Ján Gatial; Peter Kaprálik

On median point of the system of elements of A-structure

Mathematica Slovaca, Vol. 51 (2001), No. 3, 275--280

Math. Slovaca, 51 (2001), No. 3, 275–280

Mathematica Slovaca
© 2001
Mathematical Institute
Slovak Academy of Sciences

ON MEDIAN POINT OF THE SYSTEM OF ELEMENTS OF A-STRUCTURE

JÁN GATIAL — PETER KAPRÁLIK

(Communicated by Tibor Katriňák)

ABSTRACT. A generalization of the property of affine spaces that can be deduced from term "midpoint of the line segment" has been done in the paper [GATIAL, J.: *Some geometrical examples of an IMC-quasigroup*, Mat. Časopis **19** (1969), 292–298]. The main attention in this study was concentrated on automorphisms of an idempotent, medial and commutative quasigroup, which we call A-structure. In this paper we study a possibility of using a median point of the system of elements of A-structure.

For binary operation on A-structure (Q,τ) we will use the symbol $\cdot$, that is, instead of $\tau(x,y)$ we will write $x \cdot y$.

We note that a quasigroup $(Q,\cdot)$ is *medial* if all elements $a,b,c,d \in Q$ satisfy the next property

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d).$$

DEFINITION 1. Ordered k-tuple $(a_1, a_2, \dots, a_k) \in Q^k$, where $(Q,\cdot)$ is A-structure, will be called *k-gon* and $a_1, a_2, \dots, a_k$ are its *vertices*.

In affine space A^n, the *median point* of k-gon $B = (b_1, b_2, \dots, b_k)$ is a point $T(B) \in A^n$ for which

$$T(B) = \frac{b_1 + b_2 + \cdots + b_k}{k}.$$

Note. A median point of a triangle $B = (b_1, b_2, b_3)$ in an A-structure $(Q,\cdot)$ can be defined as the element $T(B) \in Q$ for which $T(B) = (b_1 \cdot b_2) \cdot \big(b_3 \cdot T(B)\big)$ (see [6]) and a median point of quadrangle $B = (b_1, b_2, b_3, b_4) \in Q^4$ as the element $T(B) \in Q$ for which $T(B) = (b_1 \cdot b_2) \cdot (b_3 \cdot b_4)$.

2000 Mathematics Subject Classification: Primary 20N05; Secondary 51E99.
Key words: quasigroup, idempotent and medial quasigroup, A-structure, median point of k-gon.

PROPOSITION 1. *For each triangle* $(b_1, b_2, b_3) \in Q^3$ *and an arbitrary permutation* (i_1, i_2, i_3) *of numbers* $1, 2, 3$, $T(b_{i_1}, b_{i_2}, b_{i_3}) = T(b_1, b_2, b_3)$.

Proof. It is deducted from medial and commutative operation $\cdot$. □

PROPOSITION 2. *For each quadrangle* $(b_1, b_2, b_3, b_4) \in Q^4$ *and an arbitrary permutation* (i_1, i_2, i_3, i_4) *of numbers* $1, 2, 3, 4$, $T(b_{i_1}, b_{i_2}, b_{i_3}, b_{i_4}) = T(b_1, b_2, b_3, b_4)$.

Proof. It is deducted from medial and commutative operation $\cdot$ (see the proof of Theorem 1). □

A median point of each quadrangle $(b_1, b_2, b_3, b_4) \in Q^4$ always exists and is unambiguously determined. The existence and unambiguousness of the median point of a triangle are not guaranteed (see [6]).

In an affine space A^n, for median point t of each pentagon $(b_1, b_2, b_3, b_4, b_5)$, it holds that

$$[(b_1 \cdot b_2) \cdot (b_3 \cdot b_4)] \cdot [(b_5 \cdot t)(t \cdot t)] = t .$$

The existence and unambiguousness of median point of a pentagon in an A-structure are not guaranteed. For example, in the A-structure $(Q, \cdot)$ where $Q = \{a, b, c, d, e\}$ and operation $\cdot$ is defined by the table

$\cdot$	a	b	c	d	e
a	a	d	e	c	b
b	d	b	a	e	c
c	e	a	c	b	d
d	c	e	b	d	a
e	b	c	d	a	e

median point of the pentagon (a, b, c, d, e) is each element from Q. The pentagon (a, b, c, d, d) has no median point.

The property of median point of $2k$-gon in affine space:

$$T(a_1, \dots, a_k, b_1, \dots, b_k) = \frac{T(a_1, \dots, a_k) + T(b_1, \dots, b_k)}{2} = T(a_1, \dots, a_k) \cdot T(b_1, \dots, b_k)$$

makes it possible to define a median point of 2^k-gon in any A-structure.

DEFINITION 2. A *median point of* 2^k-gon $(b_1, \dots, b_{2^k}) \in Q^{2^k}$, $k \in \mathbb{N}$, is a point $T(b_1, \dots, b_{2^k}) \in Q$ defined by:

$$T(b_1) = b_1 ,$$

$$T(b_1, \dots, b_{2^k}, b_{2^k+1}, \dots, b_{2^{k+1}}) = T(b_1, \dots, b_{2^k}) \cdot T(b_{2^k+1}, \dots, b_{2^{k+1}}) .$$

THEOREM 1. *For each $k \in \mathbb{N}$, each 2^k-gon $(b_1,\dots,b_{2^k}) \in Q^{2^k}$, and each permutation $(i_1,\dots,i_{2^k})$ of numbers $1,\dots,2^k$ we have*

$$T(b_{i_1},\dots,b_{i_{2^k}}) = T(b_1,\dots,b_{2^k}).$$

P r o o f. It is enough to prove that for each $i,j \in \{1,\dots,2^k\}$, $i<j$,

$$\begin{aligned}&T(b_1,\dots,b_{i-1},b_i,b_{i+1},\dots,b_{j-1},b_j,b_{j+1},\dots,b_{2^k})\\ &\qquad = T(b_1,\dots,b_{i-1},b_j,b_{i+1},\dots,b_{j-1},b_i,b_{j+1},\dots,b_{2^k}).\end{aligned}$$

For $k=1$ we have

$$T(b_1,b_2) = b_1 \cdot b_2 = b_2 \cdot b_1 = T(b_2,b_1).$$

For $i,j \in \{1,\dots,2^{k+1}\}$, $i<j$, the following cases can occur:

(1) $i,j \in \{1,\dots,2^k\}$,
(2) $i,j \in \{2^k+1,\dots,2^{k+1}\}$,
(3) $i \in \{1,\dots,2^k\}$, $j \in \{2^k+1,\dots,2^{k+1}\}$.

In case (1), there holds

$$\begin{aligned}&T(b_1,\dots,b_{i-1},b_i,b_{i+1},\dots,b_{j-1},b_j,b_{j+1},\dots,b_{2^k},b_{2^k+1},\dots,b_{2^{k+1}})\\ &= T(b_1,\dots,b_{i-1},b_i,b_{i+1},\dots,b_{j-1},b_j,b_{j+1},\dots,b_{2^k}) \cdot T(b_{2^k+1},\dots,b_{2^{k+1}})\\ &= T(b_1,\dots,b_{i-1},b_j,b_{i+1},\dots,b_{j-1},b_i,b_{j+1},\dots,b_{2^k}) \cdot T(b_{2^k+1},\dots,b_{2^{k+1}})\\ &= T(b_1,\dots,b_{i-1},b_j,b_{i+1},\dots,b_{j-1},b_i,b_{j+1},\dots,b_{2^k},b_{2^k+1},\dots,b_{2^{k+1}}).\end{aligned}$$

Case (2) can be proved in an analogous way as case (1). In case (3) we can assume that $i=1$, $j=2^{k+1}$, so

$$\begin{aligned}&T(b_1,\dots,b_{2^{k+1}})\\ &= [T(b_1,\dots,b_{2^{k-1}}) \cdot T(b_{2^{k-1}+1},\dots,b_{2^k})] \cdot [T(b_{2^k+1},\dots,b_{3.2^{k-1}}) \cdot T(b_{3.2^{k-1}+1},\dots,b_{2^{k+1}})]\\ &= [T(b_1,\dots,b_{2^{k-1}}) \cdot T(b_{3.2^{k-1}+1},\dots,b_{2^{k+1}})] \cdot [T(b_{2^{k-1}+1},\dots,b_{2^k}) \cdot T(b_{2^k+1},\dots,b_{3.2^{k-1}})]\\ &= T(b_1,\dots,b_{2^{k-1}},b_{3.2^{k-1}+1},\dots,b_{2^{k+1}}) \cdot T(b_{2^{k-1}+1},\dots,b_{2^k},b_{2^k+1},\dots,b_{3.2^{k-1}})\\ &= T(b_{2^{k+1}},b_2,\dots,b_{2^{k-1}},b_{3.2^{k-1}+1},\dots,b_{2^{k+1}-1},b_1) \cdot T(b_{2^{k-1}+1},\dots,b_{2^k},b_{2^k+1},\dots,b_{3.2^{k-1}})\\ &= [T(b_{2^{k+1}},b_2,\dots,b_{2^{k-1}}) \cdot T(b_{3.2^{k-1}+1},\dots,b_{2^{k+1}-1},b_1)] \cdot\\ &\quad \cdot [T(b_{2^{k-1}+1},\dots,b_{2^k}) \cdot T(b_{2^k+1},\dots,b_{3.2^{k-1}})]\\ &= [T(b_{2^{k+1}},b_2,\dots,b_{2^{k-1}}) \cdot T(b_{2^{k-1}+1},\dots,b_{2^k})] \cdot\\ &\quad \cdot [T(b_{2^k+1},\dots,b_{3.2^{k-1}}) \cdot T(b_{3.2^{k-1}+1},\dots,b_{2^{k+1}-1},b_1)]\\ &= T(b_{2^{k+1}},b_2,\dots,b_{2^{k+1}-1},b_1).\end{aligned}$$

□

DEFINITION 3. For each $a, b \in Q$ and $k \in \mathbb{N}$, $a^k b$ is defined by:

$$a^0 b = b\,,$$
$$a^{k+1} b = a \cdot (a^k b)\,.$$

PROPOSITION 3. *Let* $(b_1, \dots, b_{2^k}) \in Q^{2^k}$, $a \in Q$, $k, m \in \mathbb{N}$, $k > 0$. *Then for a median point of* 2^{k+m}*-gon* $\Big(b_1, \dots, b_{2^k}, \underbrace{a, \dots, a}_{2^{k+m}-2^k}\Big)$

$$T\Big(b_1, \dots, b_{2^k}, \underbrace{a, \dots, a}_{2^{k+m}-2^k}\Big) = a^m T(b_1, \dots, b_{2^k})\,.$$

Proof. (By mathematical induction.) For $m = 0$ the statement evidently holds. Let us suppose that it holds for $m - 1$. Then

$$\begin{aligned} T\Big(b_1, \dots, b_{2^k}, \underbrace{a, \dots, a}_{2^{k+m}-2^k}\Big) &= T\Big(b_1, \dots, b_{2^k}, \underbrace{a, \dots, a}_{2^{k+m-1}-2^k}\Big) \cdot T\Big(\underbrace{a, \dots, a}_{2^{k+m-1}}\Big) \\ &= \big[a^{m-1} T(b_1, \dots, b_{2^k})\big] \cdot a = a^m T(b_1, \dots, b_{2^k})\,. \end{aligned}$$

□

PROPOSITION 4. *Let* $k = 2^{k_1} + 2^{k_2}$, *where* $k_1, k_2 \in \mathbb{N}$, $0 < k_1 < k_2$, *let* $a \in Q$, *and* $(b_1^i, \dots, b_{2^{k_i}}^i) \in Q^{2^{k_i}}$ *for* $i \in \{1, 2\}$. *Then*

$$\begin{aligned} &T\Big(b_1^1, \dots, b_{2^{k_1}}^1, b_1^2, \dots, b_{2^{k_2}}^2, \underbrace{a, \dots, a}_{2^{k_2}-2^{k_1}}\Big) \\ &\qquad = \Big[a^{k_2-k_1} T(b_1^1, \dots, b_{2^{k_1}}^1)\Big] \cdot T(b_1^2, \dots, b_{2^{k_2}}^2)\,. \end{aligned}$$

Proof. From Proposition 3 and the definition of a median point, it can be deduced that

$$\begin{aligned} &\Big[a^{k_2-k_1} T(b_1^1, \dots, b_{2^{k_1}}^1)\Big] \cdot T(b_1^2, \dots, b_{2^{k_2}}^2) \\ &\qquad = T\Big(b_1^1, \dots, b_{2^{k_1}}^1, \underbrace{a, \dots, a}_{2^{k_2}-2^{k_1}}\Big) \cdot T(b_1^2, \dots, b_{2^{k_2}}^2) \\ &\qquad = T\Big(b_1^1, \dots, b_{2^{k_1}}^1, \underbrace{a, \dots, a}_{2^{k_2}-2^{k_1}}, b_1^2, \dots, b_{2^{k_2}}^2\Big) \\ &\qquad = T\Big(b_1^1, \dots, b_{2^{k_1}}^1, b_1^2, \dots, b_{2^{k_2}}^2, \underbrace{a, \dots, a}_{2^{k_2}-2^{k_1}}\Big)\,. \end{aligned}$$

□

DEFINITION 4. Let $k, m \in \mathbb{N}$, $2^m < k \le 2^{m+1}$. Then a *median point of* k-gon $(b_1, \dots, b_k) \in Q^k$ is the point $t \in Q$ satisfying the condition

$$t = T\Big(b_1, \dots, b_k, \underbrace{t, \dots, t}_{2^{m+1}-k}\Big).$$

THEOREM 2. *For each k-gon $(b_1, \dots, b_k) \in Q^k$ and each permutation $(i_1, \dots \dots, i_k)$ of numbers $1, \dots, k$ we have*

$$T(b_{i_1}, \dots, b_{i_k}) = T(b_1, \dots, b_k).$$

Proof. A median point of k-gon is defined with help of median point of 2^{m+1}-gon that does not depend on the order of the vertices. □

Definition 4 is an extension of the term a median point in an affine space A^n to any A-structure because in A^n,

$$t = T\Big(b_1, \dots, b_k, \underbrace{t, \dots, t}_{2^{m+1}-k}\Big) = \frac{b_1 + \dots + b_k + (2^{m+1} - k)t}{2^{m+1}},$$

from where

$$t = \frac{b_1 + \dots + b_k}{k}.$$

The existence and definiteness of median point of k-gon are not guaranteed. For example, in the A-structure $(\mathbb{Z}(2), \cdot)$, where $\mathbb{Z}(2) = \{\frac{p}{2^m} : p \in \mathbb{Z},\ m \in \mathbb{N}\}$ and $x \cdot y = \frac{x+y}{2}$, number 5 is the median point of the 9-gon $(1,2,3,4,5,6,7,8,9)$, but the median point of the 9-gon $(1,2,3,4,5,6,7,8,10)$ does not exist.

REFERENCES

[1] BELOUSOV, V. D.: *Systems of quasigroups with generalized identities*, Uspekhi Mat. Nauk **20** (1965), 75–146. (Russian)

[2] BELOUSOV, V. D.: *Osnovy teorii kvazigrupp i lup*, Nauka, Moskva, 1967. (Russian)

[3] DUPLÁK, J.: *On some properties of transitive quasigroups*, Zborník PdF UPJŠ **15** (1976), 29–35. (Slovak)

[4] DUPLÁK, J.: *On quasiidenteties of transitive quasigroups*, Math. Slovaca **34** (1984), 281–294.

[5] GATIAL, J.: *Some geometrical examples of an IMC-quasigroup*, Mat. Časopis **19** (1969), 292 298.

[6] GATIAL, J.: *Über die IMC-Quasigruppe und den Schwerpunkt eines Dreieckes*, Math. Nachr. **53** (1972), 119–123.

[7] GATIAL, J.—KAPRÁLIK, P.: *On some affine structures.* In: Proceedings of Seminars on Computational Geometry SCG'98, Vol. 7, Slov. Tech. Univ., Bratislava, 1998, pp. 41–44.

Received July 22, 1998
Revised April 15, 2000

Department of Mathematics
Faculty of Electrical Engineering
and Information Technology
Slovak Technical University
Ilkovičova 3
SK-812 19 Bratislava
SLOVAKIA
E-mail: kapralik@kmat.elf.stuba.sk