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Veronika Chrastinová; Václav Tryhuk

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ON THE MAYER PROBLEM I. GENERAL PRINCIPLES

VERONIKA CHRASTINOVÁ — VÁCLAV TRYHUK

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ABSTRACT. Given an underdetermined system of ordinary differential equations (i.e., the Monge system, the optimal control system) expressed by Pfaffian equations $\omega \equiv 0$ ($\omega \in \Omega$) where Ω is a module of differential 1-forms on a space $\mathbf{M}$, we determine submodules $\breve{\Omega} \subset \Omega$ which satisfy the congruence $\mathrm{d}\breve{\Omega} \simeq 0$ (mod $\breve{\Omega}$, $\Omega \wedge \Omega$) along a certain special subspace $\mathbf{E} \subset \mathbf{M}$ of the total space $\mathbf{M}$. Then $\breve{\Omega}$ and $\mathbf{E}$ may be interpreted in terms of Poincaré-Cartan forms and Euler-Lagrange equations for various Mayer problems that belong to the given Monge system. They yield a universal canonical formalism including the Weierstrass-Hilbert extremality theory. The occurrences of uncertain coefficients (Lagrange multipliers, adjoint variables) are suppressed and occasionally eliminated (e.g., for all Mayer problems arising from a Lagrange problem), the degenerate cases are not excluded.

Introduction

Given an underdetermined system of ordinary differential equations together with certain boundary conditions for the solutions $P(t)$, $0 \leq t \leq 1$, the *Lagrange problem* is concerned with such solutions that a curvilinear integral $\int P^*\alpha$ (where α is a 1-form) attains the extremal value and the *Mayer problem* deals with such solutions that a function $g(P(1))$ of the end point attains the extremal value. Everyone can be translated into the other, they may be regarded for equivalent. However, the Mayer problem corresponding to a Lagrange problem is of a very special kind (with certain additional structure) and conversely, rather strange Lagrange problems (with extraordinary extremals) are related

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to a given Mayer problem. Therefore the Mayer problem manifests itself as the more fundamental one and deserves a separate investigation.

A huge literature is devoted to this topic mainly within the framework of the optimal regulation theory, see [5] and numerous references therein. In this approach, the original underdetermined system is rewritten as the *optimal control system*

$$\mathrm{d}w^i/\,\mathrm{d}x \equiv f^i(x, w^1, \dots, w^m, u^1, \dots, u^n)\,, \qquad i = 1, \dots, m\,,$$

which involves two quite distinct families of variables: the common *phase variables* $x, w^1, \dots, w^m$ and the new *control parameters* $(u^1, \dots, u^n) \in \mathbf{K}$, where $\mathbf{K} \subset \mathbb{R}^n$ is a given compact subset. The resulting stationarity conditions are partly of the classical nature $\mathrm{d}\psi^i/\,\mathrm{d}x = \sum \psi^j \partial f^j/\partial w^i$ (where ψ^i are additional *adjoint variables*) and partly of rather unusual kind

$$\sum \psi^i f^i(x, w^1, \dots, w^m, u^1, \dots, u^n) = \max_v \sum \psi^i f^i(x, w^1, \dots, w^m, v^1, \dots, v^n)$$

$(v = (v^1, \dots, v^n) \in \mathbf{K}$, the *maximum principle*). It follows that some technical measures (e.g. the choice of the control parameters and of $\mathbf{K}$) strongly affect the final result which cannot be therefore easily transformed into classical terms if the maximum is attained at the boundary of $\mathbf{K}$. Especially the classical degenerate variational problems are misinterpreted from this point of view.

We propose a new approach based on a forgotten Cartan's observation [1]. The original differential equations are replaced by a Pfaffian system $\omega \equiv 0$ $(\omega \in \Omega)$ where Ω is a module of differential 1-forms. There exists a submodule $\check{\Omega} \subset \Omega$ along a certain subspace $\mathbf{E} \subset \mathbf{M}$ of the total space that is "infinitesimally flat" along all solutions $P(t) \in \mathbf{E}$. These $\mathbf{E}$, $\check{\Omega}$, and $P(t)$ represent the common Euler-Lagrange system, Poincaré-Cartan forms, and extremals, respectively. In succinct terms, the final achievements are as follows. For a given underdetermined system of differential equations we can characterize the family of extremals together with the Poincaré-Cartan forms. They provide the "absolute structure" independent of any particular choice of coordinates in the widest possible sense and determine the hierarchy of all reasonable variational problems (in particular, of all Mayer problems) relevant to the given underdetermined system together with the adapted Hamilton-Jacobi equations and geodesic fields needful for the extremality investigations. The algorithm is of the algebraical nature: we do not need any subtle existence of admissible variations subjected to various boundary conditions.

A certain similarity to the "royal road" by Carathéodory is worth mentioning. Recall that it comfortably provides the sufficient global extremality conditions for the nondegenerate classical variational problems and rests upon the existence of (a little artificial and hypothetical) calibration function, see thorough comments in [4]. Our approach gives analogous result but, unlike Carathéodory's method, it applies to all constrained variational problems with quite general boundary conditions as well, and leads to a universal canonical formalism which is (in principle) free of various uncertain coefficients (e.g., of the Lagrange multipliers) occurring in all common expositions. The advantage of our approach was already examined for the relatively easier case of the Lagrange variational problems which admit a simplified treatment, see the series of papers [3].

This Part I of the article is devoted to the general theory. Several easy examples of true Mayer problems will be shortly discussed in the next Part II. Regrettably, rather unusual tools must be employed. In order to explain the topic as clearly as possible, we therefore restrict ourselves to smooth category and the distinction between local and global properties is not thoroughly pointed out. Our aim is to describe certain explicit algorithm which can be applied to a large spectrum of particular problems. The use of a non-formal style enables us to explain the method on a reasonably modest space. The paper should be regarded for self-contained and easily accessible for anybody who is acquainted with the elements of differential forms and Pfaffian systems, we may also refer to monograph [2] involving thorough exposition of all fundamental but a little unorthodox concepts that will occur.

Ordinary differential equations

1. Notation.

We introduce (in general infinite-dimensional) *underlying spaces* $\mathbf{M}$ equipped with (local) coordinates $h^1, h^2, \ldots$ and the *structural ring* $\mathcal{F} = \mathcal{F}(\mathbf{M})$ (the abbreviation omitting $\mathbf{M}$ if possible) of all smooth real-valued functions $f = f(h^1, \ldots, h^m)$ on $\mathbf{M}$, where $m = m(f)$. *Admissible mappings* $\mathbf{n}\colon \mathbf{N} \to \mathbf{M}$ between such spaces will satisfy $\mathbf{n}^*\mathcal{F}(\mathbf{M}) \subset \mathcal{F}(\mathbf{N})$, in particular

$$\mathbf{n}^* h^i \equiv \bar{h}^i(k^1, \ldots, k^{m(i)}) \in \mathcal{F}(\mathbf{N}), \tag{1}$$

where $k^1, k^2, \ldots$ are coordinates on $\mathbf{N}$. If this mapping $\mathbf{n}$ is injective and a part of the family $\bar{h}^1, \bar{h}^2, \ldots$ can be taken for alternative coordinates on $\mathbf{N}$, then $\mathbf{N}$ is called a *subspace of* $\mathbf{M}$ *with the inclusion* $\mathbf{n}\colon \mathbf{N} \subset \mathbf{M}$ and the *restriction* $\mathbf{n}^*$ *to* $\mathbf{N}$ of functions and differential forms (see below). Following the common convention, the points $P = \mathbf{n}P$ and functions $f = \mathbf{n}^* f$ are occasionally identified to simplify the notation.

We also mention the $\mathcal{F}(\mathbf{M})$-module $\Phi = \Phi(\mathbf{M})$ of *differential forms* $\varphi = \sum f^i \,\mathrm{d}g^i$ ($f^i, g^i \in \mathcal{F}(\mathbf{M})$, finite sum) and the dual $\mathcal{F}(\mathbf{M})$-module $\mathcal{T} = \mathcal{T}(\mathbf{M})$ of *vector fields* $Z = \sum^{\infty} z^i \partial/\partial h^i$ ($z^i \in \mathcal{F}(\mathbf{M})$, infinite sum) with the *duality pairing* $\varphi(Z) = Z \rfloor \varphi = \sum f^i Z g^i \in \mathbb{R}$, in particular $\mathrm{d}f(Z) = Zf = \sum z^i \partial f/\partial h^i \in \mathbb{R}$ is the *directional derivative*. The *exterior differential* d and the *Lie derivative* $\mathcal{L}_Z$

satisfying

$$\mathcal{L}_Z f = Zf\,, \qquad \mathcal{L}_Z X = [Z, X]\,, \qquad \mathcal{L}_Z \varphi = Z\rfloor \mathrm{d}\varphi + \mathrm{d}(Z\rfloor\varphi) \tag{2}$$

will frequently occur. Recalling the common abbreviation of restrictions $\mathbf{n}^*\varphi = \sum \mathbf{n}^* f^i \,\mathrm{d}\mathbf{n}^* g^i = \sum f^i \,\mathrm{d}g^i \in \Phi(\mathbf{N})$, we believe that it will not cause much confusion.

Various $\mathcal{F}(\mathbf{M})$-submodules $\Theta \subset \Phi(\mathbf{M})$ may be introduced by means of $\mathcal{F}(\mathbf{M})$*-generators* $\vartheta^1, \vartheta^2, \ldots \in \Theta$, we denote $\Theta = \{\vartheta^1, \vartheta^2, \ldots\}$. Generators provide *finite expansions* of elements of Θ. Generators linearly independent *along* $\mathbf{M}$ (i.e., at every point of $\mathbf{M}$) constitute a *basis* of Θ and produce the *unique* finite expansions. Unless otherwise mentioned, the existence of bases of various submodules of $\Phi(\mathbf{M})$ will be (as a rule tacitly) postulated in order to exclude various "singular" objects. Unlike this, one can observe that the vector fields are determined by *infinite expansions* in terms of a *weak bases*.

Finite-dimensional spaces (subspaces) will also occur but they do not need any comments now.

2. Diffieties.

Diffieties represent a substitute for infinitely prolonged underdetermined systems of differential equations in the *absolute sense*, i.e., relieved of all additional structures. For the convenience of reader, two equivalent definitions will be stated. Recall once more that we restrict ourselves only to the general theory at this place and refer to the next part of the article, where the true sense of the following concepts is investigated.

Let $\mathcal{H} \subset \mathcal{T} = \mathcal{T}(\mathbf{M})$ be a one-dimensional submodule, hence $\mathcal{H}$ consists of all multiples fZ ($f \in \mathcal{F} = \mathcal{F}(\mathbf{M})$) of a (nonvanishing) vector field $Z \in \mathcal{T}$. The *orthogonal submodule* $\mathcal{H}^\perp \subset \Phi = \Phi(\mathbf{M})$ (consisting of all $\varphi \in \Phi$ such that $\varphi(\mathcal{H}) = 0$) is generated by all forms $Zf\,\mathrm{d}g - Zg\,\mathrm{d}f$ ($f, g \in \mathcal{F}$), however, better generators are available (at least locally). Let $Zx \neq 0$ for a certain $x \in \mathcal{F}$ (the *independent variable*). Denoting $X = \frac{1}{Zx}Z$, $Xx = 1$ and we obtain the forms $\omega^i = \mathrm{d}h^i - Xh^i\,\mathrm{d}x \in \mathcal{H}^\perp$ ($i = 1, 2, \ldots$) which provide "more economical" generators of $\mathcal{H}^\perp$. Clearly $\mathcal{L}_Z\mathcal{H}^\perp = Z\rfloor\,\mathrm{d}\mathcal{H}^\perp \subset \mathcal{H}^\perp$, in particular the forms of the kind

$$\omega^i_k = \mathcal{L}^k_X \omega^i = \mathrm{d}X^k h^i - X^{k+1}h^i\,\mathrm{d}x \in \mathcal{H}^\perp \qquad (i = 1, 2, \ldots,\ \ k = 0, 1, \ldots) \tag{3}$$

will frequently occur.

DEFINITION 1. We speak of a *diffiety* $\mathcal{H}^\perp$ (*with the slope* $\mathcal{H}$) if a basis of $\mathcal{H}^\perp$ can be chosen from the family ω^i_k ($i = 1, \ldots, m$, $k = 0, 1, \ldots$) where m is fixed and large enough.

In slightly different terms $\mathcal{H}^\perp$ is called a *diffiety* if there exist functions $f^1, \ldots, f^m \in \mathcal{F}$ such that a part of the family $\mathrm{d}(Z^k f^i)$ ($Z \in \mathcal{H}$, $i = 1, \ldots, m$, $k = 0, 1, \ldots$) can be used for a basis of the module Φ.

Let alternatively $\Omega \subset \Phi$ be a one-codimensional submodule, hence the *orthogonal submodule* $\Omega^\perp \subset \mathcal{T}$ (of all $Z \in \mathcal{T}$ with $\Omega(Z) = 0$) is one-dimensional. Clearly $\mathcal{L}_Z\Omega = Z \rfloor \mathrm{d}\Omega \subset \Omega$ for all $Z \in \Omega^\perp$.

DEFINITION 2. Let $Z \in \Omega^\perp$ be nonvanishing. If there exists a good filtration $\Omega_*\colon \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \bigcup \Omega_l$ of the module Ω satisfying (by definition) the conditions

$$\mathcal{L}_Z\Omega_l \subset \Omega_{l+1} \quad (\text{for all } l), \qquad \Omega_l + \mathcal{L}_Z\Omega_l \equiv \Omega_{l+1} \quad (\text{for } l \text{ large enough}) \quad (4)$$

with finite-dimensional submodules $\Omega_l \subset \Omega$, then Ω is called a *diffiety* (*with the slope* $\Omega^\perp$).

By identifying $\Omega = \mathcal{H}^\perp$ and $\mathcal{H} = \Omega^\perp$, both definitions can be proved for equivalent. The former one has a nice geometrical interpretation: a diffiety $\mathcal{H}^\perp$ can be viewed as a vector field in $\mathbb{R}^\infty$. We shall nevertheless prefer the latter dual conception: a diffiety Ω represents the system of Pfaffian equations $\omega \equiv 0$ ($\omega \in \Omega$). In more detail, we have the finite system $\omega_0^i \equiv 0$ ($i = 1, \ldots, m$) equivalent to $\mathrm{d}h^i/\mathrm{d}x \equiv Xh^i$ ($i = 1, \ldots, m$) and the remaining equations $\omega_k^i \equiv 0$ ($k > 0$) provide a mere infinite prolongation. The module Ω can be organized by various appropriately adapted filtrations satisfying (4), this is the main advantage of our approach.

3. The finite-dimensional case.

The above concepts simplify if $\mathbf{M}$ is of a finite dimension m. In terms of coordinates $h^1, \ldots, h^m$, where $h^1 = x$ is regarded for the independent variable, the forms $\mathrm{d}h^j - f^j\,\mathrm{d}x$ ($j = 2, \ldots, m$, $f^j \equiv f^j(h^1, \ldots, h^m) = Xh^j$) constitute a basis of Ω and the simple one-term filtration with $\Omega_l \equiv \Omega$ is possible. The diffiety Ω represents the determined system of ordinary differential equations $\mathrm{d}h^j/\mathrm{d}x \equiv f^j$. One can even achieve $f^j \equiv 0$ by appropriate change of coordinates $h^2, \ldots, h^m$. So we have the system $\mathrm{d}h^j/\mathrm{d}x \equiv 0$ and the diffiety $\Omega = \{\mathrm{d}h^2, \ldots, \mathrm{d}h^m\}$ has a basis consisting of total differentials of functions $h^2, \ldots, h^m$, the *first integrals.* (If $m = 1$, then obviously $\Omega = 0$ is the zero module, this is a trivial case.)

4. Subdiffieties.

We introduce subdiffieties by the following definition.

DEFINITION 3. Let $\mathbf{n}\colon \mathbf{N} \subset \mathbf{M}$ be an inclusion, $\Omega \subset \Phi(\mathbf{M})$ and $\Theta \subset \Phi(\mathbf{N})$ be diffieties. Then Θ is called a *subdiffiety of* Ω if $\Theta = \mathbf{n}^*\Omega$ is the restriction of Ω to $\mathbf{N}$.

One can observe that Θ is a subdiffiety if and only if $\Omega^\perp$ *is tangent to* $\mathbf{N}$ in the sense that $\mathbf{n}^*f = 0$ implies $\mathbf{n}^*Zf = 0$ for all $f \in \mathcal{F}(\mathbf{M})$ and $Z \in \Omega^\perp$.

Hint.
If Θ is a subdiffiety, then

$$\mathbf{n}^*(Zf\,\mathrm{d}g - Zg\,\mathrm{d}f) = \mathbf{n}^*Zf\cdot \mathrm{d}\mathbf{n}^*g - \mathbf{n}^*Zg\cdot \mathrm{d}\mathbf{n}^*f \in \Theta \qquad \text{for all} \quad f,g\in\mathcal{F}(\mathbf{M})\,.$$

Assuming $\mathbf{n}^*f = 0$, we obtain $\mathbf{n}^*Zf\cdot\mathrm{d}\mathbf{n}^*g \in \Theta$ for all g. This implies $\mathbf{n}^*Zf = 0$. The converse can be verified as well.

It follows that subdiffieties of Ω could be constructed as follows: we choose a family $f^1, f^2, \ldots \in \mathcal{F}(\mathbf{M})$ and if the subset $\mathbf{N}\subset\mathbf{M}$ of all points satisfying $Z^l f^i \equiv 0$ $(i = 1,2,\ldots,\ l = 0,1,\ldots)$ is a nonempty subspace, then the relevant restriction $\Theta = \mathbf{n}^*\Omega$ may be regarded for a candidate of a subdiffiety. One can employ the restriction $\Theta_*\colon \Theta_0 \subset \Theta_1 \subset \cdots \subset \Theta = \bigcup \Theta_l$, $\Theta_l \equiv \mathbf{n}^*\Omega_l$, of the filtration (4) as the good filtration of the subdiffiety Θ. In all particular examples to follow, this procedure will not cause any difficulties.

5. Solutions.
In a somewhat formal terms, special one-dimensional subspaces $\bar{\mathbf{n}}\colon \bar{\mathbf{N}}\subset\mathbf{M}$ will occur such that the spaces $\bar{\mathbf{N}}$ are certain open subsets of the real line $\mathbb{R}$ parametrized by the variable t and containing the universal interval $0\le t\le 1$. Then we speak of *curves* $P(t)\in\mathbf{M}$ $(0\le t\le 1)$ where P is the restriction of $\bar{\mathbf{n}}$ to the interval: $P(t) = \bar{\mathbf{n}}(t)$ if $0\le t\le 1$. Analogously (and less formally) we may speak of *curves* $P(t;\lambda^1,\ldots,\lambda^n)\in\mathbf{M}$ $(0\le t\le 1)$ *depending on parameters.*

Definition 4. Let $\Omega\subset\Phi(\mathbf{M})$ be a diffiety. The curve $P(t)\in\mathbf{M}$ $(0\le t\le 1)$ is called a *solution of* Ω if $P^*\Omega\equiv 0$. If $P(\,\cdot\,;\lambda^1,\ldots,\lambda^n)^*\Omega\equiv 0$, then we have a *solution depending on parameters.*

If $\Theta = \mathbf{n}^*\Omega\subset\Phi(\mathbf{N})$ is a subdiffiety of Ω, then every solution $P(t)\in\mathbf{N}$ of Θ clearly yields the solution $\mathbf{n}P(t)\in\mathbf{M}$. If moreover $\mathbf{N}\subset\mathbf{M}$ is a finite-dimensional space, we obtain a solution depending on $\dim\mathbf{N}-1$ parameters.

The crucial algorithm

6. Our task.
Let $\Omega\subset\Phi(\mathbf{M})$ be a diffiety. We are interested in nontrivial finite-dimensional submodules $\tilde\Omega\subset\Omega$ such that the congruence $\mathrm{d}\tilde\Omega\simeq 0$ (mod $\tilde\Omega$, $\Omega\wedge\Omega$) is valid *along* a subset $\tilde{\mathbf{M}}\subset\mathbf{M}$ (i.e., at every point of $\tilde{\mathbf{M}}$). In more explicit terms, the requirement can be expressed by either of the equivalent inclusions

$$\mathrm{d}\tilde\Omega\subset\tilde\Omega\wedge\Phi + \Omega\wedge\Omega + m(\tilde{\mathbf{M}})\Phi\wedge\Phi\,, \qquad \mathcal{L}_Z\tilde\Omega\subset\tilde\Omega + m(\tilde{\mathbf{M}})\Omega\,, \tag{5}$$

where $\Phi = \Phi(\mathbf{M})$, $Z\in\Omega^\perp$ is a nonvanishing vector field, and $m(\tilde{\mathbf{M}})\subset\mathcal{F}(\mathbf{M})$ is the common *maximal ideal of the subset* $\tilde{\mathbf{M}}\subset\mathbf{M}$ including all functions vanishing at every point of $\tilde{\mathbf{M}}$. (This is a mere preparatory task, such a space $\tilde{\mathbf{M}}$ will be replaced by a narrower subspace $\mathbf{E} \subset \tilde{\mathbf{M}}$ to obtain a subdiffiety of Ω on $\mathbf{E}$, see Section 11 below.)

7. The idea of construction.

Our aim is to search for "optimal solution" with the "maximal possible" $\tilde{\Omega}$ and $\tilde{\mathbf{M}}$. The algorithm will employ a good filtration Ω_* and a nonvanishing $Z \in \Omega^\perp$ as a mere technical tool. For the convenience of reader, we shall mention separately the simple particular case when $\tilde{\mathbf{M}} = \mathbf{M}$.

The following preparatory adaptations are of independent importance. Let $\mathcal{M}_l \equiv \Omega_l/\Omega_{l-1}$ ($l = 0, 1, \ldots, \Omega_{-1} = 0$) be factormodules. In virtue of (4_1), operator $\mathcal{L}_Z$ induces certain $\mathcal{F}(\mathbf{M})$-homomorphisms

$$\mathcal{L}_Z \colon \Omega_l \to \mathcal{M}_{l+1},\ \mathcal{M}_l \to \mathcal{M}_{l+1}$$

denoted by the same letter. Due to (4_2), both homomorphisms are surjectivities, hence the modules $\mathcal{M}_l$ are bijectively mapped on $\mathcal{M}_{l+1}$ for l large enough, say for $l \geq L$.

Definition 5. For any submodule $\Theta \subset \Omega$ of our diffiety Ω and $Z \in \Omega^\perp$ a nonvanishing vector field, let $\operatorname{Ker}\Theta \subset \Theta$ be the submodule of all $\vartheta \in \Theta$ such that $\mathcal{L}_Z\vartheta \in \Theta$.

The real choice of Z is irrelevant here, moreover $\operatorname{Ker}\Omega_{l+1} \equiv \Omega_l$ if $l \geq L$ by virtue of the definition of L. Repeatedly applying Ker to the module Ω_L, the strongly increasing sequence

$$\cdots \supset \Omega_{L+1} \supset \Omega_L = \operatorname{Ker}^0 \Omega_L \supset \operatorname{Ker} \Omega_L \supset \operatorname{Ker}^2 \Omega_L \supset \cdots \tag{6}$$

necessarily terminates with a stationarity $\operatorname{Ker}^K \Omega_L = \operatorname{Ker}^{K+1} \Omega_L$. Denoting $\tilde{\Omega} = \operatorname{Ker}^K \Omega_L$, obviously $\operatorname{Ker}\tilde{\Omega} = \tilde{\Omega}$ whence $\mathcal{L}_Z\tilde{\Omega} \subset \tilde{\Omega}$. We are done: there exists even the greatest submodule $\tilde{\Omega} \subset \Omega$ which satisfies the last inclusion. It follows that the final result does not depend on the technical tools and we may denote

$$\mathcal{R}(\Omega) = \tilde{\Omega}\,.$$

Altogether, $\mathcal{R}(\Omega)$ is the greatest finite-dimensional submodule of Ω satisfying the (equivalent) conditions

$$\mathcal{L}_Z\mathcal{R}(\Omega) \subset \mathcal{R}(\Omega) \quad (Z \in \Omega^\perp)\,, \qquad \mathrm{d}\mathcal{R}(\Omega) \simeq 0 \quad (\mathrm{mod}\ \mathcal{R}(\Omega), \Omega \wedge \Omega)\,. \tag{7}$$

8. A short digression.

Values at a point $P \in \mathbf{M}$ may be occasionally denoted by lower indices, e.g., $\varphi_P \in \Phi(\mathbf{M})_P$ is the value (lying in the cotangent space at P) of the form

$\varphi \in \Phi(\mathbf{M})$, analogously $Z_P \in \mathcal{T}(\mathbf{M})_P$ is a tangent vector at P, and so like. (Alternatively $f(P) = f_P$ for a function $f \in \mathcal{F}(\mathbf{M})$.)

Let $\Theta \subset \Phi(\mathbf{M})$ be a submodule. Recall that $\vartheta^1, \vartheta^2, \ldots \in \Theta$ is a *basis* of Θ if the values $\vartheta^1_P, \vartheta^2_P, \ldots \in \Phi(\mathbf{M})_P$ are linearly independent for every $P \in \mathbf{M}$ and $\vartheta^1, \vartheta^2, \ldots$ generate the $\mathcal{F}(\mathbf{M})$-module Θ. We shall mainly deal with finite-dimensional Θ. Then the last requirement is equivalent to the condition that $\vartheta^1_P, \vartheta^2_P, \ldots$ is a basis of Θ_P for every $P \in \mathbf{M}$ and one can see that a form $\vartheta \in \Phi(\mathbf{M})$ belongs to Θ if and only if $\vartheta_P \in \Theta_P$ for all $P \in \mathbf{M}$. If $\bar{\Theta} \subset \Theta \subset \Phi(\mathbf{M})$ are finite-dimensional submodules which have bases, then every basis of $\bar{\Theta}$ can be (locally) completed to give a basis of Θ and the additional terms provide a basis of the factor $\Theta/\bar{\Theta}$.

DEFINITION 6. Let $\mathbf{N} \subset \mathbf{M}$ be a subset, $\Theta \subset \Phi(\mathbf{M})$ a finite-dimensional submodule. A family $\vartheta^1, \vartheta^2, \ldots \in \Phi(\mathbf{M})$ is called a *basis of* Θ *along* $\mathbf{N}$ if $\vartheta^1_P, \vartheta^2_P, \ldots$ is a basis of Θ_P for every $P \in \mathbf{N}$.

In this case, if $\varphi \in \Phi(\mathbf{M})$ and $\varphi \in \Theta$ *along* $\mathbf{N}$ (i.e., $\varphi_P \in \Theta_P$ for all $P \in \mathbf{N}$), then $\varphi = \vartheta$ along $\mathbf{N}$ for appropriate $\vartheta \in \Theta$. If $\bar{\Theta} \subset \Theta \subset \Phi(\mathbf{M})$ are finite-dimensional submodules which have bases along $\mathbf{N}$, then every such a basis of $\bar{\Theta}$ can be (locally) completed to give a basis of Θ along $\mathbf{N}$ and the additional terms provide a basis of the factor $\Theta/\bar{\Theta}$ along $\mathbf{N}$.

DEFINITION 7. Let $\mathbf{N} \subset \mathbf{M}$ be a subset, $\Omega \subset \Phi(\mathbf{M})$ be a diffiety, $\Theta \subset \Omega$ be a finite-dimensional submodule. We introduce the submodule $\operatorname{Ker}_{\mathbf{N}} \Theta \subset \Theta$ of all $\vartheta \in \Theta$ such that $\mathcal{L}_Z \vartheta \in \Theta$ $(Z \in \Omega^\perp,\ Z \neq 0)$ along $\mathbf{N}$.

One can see that the real choice of Z is irrelevant and the abbreviation $\operatorname{Ker} \Theta = \operatorname{Ker}_{\mathbf{M}} \Theta$ is possible. Unfortunately, the modules appearing in this way *need not* have a basis in the common sense. However, since the behaviour of the module $\operatorname{Ker}_{\mathbf{N}} \Theta$ beyond $\mathbf{N}$ will be unimportant (see Sections 9–11 below), we may suppose only the existence of a basis along $\mathbf{N}$ to exclude the singularities.

9. The calculation of kernels.

Passing to our task, the most important technical details will be discussed in this section. For better convenience, let us again mention the calculation for the particular case $\operatorname{Ker} = \operatorname{Ker}_{\mathbf{M}}$ leading to the chain (6).

One can observe that the chain (6) is the result of the repeated application of the following step. We have finite-dimensional submodules $\bar{\Theta} \subset \Theta \subset \Omega$ of a diffiety $\Omega \in \Phi(\mathbf{M})$. The existence of bases is assumed here: let $\vartheta^1, \ldots, \vartheta^a$ be a basis of $\bar{\Theta}$ which can be completed by $\pi^1, \ldots, \pi^b$ to obtain a basis of Θ, hence $\pi^1, \ldots, \pi^b$ represent a basis of the factor $\mathcal{M} = \Theta/\bar{\Theta}$. Assuming $\mathcal{L}_Z \bar{\Theta} \subset \Theta$, we are interested in $\operatorname{Ker} \bar{\Theta}$. More explicitly, if $\mathcal{L}_Z \vartheta^i \equiv \sum a^i_j \pi^i \in \mathcal{M}$ for the induced homomorphism $\mathcal{L}_Z\colon \bar\Theta \to \mathcal{M}$, then $\vartheta = \sum b^i\vartheta^i \in \operatorname{Ker}\bar\Theta$ if and only if $\mathcal{L}_Z\vartheta = 0 \in \mathcal{M}$ which gives the system

$$\sum a^i_j b^j \equiv 0 \qquad (i = 1, \dots, b) \tag{8}$$

for the unknowns $b^1, \dots, b^a \in \mathcal{F}(\mathbf{M})$. It follows that $\operatorname{Ker}\bar\Theta$ is nontrivial if $\operatorname{rank}(a^i_j) = \text{const} < b$. (In practice, some "exceptional points" of $\mathbf{M}$ must be deleted to obtain an open subdomain where the rank is constant.) Then the module $\bar{\bar\Theta} = \operatorname{Ker}\bar\Theta$ has a (local) basis and we may continue with the triple $\bar{\bar\Theta} \subset \bar\Theta \subset \Omega$ in the same way. Starting with $\bar\Theta = \Omega_L \subset \Theta = \Omega_{L+1} \subset \Omega$, the chain (6) appears here as the final (and already well-known) result.

In full generality, in order to obtain the subset $\tilde{\mathbf{M}} \subset \mathbf{M}$ and the submodule $\tilde\Omega \subset \Omega$ satisfying (5), the bases along certain successively appearing "notable subsets" must be taken into account.

Let $\bar\Theta \subset \Theta \subset \Omega$ be submodules which have the bases denoted as above, but only *along a subset* $\mathbf{N} \subset \mathbf{M}$. Then $\mathcal{L}_Z\left(\sum b^i\vartheta^i\right)_P = 0 \in \mathcal{M}_P$ if and only if (8) is satisfied at the point $P \in \mathbf{N}$ under consideration. We are coming to the crucial point. Let $\mathbf{N}(c)$, $\mathbf{N}(\geq c)$, $\mathbf{N}(\leq c)$ be subsets of $\mathbf{N}$, where $\operatorname{rank}(a^i_j)$ is c, $\geq c$, $\leq c$, respectively. (We prefer $0 \leq c < b$. If $\operatorname{rank}(a^i_j) \geq b$, then $\operatorname{Ker}_{\mathbf{N}}\bar\Theta$ is empty. Clearly $\mathbf{N}(\geq c)$ is an open subset, $\mathbf{N}(\leq c)$ is a closed subset of "notable points" for every fixed c.) A subfamily of c linearly independent equations (8) can be resolved along $\mathbf{N}(\geq c)$. This yields the true solution also along $\mathbf{N}(c)$, hence it provides the module $\operatorname{Ker}_{\mathbf{N}(c)}\bar\Theta$. (One can observe that $\mathbf{N}$ is the union of all subsets $\mathbf{N}(c)$ for various c, and it follows that we have "the largest possible" solution.) Denoting

$$\bar{\bar\Theta} = \operatorname{Ker}_{\mathbf{N}(c)}\bar\Theta\,,$$

we may continue with the triple $\bar{\bar\Theta} \subset \bar\Theta \subset \Omega$ along the *new subset* $\mathbf{N}(c) \subset \mathbf{M}$ instead of the previous $\mathbf{N} \subset \mathbf{M}$.

10. Summary.

We have the following algorithm to determine $\tilde\Omega$ and $\tilde{\mathbf{M}}$. Starting with the triple $\bar\Theta = \Omega_L \subset \Theta = \Omega_{L+1} \subset \Omega$ along the total space $\mathbf{M}$, the above kernels can be successively calculated for various possible ranks of matrices. A finite family of chains

$$\cdots \supset \Omega_{L+1} \supset \Omega_L = \operatorname{Ker}^0\Omega_L \supset \operatorname{Ker}_{\mathbf{N}(c_1)}\Omega_L \supset \operatorname{Ker}_{\mathbf{N}(c_2)}\operatorname{Ker}_{\mathbf{N}(c_1)}\Omega_L \supset \cdots \tag{9}$$

does appear together with certain subsets $\mathbf{M} \supset \mathbf{N}(c_1) \supset \mathbf{N}(c_2) \supset \cdots$. Every such a chain terminates with the stationarity

$$\operatorname{Ker}_{\mathbf{N}(c_K)}\tilde\Omega = \tilde\Omega \qquad (\tilde\Omega = \operatorname{Ker}_{\mathbf{N}(c_K)}\cdots\operatorname{Ker}_{\mathbf{N}(c_1)}\Omega_L) \tag{10}$$

for K large enough which means that $\mathcal{L}_Z\tilde{\Omega} \subset \tilde{\Omega}$ along $\mathbf{N}(c_K)$ and we may choose

$$\tilde{\mathbf{M}} = \mathbf{N}(c_K)\,.$$

The inclusion (5_2) is clearly satisfied and one can observe that "more complete solution" does not exist. In this sense, the final result does not depend on the choice of the technical tools, however, both the final objects $\tilde{\Omega} = \Omega(c_1, \dots, c_K)$ and $\tilde{\mathbf{M}} = \tilde{\mathbf{M}}(c_1, \dots, c_K)$ depend on the choice of the ranks, of course.

11. An important adaptation.

The module $\tilde{\Omega}$ was explicitly expressed along $\tilde{\mathbf{M}}$ but the behaviour beyond $\tilde{\mathbf{M}}$ remains ambiguous. Fortunately, we do not need more precise result: only the values along rather special subsets $\mathbf{E} \subset \tilde{\mathbf{M}}$ will be employed in the following applications and then the original ambiguous module $\tilde{\Omega}$ may be replaced by any submodule $\breve{\Omega} \subset \Omega$ such that $\breve{\Omega} = \tilde{\Omega}$ along $\mathbf{E}$.

In more detail, let us assume that $\tilde{\mathbf{M}} \subset \mathbf{M}$ is a subspace, not a mere subset. (Recalling that $\tilde{\mathbf{M}}$ is defined by certain rank conditions, this requirement is satisfied in common practice. If necessary, some closed subsets of "exceptional points" of $\mathbf{M}$ should be deleted.) *We shall be interested in the largest subspace* $\mathrm{e}\colon \mathbf{E} \subset \tilde{\mathbf{M}}$ *such that the restriction* $\mathrm{e}^*\Omega \subset \Phi(\mathbf{E})$ *is a diffiety* (*a subdiffiety of* Ω). In accordance with Section 4, if $m(\tilde{\mathbf{M}})$ is the maximal ideal of the subspace $\tilde{\mathbf{M}} \subset \mathbf{M}$, such subset $\mathbf{E}$ is uniquely determined by its maximal ideal

$$m(\mathbf{E}) = m(\tilde{\mathbf{M}}) + Zm(\tilde{\mathbf{M}}) + Z^2m(\tilde{\mathbf{M}}) + \cdots \subset \mathcal{F}(\mathbf{M})\,,$$

where $Z \in \Omega^\perp$ is a nonvanishing vector field. Then, dealing with $\mathbf{E}$ instead of $\tilde{\mathbf{M}}$, the ambiguous module $\tilde{\Omega}$ can be replaced by any submodule $\breve{\Omega} \subset \Omega$ such that $\breve{\Omega} \simeq \tilde{\Omega}$ (mod $m(\mathbf{E})$) because then $\mathcal{L}_Z\breve{\Omega} \subset \breve{\Omega}$ along $\mathbf{E}$. (This follows from the inclusion

$$\mathcal{L}_Z\big(m(\mathbf{E}) \cdot \Omega\big) \subset Zm(\mathbf{E}) \cdot \Omega + m(\mathbf{E}) \cdot \mathcal{L}_Z\Omega \subset m(\mathbf{E}) \cdot \Omega\,,$$

where $Zm(\mathbf{E}) \subset m(\mathbf{E})$ is employed.) The last inclusion is equivalent to either of the (equivalent) requirements

$$\mathrm{d}\breve{\Omega} \subset \breve{\Omega} \wedge \Phi + \Omega \wedge \Omega + m(\mathbf{E})\Phi \wedge \Phi\,, \qquad \mathcal{L}_Z\breve{\Omega} \subset \breve{\Omega} + m(\mathbf{E})\Omega, \tag{11}$$

quite analogously as in (5). With this final adaptation, our algorithm is done.

In accordance with further investigations, $\mathbf{E} \subset \tilde{\mathbf{M}} \subset \mathbf{M}$ is called the *Euler-Lagrange* ($\mathcal{EL}$) *subspace* of $\mathbf{M}$ (in $\tilde{\mathbf{M}}$, corresponding to Ω), the submodule $\mathrm{e}^*\Omega \subset \Phi(\mathbf{E})$ is called the $\mathcal{EL}$ *diffiety* (on $\mathbf{E}$, corresponding to Ω), solutions $P(t) \in \mathbf{E}$ $(0 \le t \le 1)$ of the $\mathcal{EL}$ diffiety are *extremals*, and any finite-dimensional submodule $\breve{\Omega} \subset \Omega$ such that $\breve{\Omega} = \tilde{\Omega}$ along $\mathbf{E}$ will be called the *Poincaré-Cartan* ($\mathcal{PC}$) *module*.

Stationarity and extremality

12. A variational formula.

At this place, our intentions can be eventually clarified as follows. For a given diffiety $\Omega \subset \Phi(\mathbf{M})$, certain greatest possible subsets $\tilde{\mathbf{M}}$ with the relevant submodules $\tilde{\Omega} \subset \Omega$ satisfying (5) are already determined. It can be shown that the equations $\omega \equiv 0$ $(\omega \in \tilde{\Omega})$ in a certain sense represent an "autonomous infinitesimal leaf" of Ω *at every point* $P \in \tilde{\mathbf{M}}$. We are, however, not interested in isolated points but rather in the *curves* $P(t) \in \tilde{\mathbf{M}}$ $(0 \le t \le 1)$ which are solutions of Ω. This adaptation in Section 11 fulfills the above mentioned demand: we have obtained even the subdiffiety $\mathbf{e}^*\Omega \subset \Phi(\mathbf{E})$ on the greatest possible subspace $\mathbf{E} \subset \tilde{\mathbf{M}}$ and a simpler submodule $\breve{\Omega} \subset \Omega$ satisfying (11). By virtue of (11), we prove now that this module $\breve{\Omega}$ determines an "autonomous infinitesimal band" along the extremals $P(t) \in \mathbf{E}$ $(0 \le t \le 1)$. This is the crucial result which provides the true geometrical sense of our approach.

Let $Q(t;\lambda) \in \mathbf{M}$ $(0 \le t \le 1,\ -\varepsilon < \lambda < \varepsilon$, where $\varepsilon > 0$ is fixed) be a one-parameter family of curves; we shall abbreviate $P(t) = Q(t;0)$.

Definition 8. A vector field $V \in \mathcal{T}(\mathbf{M})$ such that

$$V_{P(t)}f \equiv \frac{\partial}{\partial \lambda} Q\big((t,\,\cdot\,)^* f\big)\big|_{\lambda=0} \qquad (f \in \mathcal{F}(\mathbf{M}),\ 0 \le t \le 1)$$

is called a *variation of the curve* $P(t)$ *in the family* $Q(t;\lambda)$.

This vector field V is uncertain beyond the points $P(t) \in \mathbf{M}$, nevertheless, the well-known formulae

$$Q(\,\cdot\,;\lambda)^* f = P^* f + \lambda P^* V f + o(\lambda), \qquad Q(\,\cdot\,;\lambda)^*\varphi = P^*\varphi + \lambda P^* \mathcal{L}_V \varphi + o(\lambda)$$

equivalent to the usual definitions of Vf and $\mathcal{L}_V\varphi$ $(f \in \mathcal{F}(\mathbf{M}),\ \varphi \in \Phi(\mathbf{M}))$ at the points of the curve $P(t)$ make a good sense. Assuming that $Q(\,\cdot\,;\lambda)$ is a solution of the diffiety $\Omega \in \Phi(\mathbf{M})$, $Q(\,\cdot\,;\lambda)^*\Omega = 0$ hence $P^*\Omega = 0$ and $P^*\mathcal{L}_V\Omega = 0$ is obtained for the choice $\varphi = \omega \in \Omega$. Let moreover $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ be an $\mathcal{EL}$ subspace and $P(t) = \mathbf{e}P(t) \in \mathbf{E}$ $(0 \le t \le 1)$, that means, let $P(t)$ be a solution of the relevant $\mathcal{EL}$ subdiffiety $\mathbf{e}^*\Omega \subset \Phi(\mathbf{E})$ (i.e., an extremal). Let $\omega^1, \dots, \omega^c$ be a basis along $\mathbf{E}$ of the relevant $\mathcal{PC}$ module $\breve{\Omega}$. Denoting by x the independent variable, the inclusion (11_1) ensures certain expansions

$$\mathrm{d}\omega^i \equiv \sum a^i_j \omega^j \wedge \mathrm{d}x \quad (\mathrm{mod}\ \Omega \wedge \Omega) \tag{12}$$

along $\mathbf{E}$. Using (2_3), it follows that

$$\mathcal{L}_Z \omega^i \equiv -\sum a^i_j \omega^j(Z)\,\mathrm{d}x + \mathrm{d}\omega^i(Z) \quad (\mathrm{mod}\ \Omega) \tag{13}$$

for all $Z \in \mathcal{T}(\mathbf{M})$ whence

$$0 = P^*\mathcal{L}_V\omega^i \equiv \sum b^i_j v^j \,\mathrm{d}P^*x + \mathrm{d}v^i \qquad (\, b^i_j = P^*a^i_j\,,\ \ v^i = P^*\omega^i(V)\,)$$

for the particular choice $Z = V$ (the variation). Assuming moreover $\mathrm{d}P^*x \neq 0$ (i.e., the possibility of the parametrization $t = t(x)$ is guaranteed) we obtain the system

$$\mathrm{d}v^i/\,\mathrm{d}x \equiv -\sum b^i_j v^j \qquad (\, b^i_j = P^*a^i_j\,,\ \ v^i = P^*\omega^i(V)\,) \tag{14}$$

for the values $v^i = P^*\omega^i(V)$.

CONCLUSION. *Let $V \in \mathcal{F}(\mathbf{M})$ be the variation of an extremal $P(t) \in \mathbf{E}$ $(0 \leq t \leq 1)$ in a family $Q(t,\lambda) \in \mathbf{M}$ $(0 \leq t \leq 1)$ of solutions of a diffiety $\Omega \subset \Phi(\mathbf{M})$, let ω^i $(i = 1,\dots,c)$ be a basis of the relevant $\mathcal{PC}$ module. Then the variational formula* (14) *holds true.*

13. The stationarity.

It is easy to interpret the variational formula in terms of the stationarity. As a result, a large spectrum of corresponding variational problems can be naturally introduced for the original diffiety Ω by using the $\mathcal{PC}$ module $\breve{\Omega}$. Retaining the notation of previous Section 12, we restrict ourselves to the most necessary indications and one can observe with pleasure that only quite simple consequence of (14) will be sufficient for the most important classical cases mentioned in (ι) below: *if $v^i(a) \equiv 0$ $(i = 1,\dots,c)$ for a certain value $x = a$, then $v^i \equiv 0$ identically.*

(ι) *The classical problems.*

Let $y^1,\dots,y^m, z^1,\dots,z^n, g \in \mathcal{F}(\mathbf{M})$ be functions such that

$$\breve{\Omega}_{P(0)} \subset \{\mathrm{d}y^1,\dots,\mathrm{d}y^m\}_{P(0)}\,, \qquad \mathrm{d}g_{P(1)} \in \breve{\Omega}_{P(1)} + \{\mathrm{d}z^1,\dots,\mathrm{d}z^n\}_{P(1)}\,. \tag{15}$$

Assume moreover

$$y^i\big(Q(0;\lambda)\big) \equiv 0\,, \quad z^j\big(Q(1;\lambda)\big) \equiv 0 \qquad (\,i = 1,\dots,m\,,\ \ j = 1,\dots,n\,). \tag{16}$$

Recalling the variation $V \in \mathcal{T}(\mathbf{M})$, it follows that the stationarity of the value g at the end point $P(1)$ is of the form:

$$\mathrm{d}g(V)_{P(1)} = \frac{\partial}{\partial\lambda} g\big(Q(1;\lambda)\big)\big|_{\lambda=0} = 0\,. \tag{17}$$

Indeed, the relation (16_1) implies $\mathrm{d}y^i(V) \equiv 0$ at $P(0)$, hence $v^i(a) \equiv 0$ by using (15_1) where the value $x = a$ is defined by $t(a) = 0$, therefore $v^i \equiv 0$ identically and then (15_2) together with (16_2) imply (17).

The result can be rephrased as follows. Let us consider all solutions $Q(t) \in \mathbf{M}$ $(0 \le t \le 1)$ of Ω satisfying $y^i(Q(0)) = z^j(Q(1)) = 0$ and let us try to obtain the extremal value $g(Q(1))$. This is a variational problem and (16), (17) mean that the extremal $P(t) \in \mathbf{E}$ $(0 \le t \le 1)$ satisfying moreover (15) may be regarded for a hopeful candidate of the solution. The particular choice $n = m-1$ includes the classical *Mayer problem* and much more. The common *transversality conditions* at the end points are latently involved in the requirements (15), of course.

($\iota\iota$) *Joint conditions.*

The linearity of the system (14) ensures certain identities

$$v^i(1) \equiv \sum c^i_j v^j(0)$$

or, in more detail, the identities

$$\omega^i(V)_{P(1)} \equiv \sum c^i_j \omega^j(V)_{P(0)} \tag{18}$$

with universal coefficients c^i_j for all variations V. With this preparation, *let* $k^1,\dots,k^m, g \in \mathcal{F}(\mathrm{M}\times\mathrm{M})$ *be functions of couples of points from* $\mathbf{M}$ *such that the inclusion*

$$\mathrm{d}g_{(P(0),P(1))} \in \left\{\mathrm{d}k^1,\dots,\mathrm{d}k^m, \sum c^i_j p_1^*\omega^j - p_2^*\omega^j \;:\; i = 1,\dots,c\right\}_{(P(0),P(1))} \tag{19}$$

is satisfied, where $p_i\colon \mathbf{M}\times\mathbf{M} \to \mathbf{M}$ $(i = 1,2)$ *are natural projections onto the* i*th factor. Assuming moreover* $k^i(Q(0;\lambda),Q(1;\lambda)) \equiv 0$ $(i = 1,\dots,m)$, *the stationarity of* g *at the couple of the end points*

$$\mathrm{d}g(V,V)_{(P(0),P(1))} = \frac{\partial}{\partial\lambda} g(Q(0;\lambda),Q(1;\lambda))\big|_{\lambda=0} = 0 \tag{20}$$

holds true.

(The proof is analogous as above.)

Similarly to (ι), one might again introduce a large family of variational problems where the extremals satisfying (19) represent the hopeful solution. One can observe that the functions of triples, quadruples, ... of points do not bring much novelty.

($\iota\iota\iota$) *Continuation.*

Assuming $c^i_j = \delta^i_j$, the inclusion (19) reads

$$\mathrm{d}g_{(P(0),P(1))} = \left(\sum A^i\,\mathrm{d}k^i + \sum B^j\left(p_1^*\omega^j - p_2^*\omega^j\right)\right)_{(P(0),P(1))} \tag{21}$$

for appropriate coefficients A^i, B^j. This is equivalent to the existence of a form $\omega \in \breve{\Omega}$ (namely $\omega = \sum B^j\omega^j$) and coefficients A^i such that the boundary conditions (with partial total differentials)

$$\mathrm{d}_1\left(g - \sum A^i k^i\right)_{P(0)} = \omega_{P(0)}\,, \qquad \mathrm{d}_2\left(g - \sum A^i k^i\right)_{P(1)} = \omega_{P(1)} \tag{22}$$

are satisfied. The assumption $c^i_j \equiv \delta^i_j$ can be ensured by appropriate choice of the basis $\omega^1, \dots, \omega^c$ of $\breve{\Omega}$.

(ιv) *A digression.*

We shall recall a well-known result omitting the proof.

Let $\Theta \subset \Phi(\mathbf{M})$ be a finite-dimensional submodule and

$$\operatorname{Adj}\Theta = \{\vartheta, Z\rfloor d\vartheta : \ \vartheta \in \Theta, \ \ Z \in \Theta^{\perp}\} \subset \Phi(\mathbf{M}) \tag{23}$$

the submodule generated by all the mentioned forms ϑ, $Z\rfloor d\vartheta$. Then $\operatorname{Adj}\Theta$ *is completely integrable, i.e., it has a basis* $\operatorname{Adj}\Theta = \{\mathrm{d}f^1, \dots, \mathrm{d}f^m\}$ *consisting of total differentials of certain (so called adjoint to Θ) functions. Moreover, there exists a basis*

$$\Theta = \{\vartheta^1, \dots, \vartheta^c\}, \qquad \vartheta^i = \sum F^i_j(f^1, \dots, f^m)\,\mathrm{d}f^j \tag{24}$$

expressible by adjoint functions.

(v) *Trivial variational problems.*

The formula (7) with the $\mathcal{EL}$ subspace $\mathbf{E} = \mathbf{M}$ (hence the trivial $\mathcal{EL}$ diffiety $\mathbf{e}^*\Omega = \Omega$) and the $\mathcal{PC}$ module $\breve{\Omega} = \tilde{\Omega} = \mathcal{R}(\Omega)$ deserve a short notice. Recall that this $\mathcal{R}(\Omega)$ is the greatest finite-dimensional submodule of Ω satisfying the (equivalent) conditions (7). We can see that in reality

$$\mathcal{L}_Z\mathcal{R}(\Omega) \subset \mathcal{R}(\Omega) \quad (Z \in \mathcal{R}(\Omega)^{\perp}), \qquad \mathrm{d}\mathcal{R}(\Omega) \simeq 0 \pmod{\mathcal{R}(\Omega)}, \tag{25}$$

that means, $\mathcal{R}(\Omega)$ *is a completely integrable module.*

As for the proof, let us abbreviate $\Theta = \mathcal{R}(\Omega)$. Then $\mathrm{d}\operatorname{Adj}\Theta \simeq 0 \pmod{\operatorname{Adj}\Theta}$ in virtue of complete integrability of the submodule $\operatorname{Adj}\Theta \subset \Omega$, hence (trivially) $\mathrm{d}\operatorname{Adj}\Theta \simeq 0 \pmod{\operatorname{Adj}\Theta, \ \Omega \wedge \Omega}$. Owing to the mentioned maximality of $\mathcal{R}(\Omega)$ it follows that $\Theta \supset \operatorname{Adj}\Theta$, therefore $\Theta = \operatorname{Adj}\Theta$ and we are done.

($v\iota$) *Concluding remarks.*

Let us mention the inclusion

$$\operatorname{Adj}\breve{\Omega} \subset \Omega + m(\mathbf{E})\Phi \tag{26}$$

equivalent to (11_1). If $\operatorname{Adj}\breve{\Omega} = \{\mathrm{d}f^1, \dots, \mathrm{d}f^m\}$ is a basis with differentials of adjoint functions, then (26) implies $\mathrm{d}f^i \in \Omega$ along $\mathbf{E}$. It follows that $\mathbf{e}^*\,\mathrm{d}f^i \in \mathbf{e}^*\Omega$ (equivalently $\mathbf{e}^*\,\mathrm{d}f^i \in \mathcal{R}(\mathbf{e}^*\Omega)$) and in the classical terms, *functions $f^1, \dots, f^m$ are first integrals* for the extremals. If moreover $\breve{\Omega} = \{\omega^1, \dots, \omega^c\}$ is a basis of $\breve{\Omega}$ expressible in terms of the adjoint functions, then $\mathrm{d}\omega^i \in \Omega \wedge \Omega$ by easy direct verification and therefore $a^i_j = b^i_j \equiv 0$ in formulae (12), (13), (14) rewritten for the new basis $\omega^1, \dots, \omega^c$. It follows that the assumptions $c^i_j \equiv \delta^i_j$ in ($\iota\iota\iota$) are satisfied for this choice.

14. The extremality.

Recalling the data, let $\Omega \subset \Phi(\mathbf{M})$ be a diffiety, $\mathbf{e}^*\Omega \subset \Phi(\mathbf{E})$ the $\mathcal{EL}$ subdiffiety on the $\mathcal{EL}$ subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$, and $\breve{\Omega} \subset \Omega$ the $\mathcal{PC}$ submodule satisfying (11) along $\mathbf{E}$. If $Q(t;\lambda) \in \mathbf{M}$ $(0 \leq t \leq 1)$ is a one-parameter family of solutions of the diffiety Ω such that $Q(t;0) = P(t) \in \mathbf{E}$ is even an extremal, we have already obtained certain *sufficient conditions of stationarity* of certain functions g, see (17), (20). The *sufficient conditions for the extremality* cause more difficulties. The following *increment formula* (30) will permit us to cope with this task in certain very large family of favourable cases, including the so called *nondegenerate variational problems*.

The principle is quite simple. Let $\breve{\omega} \in \Omega$ be a differential form, $\mathbf{l}\colon \mathbf{L} \subset \mathbf{E}$ be a subspace such that

$$\mathbf{l}^*(\breve{\omega} - \mathrm{d}W) = 0 \qquad (W \in \mathcal{F}(\mathbf{M})) \tag{27}$$

for appropriate $W \in \mathcal{F}(\mathbf{M})$. Let $Q(t) \in \mathbf{M}$ and $P(t), R(t) \in \mathbf{L}$ $(0 \leq t \leq 1)$ be three curves satisfying

$$W(R(0)) = W(P(0)), \qquad W(Q(1)) = W(R(1)), \tag{28}$$

where $P(t)$ is moreover a solution of Ω (hence an extremal). Then

$$W(P(1)) - W(P(0)) = \int P^*\,\mathrm{d}W = \int P^*\breve{\omega} = 0,$$

$$\begin{aligned} W(Q(1)) - W(P(0)) &= W(R(1)) - W(R(0)) \\ &= \int R^*\,\mathrm{d}W = \int R^*\breve{\omega}, \end{aligned}$$

therefore

$$W(Q(1)) - W(P(1)) = W(Q(1)) - W(P(0)) = \int R^*\breve{\omega}. \tag{29}$$

CONCLUSION. *If $P(t) \in \mathbf{L}$ $(0 \leq t \leq 1)$ is an extremal and $Q(t) \in \mathbf{M}$, $R(t) \in \mathbf{L}$ $(0 \leq t \leq 1)$ are two curves satisfying* (28), *then the values of the form $R^*\breve{\omega}$ determine the sign of the difference $W(Q(1)) - W(P(1))$.*

The result will be applied with the following additional assumptions. First of all, we suppose that $\breve{\omega} \in \breve{\Omega}$ belongs to the $\mathcal{PC}$ module. Second, $\mathbf{L} \subset \mathbf{E}$ will be the *maximal subspace* satisfying (27), the so called *Lagrangian subspace* (to $\breve{\omega}$). Then (27) provides the (generalized) *Hamilton-Jacobi* ($\mathcal{HJ}$) *equation* (better: *involutive system*) for the unknown function W and $\mathbf{L}$ may be interpreted in terms of generalized *Mayer field of extremals*. Third, a solution of Ω will be taken for the curve $Q(t)$. Four, appropriate "approximation" of $Q(t)$ satisfying even $W(Q(t)) \equiv W(R(t))$, $0 \le t \le 1$, will be taken for the curve $R(t) \in \mathbf{L}$. Fifth, clearly $R^*\breve{\omega} = \mathcal{E}\,\mathrm{d}t$ with a certain function $\mathcal{E}$ and then (29) reads

$$W(Q(1)) - W(P(1)) = \int_0^1 \mathcal{E}(t)\,\mathrm{d}t. \tag{30}$$

Therefore $\mathcal{E}$ is a substitute and a far-going generalization for the famous *Weierstrass function.*

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Department of Mathematics
Faculty of Civil Engineering
Brno University of Technology
Žižkova 17
CZ–602 00 Brno
Czech Republic
E-mail: chrastinova.v@fce.vutbr.cz
tryhuk.v@fce.vutbr.cz