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SUBSERIES IN BANACH SPACES

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ABSTRACT. We prove several theorems on subseries of an infinite series in Banach spaces along with an analogue of Gel'fand's theorem on the structure of a certain set.

1. Introduction

Investigations on subseries of an infinite series of real terms have found prominent positions in the literature during the last several decades. We may quote some of the references such as [1], [2], [3], [6], [8], [10], [12], [13], where some other references could be found. However, for vector series, the study has been mainly concentrated on conditionally and unconditionally convergent series and their rearrangements ([5], [9]) except the book [4] where the idea of subseries convergent in a normal linear space can be found.

In this paper, considering the idea of *subseries-convergence* ([4; p. 78]), we prove several theorems on subseries of a vector series including an analogue of Gel'fand's theorem on the compactness of a certain set.

2. Basic definitions and notations

Throughout X stands for a Banach space and sets are always subsets of X. The symbols $\mathbb{R}$ and $\mathbb{N}$ stand for the set of real numbers and the set of positive integers respectively. We shall follow the definition of a series in X, its convergence etc. as given in [9]. In particular, we state from [9] the following definitions.

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Definition A. A series $\sum_{i=1}^{\infty} x_i$ in X is said to be *absolutely convergent* if $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

Definition B. A series $\sum_{i=1}^{\infty} x_i$ in X is said to be *unconditionally convergent* if it converges for any rearrangement of its terms.

It is known ([9; Theorem 1.3.1]) that for an unconditionally convergent series, all its rearrangements have the same sum. However, in X, unconditional convergence does not generally imply absolute convergence, but absolute convergence always implies unconditional convergence. If X is of finite dimension, then these two concepts coincide ([9; Theorem 1.3.3]).

By $\sum_{i=1}^{\infty} x_i$ we shall always mean an infinite series in X and this will be briefly written by $\sum x_i$. By $\{\varepsilon_i\}$ we shall mean a sequence of elements ε_i where $\varepsilon_i = 0$ or 1 and *for an infinite number of* i, $\varepsilon_i = 1$.

3. Subseries-convergence of a series

We first consider the following definition.

Definition 1. ([4; p. 78]) A series $\sum x_i$ is said to be *subseries convergent* if the series $\sum \varepsilon_i x_i$ converges for any choice of coefficients $\varepsilon_i = 0$ or 1, where $\varepsilon_i = 1$ for infinity of i.

Clearly $\sum \varepsilon_i x_i$ is a subseries of $\sum x_i$. Subseries-convergence implies convergence ([4; p. 78]) but the converse is not true as shown by the following example

Example 1. In $C[0,1]$, we consider the series

$$\sum (-1)^{i-1} \frac{x}{i} = x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \dots, \qquad x \in C[0,1],$$

which is clearly convergent. Taking $\varepsilon_i = 0$ when i is even and 1 when i is odd we get $\sum \varepsilon_i \frac{x}{i} = x + \frac{x}{3} + \frac{x}{5} + \dots$ which is not convergent. So the series $\sum (-1)^{i-1} \frac{x}{i}$ is not subseries-convergent.

The following theorem gives a relation between subseries-convergence and unconditional convergence. The theorem is already known ([4; p. 78]), and proved for the first time in [11]. But because of its intrinsic interest and wide scope for application, we construct a proof for easy access.

THEOREM 1. *A series $\sum x_i$ is unconditionally convergent if and only if it is subseries-convergent.*

Proof. Suppose first that $\sum x_i$ is not subseries-convergent. Then there is a sequence $\{\varepsilon_i\}$ such that $\sum \varepsilon_i x_i$ is not convergent. By Cauchy's criterion, there is a $\delta > 0$ and an infinite sequence of indices $n_1 < \ell_1 < n_2 < \ell_2 < n_3 < \ell_3 < \dots$ such that

$$\left\| \sum_{i=n_j}^{\ell_j} \varepsilon_i x_i \right\| \geq \delta \tag{1}$$

for $j = 1, 2, \dots$. Clearly n_j can be so selected that $n_j - \ell_{j-1} > 2$ for $j = 2, 3, \dots$ and also we can assume that $n_1 > 2$.

Let Δ_j be the collection of all those terms x_i of $\sum x_i$ for which $i \in [n_j, \ell_j]$ and such that the corresponding $\varepsilon_i = 1$, and Δ^o be the collection of the remaining terms of $\sum x_i$ which do not belong to Δ_j for $j = 1, 2, \dots$, where for the series $\sum \varepsilon_i x_i$, ε_i is said to correspond to x_i in the term $\varepsilon_i x_i$.

The terms of the series occurring in the collections Δ^o and Δ_j, $j = 1, 2, \dots$, are separately ordered as per order of increasing their indices. We now form a rearranged series of $\sum x_i$ according to the following plan.

We add all the terms of Δ_1 as per order followed by the first term from Δ^o. Next we add with this all the terms of Δ_2 in order followed by the second term from Δ^o and so on. Using (1) and Cauchy criterion, this rearranged series of $\sum x_i$ does not converge. This shows that $\sum x_i$ is not unconditionally convergent.

Conversely suppose that $\sum x_i$ is not unconditionally convergent. So there exists a rearranged series $\sum x_{n_i}$ which is not convergent. By Cauchy's criterion, there is a $\delta > 0$ such that if $k \in \mathbb{N}$ is given, there is $\ell \in \mathbb{N}$, $\ell > k$, such that

$$\left\| \sum_{i=k}^{\ell} x_{n_i} \right\| \geq \delta\,. \tag{2}$$

So for $k = k_1$, there is $\ell_1 > k_1$ such that

$$\left\| \sum_{i=k_1}^{\ell_1} x_{n_i} \right\| \geq \delta\,. \tag{3}$$

Let $\Delta_1 = \{n_i :\ i = k_1, k_1+1, \dots, \ell_1\}$. Clearly positive integers m_1 and r_1 can be found such that

$$\Delta_1 \subset \{1, 2, \dots, m_1\} \subset \{n_1, n_2, \dots, n_{r_1}\}\,.$$

Let $k_2 = r_1 + 1$. Then by (2) there exists $\ell_2 > k_2$ such that

$$\left\| \sum_{i=k_2}^{\ell_2} x_{n_i} \right\| \geq \delta\,. \tag{4}$$

Let $\Delta_2 = \{n_i : i = k_2, k_2+1, \dots, \ell_2\}$. There exist positive integers $m_2 > m_1$ and $r_2 > r_1$ such that

$$\Delta_2 \subset \{m_1 + 1, m_1 + 2, \dots, m_2\} \subset \{n_1, n_2, \dots, n_{r_1}, n_{r_1+1}, \dots, n_{r_2}\}.$$

Let $k_3 = r_2 + 1$. There exists $\ell_3 > k_3$ such that

$$\left\| \sum_{i=k_3}^{\ell_3} x_{n_i} \right\| \geq \delta. \tag{5}$$

Let $\Delta_3 = \{n_i : i = k_3, k_3+1, \dots, \ell_3\}$. Proceeding in this way we obtain a disjoint non-void sequence of sets $\Delta_1, \Delta_2, \Delta_3 \dots$ of positive integers. We arrange the members in Δ_j $(j = 1, 2, \dots)$ by indexes and define $\sum \varepsilon_k x_k$ in the way that $\varepsilon_k = 1$ if $k \in \Delta_j$ for some j, and $\varepsilon_k = 0$ if k does not belong to any Δ_j. Using (3), (4) etc. and Cauchy criterion, this subseries is not convergent. So the series $\sum x_i$ is not subseries-convergent. This proves the theorem. □

Note 1. A series $\sum x_i$ is said to be *perfectly convergent* ([9; p. 7]) if the series $\sum \alpha_i x_i$ converges for any choice of coefficients $\alpha_i = \pm 1$.

It is known ([9; Theorem 1.3.2]) that a series converges unconditionally if and only if it converges perfectly. Further it has been stated ([9; Exercise 1.3.3]) that in the definition of perfect convergence, the sequence $\{\alpha_i\}$ may be replaced by a sequence $\{\theta_i\}$, $\theta_i \in T$, where T is a bounded set of complex numbers containing at least two points. But the details are not available and so Theorem 1 is justified.

Note 2. In a finite dimensional Banach space, absolute convergence is equivalent to unconditional convergence and in view of Theorem 1, absolute convergence and subseries-convergence are equivalent. In other words in a finite dimensional Banach space a series is absolutely convergent if and only if all its subseries are convergent.

4. Subseries-convergence and absolute convergence

In the next three theorems we find the relation between subseries-convergence and absolute convergence.

THEOREM 2. *If a series $\sum x_i$ is absolutely convergent, then it is subseries-convergent.*

Proof. Because absolute convergence implies unconditional convergence ([9; p. 7]), the theorem follows from Theorem 1 (see also [4; p. 78]). □

However the converse is not true as shown by the following example.

EXAMPLE 2. ([9; Example 1.3.1]) Let $X = \ell_2$ and $x_i = (0, 0, \dots, 1/i, 0, \dots)$, where the non-zero coordinate is in the ith place. Then $\sum x_i$ converges to the element $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$ of ℓ_2 and for any choice of $\{\varepsilon_i\}$, $\sum \varepsilon_i x_i$ clearly converges to an element of ℓ_2. However,

$$\sum \|x_i\| = \sum \frac{1}{i} = \infty ,$$

and so $\sum x_i$ does not converge absolutely.

In fact, for an infinite dimensional Banach space X, we have:

THEOREM 3. *Each infinite dimensional Banach space contains a subseries-convergent series which is not absolutely convergent.*

Proof. For the proof of Theorem 3 we note first the following theorem of Dvoretzy Rogers ([9; Theorem 3.1.1]).

THEOREM. *Let X be an infinite dimensional Banach space and $\{a_i\}$ a sequence of positive numbers satisfying the condition $\sum\limits_{i=1}^{\infty} a_i^2 < \infty$. Then X contains a sequence $\{x_i\}$ of vectors such that $\|x_i\| = a_i$, $i = 1, 2, \dots$, and the series $\sum x_i$ converges unconditionally.*

In this theorem if we put $a_i = \frac{1}{i}$, then this gives that any infinite dimensional Banach space contains a sequence of vectors $\{x_i\}$ such that $\sum x_i$ converges unconditionally but not absolutely. Theorem 3 now follows from Theorem 1. $\square$

5. Structure of subseries sums

We now consider the nature of the set of sums of the subseries of a subseries-convergent series.

THEOREM 4. *If $\sum x_i$ is subseries convergent, then the collection S of the sums of all subseries $\sum \varepsilon_i x_i$ forms a subset which is relatively compact ($\varepsilon_i = 1$ for infinity of i).*

Proof. We shall show that the set S is totally bounded in X. We claim that for a given $\varepsilon > 0$, there is a positive integer $n(\varepsilon)$ such that

$$\left\| \sum_{i=n(\varepsilon)}^{\infty} \varepsilon_i x_i \right\| < \varepsilon \tag{6}$$

for any sequence $\{\varepsilon_i\}$, $\varepsilon_i = 0$ or 1. If this is not true, then there is a $\delta > 0$ for which we can find a sequence of indices $n_1 < n_2 < \dots$ and sequences $\{\varepsilon_i^{(j)}\}$, $j = 1, 2, \dots$, ($\{\varepsilon_i^{(j)}\}$ depends on n_j) such that

$$\Big\| \sum_{i=n_j}^{\infty} \varepsilon_i^{(j)} x_i \Big\| \ge \delta \qquad \text{for} \quad j = 1, 2, \dots .$$

Now choose $r_j > n_j$ such that

$$\Big\| \sum_{i=n_j}^{r_j} \varepsilon_i^{(j)} x_i \Big\| \ge \delta/2 \qquad \text{for} \quad j = 1, 2, \dots .$$

Clearly we can assume that $n_j < r_j < n_{j+1}$ for $j = 1, 2, \dots$. We now construct a sequence $\{\varepsilon_i'\}$ where $\varepsilon_i' = \varepsilon_i^{(j)}$ if i belongs to $[n_j, r_j]$ for $j = 1, 2, \dots$ and 0 otherwise. Then $\sum \varepsilon_i' x_i$ is a subseries of the given series which is not convergent, a contradiction. So our claim is true.

Let S_n (n fixed) be the collection of all finite sums of the form $\sum_{i=1}^{n} \varepsilon_i x_i$ for all possible choices of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ where $\varepsilon_i = 0$ or 1. Then this collection is finite and because of (6), S_n forms a finite ε-net for the set S. So S is totally bounded. Since X is complete, $\overline{S}$ is compact. This proves the theorem. □

Gel'fand ([7], see also [9; Theorem 1.3.4]) proved that if $\sum x_i$ is unconditionally convergent, then the set of all sums of the form $\sum \alpha_i x_i$, where $\alpha_i = 1$ or -1, is compact. However, in our case (Theorem 4) the set S cannot be proved to be compact. In fact, the following example shows that S need not be closed.

EXAMPLE 3. Let $X = \ell_2$ and $x_i = \left(0, 0, \dots, \frac{1}{i}, 0, \dots\right)$ where the non-zero coordinate is in the ith place as in Example 2. Then $\sum x_i$ is subseries-convergent. Now choose $y_n = \sum \varepsilon_i^{(n)} x_i$ where $\varepsilon_i^{(n)} = 0$ for $i = 1, 2, \dots, n$ and 1 otherwise. Then $\{y_n\}$ is a sequence in S. Let $y = \sum \varepsilon_i x_i$ where $\varepsilon_i = 0$ for all i, i.e. y is the null element in ℓ_2. Then

$$\begin{aligned} \|y_n - y\| &= \Big\| \sum \varepsilon_i^{(n)} x_i - \sum \varepsilon_i x_i \Big\| = \Big\| \sum \varepsilon_i^{(n)} x_i \Big\| \\ &= \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^{1/2} \to 0 \qquad \text{as} \quad n \to \infty . \end{aligned}$$

Since $y \notin S$, S is not closed.

For finite sums, we obtain a similar theorem.

THEOREM 5. ([9; Example 1.3.5]) *If $\sum x_i$ is subseries-convergent, then the set*

$$G = \left\{ \sum_{i=1}^{n} \varepsilon_i x_i : \ \varepsilon_i = 0 \ \text{or} \ 1 \ \text{for} \ i = 1, 2, \dots, n, \ n \in \mathbb{N} \right\}$$

is relatively compact.

Proof. Suppose that $\sum x_i$ is subseries-convergent. Let $\varepsilon > 0$ be given. As in Theorem 4, we can prove that there is a positive integer $m = m(\varepsilon)$ such that

$$\left\| \sum_{i=m+1}^{r} \varepsilon_i x_i \right\| < \varepsilon/2 \tag{7}$$

for every subseries $\sum \varepsilon_i x_i$ and every positive integer $r > m$.

Now let

$$T = \left\{ \sum_{i=1}^{k} \varepsilon_i x_i : \ \varepsilon_i = 0 \ \text{or} \ 1 \ \text{for} \ i = 1, 2, \dots, k, \ k \in \{1, 2, \dots, m\} \right\}.$$

Then T is a finite subset of G. Let $\alpha \in G$ and $\alpha \notin T$. Then α must be of the form $\alpha = \sum\limits_{i=1}^{r} \varepsilon_i x_i$, $r > m$. So

$$\alpha = \sum_{i=1}^{r} \varepsilon_i x_i = \beta + \sum_{i=m+1}^{r} \varepsilon_i x_i$$

where

$$\beta = \sum_{i=1}^{m} \varepsilon_i x_i \in T.$$

By (7), $\|\alpha - \beta\| = \left\| \sum\limits_{i=m+1}^{r} \varepsilon_i x_i \right\| < \varepsilon/2 < \varepsilon$. If however $\alpha \in \overline{G}$ but $\alpha \notin G$, then there is a $\nu \in G$ such that $\|\alpha - \nu\| < \varepsilon/2$ and so as above we can find a $\beta \in T$ such that $\|\beta - \nu\| < \varepsilon/2$. Then $\|\alpha - \beta\| < \varepsilon$.

This shows that T is an ε-net for $\overline{G}$. Hence $\overline{G}$ is totally bounded. Since X is complete and $\overline{G}$ is closed, $\overline{G}$ is compact. This proves the theorem. □

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