) *$X^{\perp}$ is the intersection of all prime ideals not containing X.*
(2) *$X^{\perp}$ is an ideal of $\mathcal{A}$.*
(3) *If $X^{\perp}$ is a proper prime ideal, then it is minimal.*

Proof. By the first part of the proof of the previous proposition, $X^{\perp}$ is included in the intersection of all prime ideals not containing X. However, it is a subset of the intersection of all such minimal prime ideals which is equal to $X^{\perp}$.

The statements (2) and (3) are evident, since any polar is the intersection of minimal prime ideals. □

As proved in [12], $\mathcal{I}(\mathcal{A})$ is a Brouwerian lattice in which the pseudocomplement of an ideal I is given as follows:

$$I^{*} = \{a \in A : \ (\forall x \in I)(|a| \wedge |x| = 0)\}.$$

Hence it can be easily seen that $I^{*} = I^{\perp}$ whenever I is an ideal. Conversely, any polar P in $\mathcal{A}$ is the pseudocomplement of some ideal of $\mathcal{A}$; in fact, $P = (P^{\perp})^{*}$. Summarizing, the polars in $\mathcal{A}$ are precisely the pseudocomplements in the lattice $\mathcal{I}(\mathcal{A})$. Therefore, by the Glivenko-Frink Theorem (e.g. [7]), it follows that:

THEOREM 3.5. *The set $\mathcal{P}(\mathcal{A})$ of all polars in a DRℓ-monoid $\mathcal{A}$, ordered by set inclusion, is a complete Boolean algebra.*

By [17; Theorem 8], if I is a prime ideal of a representable commutative $DR\ell$-semigroup, then either $I^{\perp\perp} = A$ or I is minimal prime.

PROPOSITION 3.6. *If I is a proper prime ideal of a DRℓ-monoid $\mathcal{A}$, then either $I^{\perp\perp} = A$ or $I^{\perp\perp} = I$. In the latter case, I is minimal prime.*

Proof. Suppose $I^{\perp\perp} \neq A$, that is, $I^{\perp} \neq \{0\}$. Let $x \in I^{\perp\perp} \setminus I$; then $|x| \wedge |y| = 0$ for any $y \in I^{\perp}$. As $x \notin I$, we have $y \in I$. However, $I \cap I^{\perp} = \{0\}$ yields $y = 0$. Consequently $I^{\perp} = \{0\}$, which is a contradiction. Thus $I^{\perp\perp} = I$. The rest follows immediately from Corollary 3.4(3). □

PROPOSITION 3.7. *Let I be a linearly ordered ideal of a DRℓ-monoid $\mathcal{A}$. Then $I^{\perp}$ is a prime ideal.*

Proof. Suppose $x, y \notin I^{\perp}$, that is, $|x| \wedge |a| > 0$ for some $a \in I$ and $|y| \wedge |b| > 0$ for some $b \in I$. Since I is linearly ordered, it follows $0 < |x| \wedge |a| \wedge |y| \wedge |b| = |x| \wedge |y| \wedge |a| \wedge |b|$. But $|a| \wedge |b| \in I$, and so $|x| \wedge |y| \notin I^{\perp}$, proving that $I^{\perp}$ is a prime ideal of $\mathcal{A}$. □

In conclusion, we examine minimal prime ideals and polars of $DR\ell$-monoid satisfying the identities $(*)$.

LEMMA 3.8. *An ideal I of a DRℓ-monoid $\mathcal{A}$ with $(*)$ is totally ordered if and only if $x \wedge y = 0$ entails $x = 0$ or $y = 0$ for all $x, y \in I$.*

Proof. The part " $\Longrightarrow$ " is obvious. Conversely, $(*)$ provides $(x \rightharpoonup y)^+ = 0$ or $(y \rightharpoonup x)^+ = 0$, whence $x \leqslant y$ or $x \geqslant y$ for any $x, y \in I$. □

THEOREM 3.9. *Let $I \neq \{0\}$ be an ideal of a DRℓ-monoid $\mathcal{A}$ satisfying $(*)$. Then the following conditions are equivalent:*

(1) *I is linearly ordered.*
(2) *$I^\perp$ is a prime ideal.*
(3) *$I^\perp$ is a minimal prime ideal.*
(4) *$I^\perp$ is a maximal polar.*
(5) *$I^{\perp\perp}$ is a minimal polar.*
(6) *$I^{\perp\perp}$ is linearly ordered.*

Proof. We have already proved (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) (see Corollary 3.4 and Proposition 3.7).

(3) $\Longrightarrow$ (4): Assume $I^\perp \subseteq P$ for some $P \in \mathcal{P}(\mathcal{A})$, $P \neq A$. Since $I^\perp$ is prime, so is P. Further, considering $P = P^{\perp\perp}$, P is minimal prime, and therefore $P = I^\perp$.

(4) $\Longrightarrow$ (5): It holds that $P \subseteq I^{\perp\perp}$ if and only if $P^\perp \supseteq I^\perp$ for each $P \in \mathcal{P}(\mathcal{A})$, $P \neq \{0\}$. However, $P^\perp \supseteq I^\perp$ yields $P^\perp = I^\perp$, and thus $P = I^{\perp\perp}$.

(5) $\Longrightarrow$ (6): Let $x, y \in I^{\perp\perp}$ and $x \wedge y = 0$. If $x \neq 0$, then $x^\perp \neq A$ and hence $x^{\perp\perp} \neq \{0\}$. Further, $x \in I^{\perp\perp}$ implies $x^\perp \supseteq I^\perp$ whence $x^{\perp\perp} \subseteq I^{\perp\perp}$. Since $I^{\perp\perp}$ is minimal, we have $x^{\perp\perp} = I^{\perp\perp}$. Thus $y \in x^{\perp\perp} = I^{\perp\perp}$. But also $y \in x^\perp$. Hence $y \in x^\perp \cap x^{\perp\perp} = \{0\}$, showing $y = 0$. Thus, by the preceding lemma, $I^{\perp\perp}$ is a chain.

(6) $\Longrightarrow$ (1): It is clear as $I \subseteq I^{\perp\perp}$. □

Remark 3.10. We remark that one also defines (using the property in Corollary 3.2 and the condition (2) of Theorem 2.1) polars and prime elements in algebraic, distributive lattices (see [14], [19]). The conditions (2) through (5) of Theorem 3.9 are equivalent in lower-bounded Brouwerian lattices by [14; Lemma 2.1] and in certain algebraic, distributive lattices by [19; Proposition 5.2] (if $\mathcal{A}$ fulfils $(*)$, then $\mathcal{I}(\mathcal{A})$ is such a lattice). In addition, (2)–(5) are equivalent to the condition that I (respectively, $I^{\perp\perp}$) is a basic element in the ideal lattice $\mathcal{I}(\mathcal{A})$, that is, $J = \{0\}$ or $K = \{0\}$ whenever $J \cap K = \{0\}$ for $J, K \subseteq I$. It follows from Lemma 3.8 that an ideal I of a *DRℓ*-monoid $\mathcal{A}$ satisfying $(*)$ is basic in $\mathcal{I}(\mathcal{A})$ exactly if I is linearly ordered. Therefore, under the premises of Theorem 3.9, the statements (1)–(6) are equivalent.

Let us notice further that Proposition 3.3 is also in fact only a particular case of a similar statement from [19]. Indeed, by [19; Lemma 2.4], in an algebraic distributive lattice L, the pseudocomplement a^* of $a \in L$ equals to the intersection of all minimal prime elements of L not exceeding a.

COROLLARY 3.11. *Given a $DR\ell$-monoid $\mathcal{A}$ satisfying $(*)$, the following statements are equivalent:*

(1) *$\mathcal{A}$ is linearly ordered.*
(2) *$\mathcal{I}(\mathcal{A})$ is linearly ordered.*
(3) *Any ideal is prime.*
(4) *$\{0\}$ is a prime ideal.*

Proof. By Proposition 2.11 we have (1) $\implies$ (2) $\implies$ (3) and the implication (3) $\implies$ (4) is evident. Finally, (4) $\implies$ (1) follows by Theorem 3.9 as $\mathcal{A}$ is totally ordered if and only if $\{0\} = A^{\perp}$ is prime. □

4. Pseudo BL-algebras

In this section, we apply the previous results to pseudo BL-algebras that constitute a noncommutative abstraction of BL-algebras (see [8], [2] and [3]) and that can be regarded as a special case of $DR\ell$-monoids (see [13]). Recall the notion of a pseudo BL-algebra and some further concepts.

An algebra $\langle A; \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1\rangle$ of the type $(2,2,2,2,2,0,0)$ is called a *pseudo BL-algebra* if and only if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded lattice, $\langle A; \odot, 1\rangle$ is a monoid and the following conditions are satisfied, for all $x, y, z \in A$:

(1) $x \odot y \leqslant z \iff x \leqslant y \to z \iff y \leqslant x \rightsquigarrow z$,
(2) $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$,
(3) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

A subset F of a pseudo BL-algebra $\mathcal{A}$ is said to be a *filter* of $\mathcal{A}$ if

(i) $1 \in F$,
(ii) $x \odot y \in F$ for all $x, y \in F$,
(iii) F contains together with any x also all y such that $y \geqslant x$.

A filter is *prime* (*regular*) if it is finitely (completely) meet-irreducible in the lattice of filters $\mathcal{F}(\mathcal{A})$.

The *copolar* of $X \subseteq A$ is the set

$$X^{\perp} = \{a \in A : \ (\forall x \in X)(a \vee x = 1)\}.$$

A subset X of $\mathcal{A}$ is called a *copolar* in $\mathcal{A}$ if $X = Y^{\perp}$ for some $Y \subseteq A$.

By [13], pseudo BL-algebras are categorically equivalent to bounded $DR\ell$-monoids satisfying the identities $(*)$. In fact, if $\langle A; \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1\rangle$ is a pseudo BL-algebra and if we put $1_d = 0$, $0_d = 1$, $x \vee_d y = x \wedge y$, $x \wedge_d y = x \vee y$, $x \rightharpoonup y = y \rightarrow x$ and $x \leftharpoondown y = y \rightsquigarrow x$ for any $x, y \in A$, then $\langle A; \odot, 0_d, \vee_d, \wedge_d, \rightharpoonup, \leftharpoondown\rangle$ is a bounded $DR\ell$-monoid whose greatest element is 1_d. Of course, this $DR\ell$-monoid fulfils $(*)$.

Conversely, if $\langle A; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown\rangle$ is bounded $DR\ell$-monoid with the greatest element 1 satisfying $(*)$ and if we define the operations $\vee_d$, $\wedge_d$, $\rightarrow$, $\rightsquigarrow$ as above, then $\langle A; \vee_d, \wedge_d, +, \rightarrow, \rightsquigarrow, 0_d, 1_d\rangle$ becomes a pseudo BL-algebra.

Considering the duality between the mentioned classes of algebras, the following consequences of Theorem 2.12, Proposition 3.3, Proposition 3.6, and Theorem 3.9 are obtained.

Just as in the case of $DR\ell$-monoids, for any filter F of a pseudo BL-algebra $\mathcal{A}$ we define

$$\Theta_1(F) = \{\langle x, y\rangle \in A^2 : \ (x \rightarrow y) \wedge (y \rightarrow x) \in F\}$$

and

$$\Theta_2(F) = \{\langle x, y\rangle \in A^2 : \ (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \in F\}.$$

THEOREM 4.1. *If $\mathcal{A}$ is a pseudo BL-algebra, then for any filter F of $\mathcal{A}$, the following conditions are equivalent:*

(1) *F is prime.*
(2) $(\forall G, H \in \mathcal{F}(\mathcal{A}))(G \cap H \subseteq F \implies (G \subseteq F \text{ or } H \subseteq F))$.
(3) $(\forall x, y \in A)(x \vee y \in F \implies (x \in F \text{ or } y \in F))$.
(4) $(\forall x, y \in A)(x \vee y = 1 \implies (x \in F \text{ or } y \in F))$.
(5) $(\forall x, y \in A)(x \rightarrow y \in F \text{ or } y \rightarrow x \in F)$.
(6) $(\forall x, y \in A)(x \rightsquigarrow y \in F \text{ or } y \rightsquigarrow x \in F)$.
(7) *$\ell(\mathcal{A})/\Theta_1(F)$ is linearly ordered.*
(8) *$\ell(\mathcal{A})/\Theta_2(F)$ is linearly ordered.*
(9) *The set of all filters including F is linearly ordered by set inclusion.*
(10) *F is the intersection of some chain of regular filters.*

Remark 4.2. In [2], the concept of a prime filter was established be means of the condition (3) and it was shown that (3), (5), (6), (7) and (8) are equivalent.

COROLLARY 4.3. *The set of all prime filters and so also the set of all regular filters of any pseudo BL-algebra is a root-system.*

PROPOSITION 4.4. *Let $\mathcal{A}$ be a pseudo BL-algebra and $X \subseteq A$. Then $X^\perp$ is equal to the intersection of all minimal prime filters M of $\mathcal{A}$ such that $X \not\subseteq M$. Consequently, any copolar $X^\perp$ is a filter, and moreover, $X^\perp$ is a minimal prime filter whenever $X^\perp$ is proper prime.*

PROPOSITION 4.5. *If F is a proper prime filter of a pseudo BL-algebra $\mathcal{A}$, then either $F^{\perp\perp} = A$ or $F^{\perp\perp} = F$ and F is minimal prime.*

THEOREM 4.6. *Let $F \neq \{1\}$ be a filter of a pseudo BL-algebra $\mathcal{A}$. Then the following conditions are equivalent:*

(1) *F is linearly ordered.*
(2) *$F^{\perp}$ is a prime filter.*
(3) *$F^{\perp}$ is a minimal prime filter.*
(4) *$F^{\perp}$ is a maximal copolar.*
(5) *$F^{\perp\perp}$ is a minimal copolar.*
(6) *$F^{\perp\perp}$ is linearly ordered.*

Remark 4.7. The equivalence of the statements (1) and (2) was also proved in [2].

COROLLARY 4.8. *The following statements are equivalent in any pseudo BL-algebra $\mathcal{A}$:*

(1) *$\mathcal{A}$ is linearly ordered.*
(2) *$\mathcal{F}(\mathcal{A})$ is linearly ordered.*
(3) *Every filter of $\mathcal{A}$ is prime.*
(4) *$\{1\}$ is a prime filter.*

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