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UNBOUNDED OSCILLATION OF THE SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this paper is to present sufficient conditions for all unbounded solutions of the second order neutral differential equation

$$\big(x(t)+px(t-\tau)\big)''-q(t)x\big(\sigma(t)\big)=0$$

to be oscillatory. Some new sufficient conditions for unbounded oscillation are derived along with some comparison result.

Introduction

Let us consider the second order neutral differential equation of the form:

$$\big(x(t)+px(t-\tau)\big)''-q(t)x\big(\sigma(t)\big)=0\,,\qquad t\geqslant t_0\,, \tag{1}$$

under the assumption:

(i) $p\geqslant 0$ and $\tau>0$ are constants;
(ii) $\sigma\in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $\sigma(t)>t$, $\sigma'(t)>0$;
(iii) $q\in C(\mathbb{R}_+,\mathbb{R}_+)$.

In what follows, by a *solution* of (1) is meant a continuous function $x(t)$ defined on an interval $[t_x,\infty)$, $t_x\geq t_0$, such that $x(t)+px(t-\tau)$ is a twice continuously differentiable and $x(t)$ satisfies (1) for all sufficiently large t and $\sup\{|x(t)|:\ t\geq T\}>0$ for any $T\geq t_x$. Such a solution is called *oscillatory* if it has a sequence of zeros tending to infinity; otherwise it is called *nonoscillatory*. Equation (1) is said to be *oscillatory* if all its solutions are oscillatory.

In the past two decades research on the oscillation theory of differential equations of neutral types has attracted the attention of many authors. The problems of asymptotic and oscillatory behavior of solutions of neutral differential

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equations is of both theoretical and practical interest. For recent contributions regarding the theoretical part and providing the systematic treatment of oscillation and nonoscillation theory of neutral differential equations we refer to the books by D. D. Bainov and D. P. Mishev [1], by I. Győri and G. Ladas [8], and by L. H. Erbe, Q. Kong and B. G. Zhang [6]. The books summarize some important work in this area and the main effort is devoted to the first and second order differential equations. The results presented in the mentioned books cover in sufficient way the theory of equations involving delayed arguments ($\sigma(t) < t$). However the known results touch only little upon advanced types of neutral equations (see [6], [7] and [11]).

The aim of this paper is to establish unbounded oscillation criteria for the second order neutral differential equations of unstable type.

For the corresponding bounded oscillation criteria for the second order delay differential equations of the form

$$\big(x(t) - px(t-\tau)\big)'' - q(t)x\big(\sigma(t)\big) = 0$$

the reader is referred to [5] and [6].

In this paper all functional inequalities that we write are assumed to hold eventually, that is, for sufficiently large t.

Main results

In the sequel we shall use the following function

$$r(t) = \sigma^{-1}\big[\sigma(t) - \tau\big]\,, \tag{2}$$

where $\sigma^{-1}(t)$ is the inverse function to $\sigma(t)$. It is easy to see that $r(t) < t$ and $r'(t) > 0$. We begin with the following unbounded oscillation criterion:

THEOREM 1. *Let $r(t)$ be defined by (2) and $r''(t) \geq 0$. Assume that*

$$\limsup_{t\to\infty} \int\limits_t^{\sigma(t)} \big(\sigma(t) - s\big)q^*(s)\,\mathrm{d}s > 1 + p\,, \tag{3}$$

where $q^*(t) = \min\big\{q(t), q[r(t)](r'(t))^2\big\}$. Then every unbounded solution of equation (1) is oscillatory.

Proof. Assume for the sake of contradiction, that equation (1) has an eventually positive unbounded solution $x(t)$. That is, there exists a $t_0 \geqslant 0$ such that $x(t) > 0$ for $t \geqslant t_0$. Set

$$z(t) = x(t) + px(t-\tau)\,.$$

Then $z(t) > 0$ for $t \geqslant t_1 = t_0 + \tau$, $z(t)$ is unbounded and

$$z''(t) = q(t)x[\sigma(t)] .$$

Thus $z''(t) > 0$ and this implies that $z'(t)$ is of constant sign. But if we admit $z'(t) < 0$, then $z(t)$ would be bounded. Therefore, $z'(t) > 0$ for $t \geqslant t_2 \geqslant t_1$. We put

$$y(t) = z(t) + pz[r(t)] .$$

Now, using properties of $z(t)$ and $r(t)$, we see that $y(t) > 0$, $y'(t) > 0$ for all large t and moreover

$$\begin{aligned} y''(t) &= z''(t) + pz''[r(t)]\,(r'(t))^2 + pz'[r(t)]\,r''(t) \\ &\geqslant q(t)x(\sigma(t)) + pq[r(t)]\,(r'(t))^2 x[\sigma(t) - \tau] \\ &\geqslant q^*(t)\big[x(\sigma(t)) + px(\sigma(t) - \tau)\big] \\ &= q^*(t)z(\sigma(t)) \\ &\geqslant \frac{q^*(t)}{1+p}\,[z(\sigma(t)) + pz[r(\sigma(t))]] \\ &= \frac{q^*(t)}{1+p}y(\sigma(t)) . \end{aligned}$$

Hence, $y(t)$ is a positive solution of the differential inequality

$$y''(t) \geqslant \frac{q^*(t)}{1+p}y(\sigma(t)) , \qquad t \geqslant t_3 . \tag{4}$$

An integration of (4) from t to u yields

$$y'(u) - y'(t) \geqslant \int_t^u \frac{q^*(s)}{1+p}y(\sigma(s))\,\mathrm{d}s .$$

Now integrating in u from t to $\sigma(t)$ we are lead to

$$y(\sigma(t)) - y(t) - y'(t)(\sigma(t) - t) \geqslant \int_t^{\sigma(t)} (\sigma(t) - s)\frac{q^*(s)}{1+p}y(\sigma(s))\,\mathrm{d}s .$$

Consequently, using the monotonicity of $y(t)$ one gets

$$0 > y(\sigma(t)) \left[\int_t^{\sigma(t)} (\sigma(t) - s)\frac{q^*(s)}{1+p}\,\mathrm{d}s - 1 \right] .$$

We have arrived at a contradiction with (3). The proof is complete now. □

Remark 1. For differential equation without neutral term ($p = 0$) Theorem 1 provides the well-known result of Chanturia and Koplatadze [2].

EXAMPLE 1. Consider the following neutral differential equation of the form

$$\big(x(t)+px(t-\tau)\big)'' - \frac{1}{t^2}x\big(\lambda^2 t\big) = 0\,, \qquad \lambda > 1\,,\ \ t \geqslant 1\,. \tag{5}$$

Now $r(t) = t-\tau/\lambda^2$ and $q^*(t) = 1/t^2$. Simple computation shows that condition (3) for equation (5) reduces to

$$\lambda^2 - 2\ln\lambda - 2 - p > 0\,. \tag{6}$$

A natural question arises what happen if (3) is violated. The following comparison theorem provides the partial answer.

THEOREM 2. *Let $r(t)$ and $q^*(t)$ be the same as in Theorem 1. Assume that there exists a couple of the functions*

$$a \in C^1\big((t_0,\infty)\big)\,, \qquad a(t) > 0\,,\ \ a'(t) \leqslant 0\,, \tag{7}$$

$$\beta \in C^1\big((t_0,\infty)\big)\,, \qquad \beta(t) > t\,,\ \ \beta'(t) > 0\,, \tag{8}$$

such that

$$\sigma(t) \geqslant \beta\big(\beta(t)\big)\,, \tag{9}$$

$$q^*(t) \geqslant (1+p)a(t)a\big(\beta(t)\big)\beta'(t)\,. \tag{10}$$

If the first order differential inequality

$$v'(t) - a\big(\beta(t)\big)\beta'(t)v\big(\beta(t)\big) \geqslant 0 \tag{11}$$

has no eventually positive solutions, then all unbounded solutions of (1) *are oscillatory.*

Proof. Assume that $x(t) > 0$ is an unbounded solution of (1). Let $z(t)$ and $y(t)$ be the same functions as in the proof of Theorem 1. Then proceeding exactly as in the proof of Theorem 1 we are led to (4). We put

$$b(t) = y'(t) + a(t)y\big(\beta(t)\big)\,.$$

Then $b(t)$ is positive and

$$\begin{aligned} &b'(t) - a(t)\beta'(t)b\big(\beta(t)\big) \\ &\qquad = y''(t) + a'(t)y\big(\beta(t)\big) - a(t)a\big(\beta(t)\big)\beta'(t)y\big(\beta\big(\beta(t)\big)\big)\,. \end{aligned}$$

Hence, taking into account (4), (7), (8), (9), (10) one gets for all large t

$$\begin{aligned} b'(t) - \frac{a'(t)}{a(t)}b(t) - a(t)\beta'(t)b\big(\beta(t)\big) \\ &= y''(t) - \frac{a'(t)}{a(t)}y'(t) - a(t)a\big(\beta(t)\big)\beta'(t)y\big(\beta\big(\beta(t)\big)\big) \\ &\geqslant y''(t) - a(t)a\big(\beta(t)\big)\beta'(t)y\big(\beta\big(\beta(t)\big)\big) \\ &\geqslant y''(t) - \frac{q^*(t)}{1+p}y\big(\sigma(t)\big) \geqslant 0\,. \end{aligned}$$

Setting

$$b(t) = a(t)v(t)$$

we conclude that $v(t)$ is a positive solution of (11). This is a contradiction and the proof is complete. □

COROLLARY 1. *Let $r(t)$ and $q^*(t)$ be the same as in Theorem 1. Assume that (7) - (10) hold. If*

$$\liminf_{t\to\infty} \int_t^{\beta(t)} a(\beta(s))\beta'(s)\, ds > \frac{1}{e}, \tag{12}$$

then (1) does not allow unbounded nonoscillatory solutions.

Proof. It is known (see [10; Theorem 2.4.1]) that (12) is sufficient for (11) to have no eventually positive solutions. So, the assertion of this corollary follows from Theorem 2. □

Remark 2. Theorem 2 permits to apply any sufficient condition for absence of eventually positive solutions of (11) to get unbounded oscillation criterion for equation (1) provided that (7) - (10) hold.

We have established the relationship between unbounded oscillation of the second order neutral equation of unstable type and the absence of an eventually positive solution of the first order differential inequality without neutral term. The following example is intended to show that Theorems 1 and 2 are independent.

EXAMPLE 2. We consider once more neutral differential equation (5). We let $a(t) = \frac{1}{t\sqrt{1+p}}$ and $\beta(t) = \lambda t$. Then all conditions of Corollary 1 are satisfied and (12) takes the form

$$\lambda > e^{\frac{\sqrt{1+p}}{e}}. \tag{13}$$

Thus (13) is sufficient for unbounded oscillation of (5).

Remark 3. For equation (5) we shall compare unbounded oscillation criteria (6) and (13). It is easy to verify that for $p = 0.5$ conditions (6) and (13) are satisfied for $\lambda > 1.962$ and $\lambda > 1.57$, respectively. On the other hand, for $p = 24$ conditions (6) and (13) are satisfied for $\lambda > 5.42$ and $\lambda > 6.29$, respectively.

So we conclude that neither Corollary 1 is included in Theorem 1 nor Theorem 1 in Corollary 1.

The following results provides unbounded oscillation criteria for special types of equation (1).

COROLLARY 2. *Let $r(t)$ and $q^*(t)$ be the same as in Theorem 1. If $\sigma > 0$ and $q^*(t) \geq \frac{(2+\varepsilon)^2}{\sigma^2 \mathrm{e}^2}(1+p)$ for some $\varepsilon > 0$, then all unbounded solutions of*

$$\big(x(t) + px(t-\tau)\big)'' - q(t)x(t+\sigma) = 0$$

are oscillatory.

Proof. We set $a(t) = \frac{2+\varepsilon}{\sigma \mathrm{e}}$ and $\beta(t) = t + \frac{\sigma}{2}$ in Corollary 1. □

COROLLARY 3. *If $\sigma > 0$ and $q > \frac{4}{\sigma^2 \mathrm{e}^2}(1+p)$, then all unbounded solutions of*

$$\big(x(t) + px(t-\tau)\big)'' - qx(t+\sigma) = 0 \tag{14}$$

are oscillatory.

Proof. It is easy to see that $q^*(t) = q$. Since $q > \frac{4}{\sigma^2 \mathrm{e}^2}(1+p)$, there exists $\varepsilon > 0$ such that $q > \frac{(2+\varepsilon)^2}{\sigma^2 \mathrm{e}^2}(1+p)$. Hence we put $a(t) = \frac{2+\varepsilon}{\sigma \mathrm{e}}$ and $\beta(t) = t + \frac{\sigma}{2}$ in Corollary 1. □

Remark 4. For differential equation (14) we get the same result as Grace in [5].

COROLLARY 4. *Let $r(t)$ and $q^*(t)$ be the same as in Theorem 1. Suppose that $q^*(t) \geq \frac{(1+p)}{t^2}$ and $\lambda > \mathrm{e}^{\frac{1}{\mathrm{e}}}$. Then all unbounded solutions of*

$$\big(x(t) + px(t-\tau)\big)'' - q(t)x\big(\lambda^2 t\big) = 0$$

are oscillatory.

Proof. Put $a(t) = \frac{1}{t}$ and $\beta(t) = \lambda t$ in Corollary 1. □

In this paper we have established two independent results for unbounded oscillation of delay-advanced differential equations of neutral type. Recently the advanced equations became a discussed theme in qualitative theory of functional differential equations. This paper complements the recent works in this direction [2], [3], [7]. Moreover the presented results complement and generalize those in the well-known monograph by T. A. Chanturia and R. G. Koplatadze [2].

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