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OSCILLATION OF FIRST ORDER LINEAR RETARDED EQUATIONS

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ABSTRACT. This paper deals with oscillatory properties of linear differential equations with distributed delay. Existence of positive solution is also treated.

1. Introduction

We consider first order linear functional equation of the form

$$\dot{x}(t) + \int\limits_0^{\sigma(t)} x(t-s)\, \mathrm{d}r(t,s) = 0\,, \qquad t \ge t_0\,, \tag{1}$$

where the integral is in the sense of Riemann-Stieltjes, under the standing hypotheses:

(H_1) $\sigma\colon [t_0,\infty) \to (0,\infty)$ is continuous and $\lim\limits_{t\to\infty} \big(t - \sigma(t)\big) = \infty$;

(H_2) $r(t,0) = 0$ for $t \in [t_0,\infty)$;

(H_3) $r\big(t,\sigma(t)\big)\colon [t_0,\infty) \to \mathbb{R}$ is continuous;

(H_4) $r(t,s)$ is nondecreasing with respect to s for $s \in \big[0,\sigma(t)\big]$.

By a *solution* of equation (1) we mean a continuously differentiable function x for all t large enough, which eventually satisfies equation (1). A solution of equation (1) is called *oscillatory* if it has arbitrarily large zeros and otherwise it is called *nonoscillatory*. In this paper no solutions are considered for which $x \equiv 0$ eventually.

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Our aim is to obtain sufficient conditions for the oscillation of all solutions of equation (1). We also establish conditions under which equation (1) has at least one nonoscillatory solution.

The problem of oscillation and nonoscillation for retarded differential equations has received considerable attention in recent years; see e.g. [4], [5], [8] [11] and the references cited therein. However, most of the works on the subject has been focused on the differential equations with a discrete delay and very little has been published on the equations with a distributed delay. For some results we refer to [1], [2], [5], [6].

The present paper is an attempt to make a study of oscillatory properties of the differential equation of the form (1). Our study here is related to the book of Myškis [6] in which a systematic investigation of the properties of equation (1) is carried out.

2. Main theorems

The following lemma will be useful in the proof of the main result.

LEMMA 1. *Suppose that there exists a continuous nondecreasing function* $\alpha(t)$ *such that*

$$0 < \alpha(t) \le \sigma(t) \qquad \text{for} \quad t \ge t_0\,,$$

$$\liminf_{t\to\infty} \int\limits_{t-\alpha(t)}^{t} \big[r\big(s,\sigma(s)\big) - r\big(s,\alpha(s)\big)\big]\,\mathrm{d}s > 0 \tag{2}$$

and let $x(t)$ *be a nonoscillatory solution of equation* (1). *Then there exists a* $T_0 \ge t_0$ *such that* $x\big(t-\alpha(t)\big)/x(t)$ *is bounded on* $[T_0,\infty)$.

P r o o f. Without loss of generality we may assume that $x(t)$ is positive for all large t. Then in view of conditions (H_1)–(H_4), equation (1) and inequalities $t-\sigma(t) \le t-s \le t$, $x(t)$ must eventually be monotonically decreasing. We have

$$-\dot{x}(t) \ge \int\limits_{\alpha(t)}^{\sigma(t)} x(t-s)\,\mathrm{d}r(t,s) \ge \big[r\big(t,\sigma(t)\big) - r\big(t,\alpha(t)\big)\big]x\big(t-\alpha(t)\big) \tag{3}$$

for all large t. According to condition (2) there exists a $T_0 \ge t_0$ and $\varepsilon > 0$ such that

$$\int\limits_{t-\alpha(t)}^{t} \big[r\big(s,\sigma(s)\big) - r\big(s,\alpha(s)\big)\big]\,\mathrm{d}s \ge 2\varepsilon \qquad \text{for} \quad t \in [T_0,\infty)\,.$$

Then for arbitrary $t \geq T_0$ we can find a $t^\star > t$ such that

$$\int_{t^\star-\alpha(t^\star)}^{t} \left[r\big(s,\sigma(s)\big) - r\big(s,\alpha(s)\big)\right] \mathrm{d}s \geq \varepsilon$$

and

$$\int_{t}^{t^\star} \left[r\big(s,\sigma(s)\big) - r\big(s,\alpha(s)\big)\right] \mathrm{d}s \geq \varepsilon\,.$$

With regard to inequality (3) and decreasing character of $x(t)$ we get

$$\begin{aligned} x\big(t^\star - \alpha(t^\star)\big) - x(t) &\geq x\big(t-\alpha(t)\big) \int_{t^\star-\alpha(t^\star)}^{t} \left[r\big(s,\sigma(s)\big) - r\big(s,\alpha(s)\big)\right] \mathrm{d}s \\ &\geq \varepsilon x\big(t-\alpha(t)\big)\,, \end{aligned}$$

$$\begin{aligned} x(t) - x(t^\star) &\geq x\big(t^\star - \alpha(t^\star)\big) \int_{t}^{t^\star} \left[r\big(s,\sigma(s)\big) - r\big(s,\alpha(s)\big)\right] \mathrm{d}s \\ &\geq \varepsilon x\big(t^\star-\alpha(t^\star)\big)\,, \end{aligned}$$

$t \geq T_0$. Combining the above inequalities we obtain

$$x(t) > \varepsilon^2 x\big(t-\alpha(t)\big) \qquad \text{for} \quad t \in [T_0,\infty)\,.$$

The proof is complete. □

THEOREM 1. *Suppose that* (2) *holds and*

$$\limsup_{t\to\infty} \int_{t-\sigma(t)}^{t} r\big(s,\sigma(s)\big)\, \mathrm{d}s < \infty\,, \tag{4}$$

$$-\lambda + \liminf_{t\to\infty} \frac{\int\limits_0^{\sigma(t)} \exp\Big(\lambda \int\limits_{t-s}^{t} r\big(\xi,\sigma(\xi)\big)\,\mathrm{d}\xi\Big)\, \mathrm{d}r(t,s)}{r\big(t,\sigma(t)\big)} > 0 \tag{5}$$

for all $\lambda > 0$. *Then all solutions of equation* (1) *are oscillatory.*

P r o o f . Assume, for the sake of contradiction, that equation (1) has a solution x which is positive for all large t. Set

$$\Lambda = \left\{\lambda > 0 :\ \dot{x}(t) + \lambda r\big(t,\sigma(t)\big)x(t) \leq 0 \ \text{for all large}\ t\right\}.$$

By using the decreasing nature of x on $[t_1,\infty)$, where $t_1 \ge t_0$ is sufficiently large, it follows from equation (1) that

$$\dot{x}(t) + x(t) \int_0^{\sigma(t)} \mathrm{d}r(t,s) \le 0\,,$$

$$\dot{x}(t) + r\big(t,\sigma(t)\big)x(t) \le 0\,, \qquad t \ge t_1\,,$$

which means that $1 \in \Lambda$ and Λ is a subinterval of $(0,\infty)$. Next we will prove that $\sup\Lambda < \infty$. By Lemma 1, $x\big(t-\alpha(t)\big)/x(t)$ is bounded on $[T_0,\infty)$, $T_0 \ge t_1$. So

$$x\big(t-\alpha(t)\big)/x(t) \le c \qquad \text{for} \quad t \ge T_0\,, \tag{6}$$

where constant $c > 0$ can be choosen so large that $\exp(kc) > c$, where

$$0 < k \le \int_{t-\alpha(t)}^{t} \big[r\big(s,\sigma(s)\big) - r\big(s,\alpha(s)\big)\big]\,\mathrm{d}s \qquad \text{for} \quad t \ge t_2 \ge T_0\,.$$

We now claim that $\sup\Lambda \le c$. Otherwise $c \in \Lambda$ and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)\exp\left(c\int_{t_2}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right)\right]$$
$$= \big[\dot{x}(t) + cr\big(t,\sigma(t)\big)x(t)\big]\exp\left(c\int_{t_2}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right) \le 0\,,$$

since $c \in \Lambda$. This implies that the function

$$x(t)\exp\left(c\int_{t_2}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right)$$

is decreasing on $[t_2,\infty)$. Hence

$$x\big(t-\alpha(t)\big)\exp\left(c\int_{t_2}^{t-\alpha(t)} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right) \ge x(t)\exp\left(c\int_{t_2}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right),$$

$$x\big(t-\alpha(t)\big) \ge x(t)\exp\left(c\int_{t-\alpha(t)}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right)$$
$$\ge x(t)\exp(kc) > cx(t)\,, \qquad t \ge t_3 \ge t_2\,.$$

Then $x(t-\alpha(t)) > cx(t)$ for $t \geq t_3$, which is a contradiction to (6). Thus $\sup\Lambda \leq c < \infty$. Set $\lambda^\star = \sup\Lambda$ and let μ be an arbitrary number in the interval $(0, \lambda^\star)$. Then

$$\lambda^\star - \mu = \beta \in \Lambda$$

and there exists a $T_1 \geq t_2$ such that

$$\dot{x}(t) + \beta r(t, \sigma(t))x(t) \leq 0 \qquad \text{for} \quad t \geq T_1 .$$

So for any t, s with $t \geq T_1$ and $0 \leq s \leq \sigma(t)$ we have

$$\begin{aligned}\frac{x(t-s)}{x(t)} &= \exp\left(-\ln\frac{x(t)}{x(t-s)}\right) = \exp\left(-\int_{t-s}^{t}\frac{\dot{x}(\xi)}{x(\xi)}\,\mathrm{d}\xi\right)\\ &\geq \exp\left(\beta\int_{t-s}^{t} r(\xi, \sigma(\xi))\,\mathrm{d}\xi\right),\end{aligned}$$

that is

$$x(t-s) \geq x(t)\exp\left(\beta\int_{t-s}^{t} r(\xi, \sigma(\xi))\,\mathrm{d}\xi\right).$$

From equation (1) and the above inequality it follows that for $t \geq T_1$,

$$\begin{aligned}0 &\geq \dot{x}(t) + \left[\int_0^{\sigma(t)}\exp\left(\beta\int_{t-s}^{t} r(\xi,\sigma(\xi))\,\mathrm{d}\xi\right)\mathrm{d}r(t,s)\right]x(t)\\ &= \dot{x}(t) + \frac{\int_0^{\sigma(t)}\exp\left(\beta\int_{t-s}^{t} r(\xi,\sigma(\xi))\,\mathrm{d}\xi\right)\mathrm{d}r(t,s)}{r(t,\sigma(t))}\,r(t,\sigma(t))x(t).\end{aligned} \tag{7}$$

According to (7) we claim that

$$\liminf_{t\to\infty}\frac{\int_0^{\sigma(t)}\exp\left(\beta\int_{t-s}^{t} r(\xi,\sigma(\xi))\,\mathrm{d}\xi\right)\mathrm{d}r(t,s)}{r(t,\sigma(t))} \leq \lambda^\star . \tag{8}$$

Otherwise, there exist a $\lambda_1^\star > \lambda^\star$ and $T_2 \geq T_1$ such that

$$\frac{\int_0^{\sigma(t)}\exp\left(\beta\int_{t-s}^{t} r(\xi,\sigma(\xi))\,\mathrm{d}\xi\right)\mathrm{d}r(t,s)}{r(t,\sigma(t))} \geq \lambda_1^\star \qquad \text{for all} \quad t \geq T_2 ,$$

and therefore (7) yields that

$$0 \ge \dot{x}(t) + \lambda_1^\star r\bigl(t,\sigma(t)\bigr)x(t) \qquad \text{for} \quad t \ge T_2 .$$

Hence $\lambda_1^\star \in \Lambda$, which contradicts the hypothesis that $\lambda_1^\star > \lambda^\star$. Thus (8) has been established. Finally (8) implies that

$$-\lambda^\star + \liminf_{t\to\infty} \frac{\int\limits_0^{\sigma(t)} \exp\Bigl[(\lambda^\star - \mu) \int\limits_{t-s}^{t} r\bigl(\xi,\sigma(\xi)\bigr)\,\mathrm{d}\xi\Bigr]\,\mathrm{d}r(t,s)}{r\bigl(t,\sigma(t)\bigr)} \le 0 .$$

As $\mu \to 0$ and with regard to (4) we obtain

$$-\lambda^\star + \liminf_{t\to\infty} \frac{\int\limits_0^{\sigma(t)} \exp\Bigl(\lambda^\star \int\limits_{t-s}^{t} r\bigl(\xi,\sigma(\xi)\bigr)\,\mathrm{d}\xi\Bigr)\,\mathrm{d}r(t,s)}{r\bigl(t,\sigma(t)\bigr)} \le 0 ,$$

which contradicts (5) and completes the proof of the theorem. □

COROLLARY 1. *Suppose that* (2), (4) *hold and*

$$\liminf_{t\to\infty} \frac{\int\limits_0^{\sigma(t)} \int\limits_{t-s}^{t} r\bigl(\xi,\sigma(\xi)\bigr)\,\mathrm{d}\xi\,\mathrm{d}r(t,s)}{r\bigl(t,\sigma(t)\bigr)} > \frac{1}{e} .$$

Then all solutions of equation (1) *are oscillatory.*

P r o o f . Since

$$\min_{\lambda>0} \frac{1}{\lambda}\,\mathrm{e}^{\lambda a} = \mathrm{e}\,a , \qquad a > 0 ,$$

we have

$$\min_{\lambda>0} \frac{1}{\lambda} \exp\left(\lambda \int_{t-s}^{t} r\bigl(\xi,\sigma(\xi)\bigr)\,\mathrm{d}\xi\right) = \mathrm{e} \int_{t-s}^{t} r\bigl(\xi,\sigma(\xi)\bigr)\,\mathrm{d}\xi ,$$

$t \geq T_0$, $0 \leq s \leq \sigma(t)$. Then for $\lambda > 0$ we get

$$
\begin{aligned}
&-\lambda + \liminf_{t\to\infty} \frac{\int\limits_0^{\sigma(t)} \exp\Big(\lambda \int\limits_{t-s}^{t} r\big(\xi,\sigma(\xi)\big)\,\mathrm{d}\xi\Big)\,\mathrm{d}r(t,s)}{r\big(t,\sigma(t)\big)} \\
&= \lambda \left[-1 + \liminf_{t\to\infty} \frac{\int\limits_0^{\sigma(t)} \frac{1}{\lambda}\exp\Big(\lambda \int\limits_{t-s}^{t} r\big(\xi,\sigma(\xi)\big)\,\mathrm{d}\xi\Big)\,\mathrm{d}r(t,s)}{r\big(t,\sigma(t)\big)}\right] \\
&\geq \lambda \left[-1 + \mathrm{e}\liminf_{t\to\infty} \frac{\int\limits_0^{\sigma(t)} \int\limits_{t-s}^{t} r\big(\xi,\sigma(\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}r(t,s)}{r\big(t,\sigma(t)\big)}\right] > 0\,,
\end{aligned}
$$

so the condition (5) is satisfied and we can apply Theorem 1. □

LEMMA 2. *If v is a positive solution of the inequality*

$$\dot v(t) + \int\limits_0^{\sigma(t)} v(t-s)\,\mathrm{d}r(t,s) \leq 0\,, \tag{9}$$

then there exists a positive solution x of equation (1) *with*

$$x(t) \leq v(t) \qquad \text{for all large} \quad t\,.$$

P r o o f. Assume that v is a positive solution of (9) on $[t_1,\infty)$. Then

$$\dot v(t) \leq -\int\limits_0^{\sigma(t)} v(t-s)\,\mathrm{d}r(t,s)$$

and because of (H_1)–(H_4), we have

$$\dot v(t) < 0 \qquad \text{for} \quad t \geq T_0\,,$$

i.e. v is strictly decreasing on $[T_0,\infty)$, where $T_0 \geq t_1$ is sufficiently large. From (9) we obtain

$$v(t) \geq \int\limits_t^{\infty} \int\limits_0^{\sigma(u)} v(u-s)\,\mathrm{d}r(u,s)\,\mathrm{d}u\,, \qquad t \geq T_0\,.$$

We define the set $S \subset C_{\mathrm{loc}}\big([T_0,\infty),\mathbb{R}\big)$ of functions x which satisfy the inequalities

$$0 \le x(t) \le v(t) \qquad \text{for} \quad t \ge T_0 .$$

$F\colon S \to C_{\mathrm{loc}}\big([T_0,\infty),\mathbb{R}\big)$ is the operator which is defined by

$$F(x)(t) = \begin{cases} \int\limits_t^\infty \int\limits_0^{\sigma(u)} x(u-s)\, \mathrm{d}r(u,s)\, \mathrm{d}u & \text{for } t \ge T_1 , \\ v(t) - v(T_1) + F(x)(T_1) & \text{for } t \in [T_0, T_1) , \end{cases}$$

where $T_1 > T_0$ is such that $t - \sigma(t) \ge T_0$ for $t \ge T_1$.

If $x \in S$, we have

$$\begin{aligned} 0 \le F(x)(t) &= \int\limits_t^\infty \int\limits_0^{\sigma(u)} x(u-s)\, \mathrm{d}r(u,s)\, \mathrm{d}u \\ &\le \int\limits_t^\infty \int\limits_0^{\sigma(u)} v(u-s)\, \mathrm{d}r(u,s)\, \mathrm{d}u \le v(t) , \qquad t \ge T_1 . \end{aligned}$$

Thus $F(S) \subset S$. We note that S is a nonempty closed convex subset of $C_{\mathrm{loc}}\big([T_0,\infty),\mathbb{R}\big)$ and the operator F is continuous. The functions belonging to the set $F(S)$ are equicontinuous on every compact subinterval of $[T_0,\infty)$. Then according to the Schauder-Tychonoff fixed point theorem (cf., e.g. [3; p. 231]), F has an element $x \in S$ such that $x = F(x)$. It is easy to see that x satisfies equation (1) on $[T_1,\infty)$.

Now we show that x is positive on $[T_1,\infty)$. Obviously $v(t) > v(T_1)$ on $[T_0,T_1)$, x is nonnegative on $[T_1,\infty)$, $x(T_1) = F(x)(T_1) > 0$ and moreover from equation (1) it follows that x is decreasing on $[T_1,\infty)$. Let $T_2 \in (T_1,\infty)$ be the first point in which $x(T_2) = 0$. Then by equation (1) we have

$$\dot{x}(T_2) = - \int\limits_0^{\sigma(T_2)} x(T_2 - s)\, \mathrm{d}r(T_2,s) < 0 .$$

By decreasing character of x we always have $x = 0$ on $[T_2,\infty)$, which gives $\dot{x}(T_2) = 0$. This contradiction proves that x has no zeros on $[T_2,\infty)$ and so x is positive on $[T_1,\infty)$. The proof is complete. $\square$

THEOREM 2. *Suppose that for some $T_0 > t_0$,*

$$-\lambda + \sup_{t \ge T_0} \frac{\int\limits_0^{\sigma(t)} \exp\Big(\lambda \int\limits_{t-s}^{t} r\big(\xi,\sigma(\xi)\big)\, \mathrm{d}\xi\Big)\, \mathrm{d}r(t,s)}{r\big(t,\sigma(t)\big)} \le 0 \tag{10}$$

for some $\lambda > 0$. *Then there exists a positive solution* x *of equation* (1) *with*

$$x(t) \le \exp\left(-\lambda \int_{t_0}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right) \qquad \textit{for all large } t .$$

Proof. Set

$$v(t) = \exp\left(-\lambda \int_{t_0}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right), \qquad t \ge t_0 .$$

Then by condition (10) we obtain for $t \ge T_0$,

$$\begin{aligned}
&\dot v(t) + \int_0^{\sigma(t)} v(t-s)\,\mathrm{d}r(t,s)\\
&= -\lambda r\big(t,\sigma(t)\big)v(t) + \int_0^{\sigma(t)} \exp\left(-\lambda \int_{t_0}^{t-s} r\big(\xi,\sigma(\xi)\big)\,\mathrm{d}\xi\right)\mathrm{d}r(t,s)\\
&= v(t)\left[-\lambda r\big(t,\sigma(t)\big) + \int_0^{\sigma(t)} \exp\left(\lambda \int_{t-s}^{t} r\big(\xi,\sigma(\xi)\big)\,\mathrm{d}\xi\right)\mathrm{d}r(t,s)\right]\\
&\le r\big(t,\sigma(t)\big)v(t)\left[-\lambda + \sup_{t\ge T_0} \frac{\int_0^{\sigma(t)} \exp\left(\lambda \int_{t-s}^{t} r\big(\xi,\sigma(\xi)\big)\,\mathrm{d}\xi\right)\mathrm{d}r(t,s)}{r\big(t,\sigma(t)\big)}\right] \le 0 ,
\end{aligned}$$

and consequently v is a positive solution of inequality (9) on $[T_0,\infty)$. The conclusion of the theorem follows by applying Lemma 2. □

COROLLARY 2. *Suppose that for some* $T_0 > t_0$,

$$\sup_{t\ge T_0} \int_{t-\sigma(t)}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s \le \frac{1}{\mathrm{e}} . \tag{11}$$

Then there exists a positive solution x *of equation* (1) *with*

$$x(t) \le \exp\left(-\,\mathrm{e} \int_{t_0}^{t} r\big(s,\sigma(s)\big)\,\mathrm{d}s\right) \qquad \textit{for all large } t .$$

Proof. The condition (11) implies (10) with $\lambda = \mathrm{e}$. Hence we can apply Theorem 2. □

COROLLARY 3. *Consider the equation*

$$\dot{x}(t) + p\int_0^\sigma x(t-s)\,\mathrm{d}s = 0\,, \qquad t \ge t_0\,, \tag{12}$$

where $p \in (0,\infty)$, $\sigma \in (0,\infty)$. *All solutions of equation* (12) *are oscillatory if and only if*

$$\mathrm{e}^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1 > 0 \qquad \textit{for all} \quad \lambda > 0\,. \tag{13}$$

Proof. Assume that (13) holds. It is easy to see that conditions (2) and (4) are satisfied. Further for an arbitrary $\lambda > 0$ we get

$$-\lambda + \liminf_{t\to\infty} \frac{\int_0^\sigma \exp\Bigl(\lambda \int_{t-s}^t p\sigma\,\mathrm{d}\xi\Bigr)\,\mathrm{d}s}{\sigma} = -\lambda + \frac{1}{\sigma}\int_0^\sigma \exp(p\sigma\lambda s)\,\mathrm{d}s$$

$$= -\lambda + \frac{1}{p\sigma^2\lambda}\bigl(\mathrm{e}^{p\sigma^2\lambda} - 1\bigr) = \frac{1}{p\sigma^2\lambda}\bigl(\mathrm{e}^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1\bigr) > 0\,.$$

So condition (5) is fulfilled and we can apply Theorem 1.

Let us now suppose that

$$\mathrm{e}^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1 \le 0 \qquad \text{for some} \quad \lambda > 0\,.$$

We obtain

$$-\lambda + \sup_{t\ge\sigma} \frac{\int_0^\sigma \exp\Bigl(\lambda \int_{t-s}^t p\sigma\,\mathrm{d}\xi\Bigr)\,\mathrm{d}s}{\sigma} = \frac{1}{p\sigma^2\lambda}\bigl(\mathrm{e}^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1\bigr) \le 0\,.$$

Thus condition (10) is satisfied and Theorem 2 ensures the existence of a nonoscillatory solution of equation (12). This completes the proof. □

In the next remark we establish an explicit sufficient condition for all solutions of equation (12) to be oscillatory.

Remark. Suppose that

$$\mathrm{e}\,p\sigma^2 \ge 2\,. \tag{14}$$

Then all solutions of equation (12) are oscillatory.

To prove above assertion, consider function

$$G(\lambda) = \mathrm{e}^{a\lambda} - a\lambda^2 - 1\,, \qquad \text{where} \quad a = p\sigma^2\,,\ \lambda \ge 0\,.$$

We obtain $G(0) = 0$, $G'(\lambda) = a\bigl(\mathrm{e}^{a\lambda} - 2\lambda\bigr)$. Using the inequality $\mathrm{e}^{a\lambda} \ge a\,\mathrm{e}\,\lambda$ for $\lambda \ge 0$ and (14) we get

$$G'(\lambda) \ge a\lambda(a\,\mathrm{e} - 2) \ge 0 \qquad \text{for} \quad \lambda > 0 \ \text{and}\ G'(0) = a > 0\,.$$

The above relations implicate that

$$G(\lambda) > 0 \qquad \text{for} \quad \lambda > 0\,,$$

i.e. condition (13) is satisfied.

EXAMPLE. Consider the equation

$$\dot{x}(t) + \frac{1}{2} \int_0^{\pi} x(t-s)\, \mathrm{d}s = 0\,, \qquad t \geq \pi\,. \tag{15}$$

Since condition (14) is fulfilled, all solutions of equation (15) are oscillatory. For example such solutions are functions $\sin t$ and $\cos t$.

REFERENCES

[1] BAINOV, D.—MISHEV, D.: *Oscillation Theory for Neutral Differential Equations with Delay*, Adam Hilger, Briston-Philadelphia-New York, 1991.

[2] BAINOV, D.—PETROV, V.: *Asymptotic properties of the nonoscillatory solutions of second-order neutral equations with a deviating argument*, J. Math. Anal. Appl. **194** (1995), 343-351.

[3] EDWARDS, R. E.: *Functional Analysis*, Mir, Moscow, 1969 (Russian translation).

[4] GYÖRI, I.—LADAS, G.: *Oscillation Theory of Delay Differential Equations*, Clarendon Press, Oxford, 1991.

[5] LADDE, G. S.—LAKSHMIKANTHAM, V.—ZHANG, B. G.: *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York-Basel, 1987.

[6] MYŠKIS, A. D.: *Linear Differential Equations with a Retarded Argument*, Nauka, Moscow, 1972. (Russian)

[7] OHRISKA, J.: *Second order self-adjoint differential equation and oscillation.* In: Proceedings Inter. Scient. Conf. Math., Žilina, 1998, pp. 209–216.

[8] OHRISKA, J.: *Oscillation of second order delay and ordinary differential equation*, Czechoslovak Math. J. **34** (1984), 107–112.

[9] OLACH, R.: *Observation of a feedback mechanism in a population model*, Nonlinear Anal. **41** (2000), 539–544.

[10] PHILOS, CH. G.: *Oscillations of first order linear retarded differential equations*, J. Math. Anal. Appl. **157** (1991), 17–33.

[11] WERBOWSKI, J.: *Oscillations of first-order differential inequalities with deviating arguments*, Ann. Math. Pura Appl. **140** (1985), 383–392.

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