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Young Bae Jun; Hee Sik Kim
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ON FUZZY TOPOLOGICAL d-ALGEBRAS

YOUNG BAE JUN* — HEE SIK KIM**

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ABSTRACT. In this paper we introduce the concept of fuzzy topological d-algebras and apply some of Foster's results on homomorphic images and inverse images to fuzzy topological d-algebras.

1. Introduction

Y. Imai and K. Iséki [4] and K. Iséki [5] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2], [3], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [11] introduced a new notion, called a d-algebra, which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. In [7], Y. B. Jun, J. Neggers and H. S. Kim introduced the notions of fuzzy d-subalgebra, fuzzy d-ideal, fuzzy $d^\sharp$-ideal and fuzzy d^*-ideal, and investigated relations among them. The concept of a fuzzy set, which was introduced in [13], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D. H. Foster (cf. [1]) combined the structure of a fuzzy topological spaces with that of a fuzzy group, introduced by A. Rosenfeld (cf. [12]), to formulate the elements of a theory of fuzzy topological groups. In 1993, Y. B. Jun [6] combined the structure of a fuzzy topological spaces with that of a fuzzy BCK-algebras to formulate the elements of a theory of fuzzy topological BCK-algebras. In the present paper, we introduce the concept of fuzzy topological d-algebras and apply some of Foster's results on homomorphic images and inverse images to fuzzy topological d-algebras.

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2. Preliminaries

A *d-algebra* ([11]) is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y, z in X.

A non-empty subset N of a d-algebra X is called a *d-subalgebra* of X if $x * y \in N$ for any $x, y \in N$.

A mapping $\alpha\colon X \to Y$ of d-algebras is called a *d-homomorphism* if $\alpha(x*y) = \alpha(x) * \alpha(y)$ for all $x, y \in X$.

We now review some fuzzy logic concepts (see [1] and [13]). Let X be a set.

A *fuzzy set* A in X is characterized by a membership function $\mu_A\colon X \to [0,1]$. Let α be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B.

The *inverse image* of B, denoted $\alpha^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{\alpha^{-1}(B)}$ defined by $\mu_{\alpha^{-1}(B)}(x) = \mu_B\big(\alpha(x)\big)$ for all $x \in X$. Conversely, let A be a fuzzy set in X with membership function μ_A. Then the *image* of A, denoted by $\alpha(A)$, is the fuzzy set in Y such that

$$\mu_{\alpha(A)}(y) = \begin{cases} \sup\limits_{z\in\alpha^{-1}(y)} \mu_A(z) & \text{if } \alpha^{-1}(y) = \{x :\ \alpha(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

A *fuzzy topology* on a set X is a family $\mathcal{T}$ of fuzzy sets in X which satisfies the following conditions:

(i) for all $c \in [0,1]$, $k_c \in \mathcal{T}$, where k_c has a constant membership function,
(ii) if $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
(iii) if $A_j \in \mathcal{T}$ for all $j \in J$, then $\bigcup\limits_{j\in J} A_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space* and members of $\mathcal{T}$ are called *open fuzzy sets*.

Let A be a fuzzy set in X and $\mathcal{T}$ a fuzzy topology on X. Then the *induced fuzzy topology* on A is the family of fuzzy subsets of A which are the intersection with A of $\mathcal{T}$-open fuzzy sets in X. The induced fuzzy topology is denoted by $\mathcal{T}_A$, and the pair $(A, \mathcal{T}_A)$ is called a *fuzzy subspace* of $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two fuzzy topological spaces. A mapping α of $(X, \mathcal{T})$ into $(Y, \mathcal{U})$ is *fuzzy continuous* if for each open fuzzy set U in $\mathcal{U}$ the inverse image $\alpha^{-1}(U)$ is in $\mathcal{T}$. Conversely, α is *fuzzy open* if for each open fuzzy set V in $\mathcal{T}$, the image $\alpha(V)$ is in $\mathcal{U}$.

Let $(A,\mathcal{T}_A)$ and $(B,\mathcal{U}_B)$ be fuzzy subspaces of fuzzy topological spaces $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ respectively, and let α be a mapping from $(X,\mathcal{T})$ to $(Y,\mathcal{U})$. Then α is a mapping of $(A,\mathcal{T}_A)$ *into* $(B,\mathcal{U}_B)$ if $\alpha(A)\subseteq B$. Furthermore α is *relatively fuzzy continuous* if for each open fuzzy set V' in $\mathcal{U}_B$, the intersection $\alpha^{-1}(V')\cap A$ is in $\mathcal{T}_A$. Conversely, α is *relatively fuzzy open* if for each open fuzzy set U' in $\mathcal{T}_A$, the image $\alpha(U')$ is in $\mathcal{U}_B$.

Lemma 2.1. ([1]) *Let $(A,\mathcal{T}_A)$, $(B,\mathcal{U}_B)$ be fuzzy subspaces of fuzzy topological spaces $(X,\mathcal{T})$, $(Y,\mathcal{U})$ respectively, and let α be a fuzzy continuous mapping of $(X,\mathcal{T})$ into $(Y,\mathcal{U})$ such that $\alpha(A)\subset B$. Then α is a relatively fuzzy continuous mapping of $(A,\mathcal{T}_A)$ into $(B,\mathcal{U}_B)$.*

3. Fuzzy topological d-algebras

Definition 3.1. ([7]) A fuzzy set D in a d-algebra X with membership function μ_D is called a *fuzzy d-algebra* of X if

$$\mu_D(x*y)\geq \min\{\mu_D(x),\,\mu_D(y)\}\qquad \text{for all}\quad x,y\in X.$$

Example 3.2. ([7]) Let $X=\{0,a,b,c\}$ be a set with the following Cayley table (Table 1) as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Table 1.

Then $(X,*,0)$ is a d-algebra. Define a fuzzy set D in X with membership function μ_D by $\mu_D(0)=\mu_D(a)=\mu_D(c)=t_1$ and $\mu_D(b)=t_2$ for $t_1>t_2$. Then D is a fuzzy d-algebra of X.

Example 3.3. ([7]) Let $X=\{0,a,b,c\}$ be a set with the following Cayley table (Table 2) as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Table 2.

Then $(X,*,0)$ is a d-algebra. Define a fuzzy set D in X with membership function μ_D by $\mu_D(0)=\mu_D(a)=t_1>t_2=\mu_D(b)=\mu_D(c)$, where $t_1,t_2\in[0,1]$. Then D is a fuzzy d-algebra of X.

PROPOSITION 3.4. *Let α be a d-homomorphism of a d-algebra X into a d-algebra Y and G a fuzzy d-algebra of Y with membership function μ_G. Then the inverse image $\alpha^{-1}(G)$ of G is a fuzzy d-algebra of X.*

Proof. Let $x,y\in X$. Then

$$\begin{aligned}\mu_{\alpha^{-1}(G)}(x*y)&=\mu_G\big(\alpha(x*y)\big)=\mu_G\big(\alpha(x)*\alpha(y)\big)\\&\geq\min\{\mu_G\big(\alpha(x)\big),\,\mu_G\big(\alpha(y)\big)\}\\&=\min\{\mu_{\alpha^{-1}(G)}(x),\,\mu_{\alpha^{-1}(G)}(y)\}.\end{aligned}$$

This completes the proof. □

For images, we need the following definition ([12]).

DEFINITION 3.5. A fuzzy set D in a d-algebra X with membership function μ_D is said to have the *sup property* if, for any subset $T\subseteq X$, there exists $t_0\in T$ such that

$$\mu_D(t_0)=\sup_{t\in T}\mu_D(t).$$

PROPOSITION 3.6. *Let α be a d-homomorphism of a d-algebra X onto a d-algebra Y and let D be a fuzzy d-algebra of X with the sup property. Then the image $\alpha(D)$ of D is a fuzzy d-algebra of Y.*

Proof. For $u,v\in Y$, let $x_0\in\alpha^{-1}(u)$, $y_0\in\alpha^{-1}(v)$ such that

$$\mu_D(x_0)=\sup_{t\in\alpha^{-1}(u)}\mu_D(t),\qquad \mu_D(y_0)=\sup_{t\in\alpha^{-1}(v)}\mu_D(t).$$

Then, by the definition of $\mu_{\alpha(D)}$, we have

$$\begin{aligned}\mu_{\alpha(D)}(u*v) &= \sup_{t\in\alpha^{-1}(u*v)} \mu_D(t)\\ &\geq \mu_D(x_0 * y_0)\\ &\geq \min\{\mu_D(x_0),\, \mu_D(y_0)\}\\ &= \min\Big\{\sup_{t\in\alpha^{-1}(u)} \mu_D(t),\ \sup_{t\in\alpha^{-1}(v)} \mu_D(t)\Big\}\\ &= \min\{\mu_{\alpha(D)}(u),\, \mu_{\alpha(D)}(v)\},\end{aligned}$$

completing the proof. □

For any d-algebra X and any element $a \in X$ we denote by R_a the *right translation* of X defined by $R_a(x) = x * a$ for all $x \in X$. It is clear that $R_x(0) = 0 = R_x(x)$ for all $x \in X$.

DEFINITION 3.7. Let X be a d-algebra and $\mathcal{T}$ a fuzzy topology on X. Let D be a fuzzy d-algebra of X with induced topology $\mathcal{T}_D$. Then D is called a *fuzzy topological d-algebra* of X if for each $a \in X$ the mapping R_a: $(D,\mathcal{T}_D) \to (D,\mathcal{T}_D)$ is relatively fuzzy continuous.

THEOREM 3.8. *Given d-algebras X, Y and a d-homomorphism $\alpha\colon X \to Y$, let $\mathcal{U}$ and $\mathcal{T}$ be the fuzzy topologies on Y and X respectively, such that $\mathcal{T} = \alpha^{-1}(\mathcal{U})$. Let G be a fuzzy topological d-algebra of Y with membership function μ_G. Then $\alpha^{-1}(G)$ is a fuzzy topological d-algebra of X with membership function $\mu_{\alpha^{-1}(G)}$.*

P r o o f. We have to show that, for each $a \in X$, the mapping

$$R_a\colon \big(\alpha^{-1}(G),\, \mathcal{T}_{\alpha^{-1}(G)}\big) \longrightarrow \big(\alpha^{-1}(G),\, \mathcal{T}_{\alpha^{-1}(G)}\big)$$

is relatively fuzzy continuous. Let U be an open fuzzy set in $\mathcal{T}_{\alpha^{-1}(G)}$ on $\alpha^{-1}(G)$. Since α is a fuzzy continuous mapping of $(X,\mathcal{T})$ into $(Y,\mathcal{U})$, it follows from Lemma 2.1 that α is a relatively fuzzy continuous mapping of $\big(\alpha^{-1}(G),\mathcal{T}_{\alpha^{-1}(G)}\big)$ into $\big(G,\mathcal{U}_G\big)$. Note that there exists an open fuzzy set $V \in \mathcal{U}_G$ such that $\alpha^{-1}(V) = U$. The membership function of $R_a^{-1}(U)$ is given by

$$\begin{aligned}\mu_{R_a^{-1}(U)}(x) = \mu_U\big(R_a(x)\big) &= \mu_U(x*a) = \mu_{\alpha^{-1}(V)}(x*a)\\ &= \mu_V\big(\alpha(x*a)\big) = \mu_V\big(\alpha(x)*\alpha(a)\big).\end{aligned}$$

Since G is a fuzzy topological d-algebra of Y, the mapping

$$R_b\colon (G,\mathcal{U}_G) \to (G,\mathcal{U}_G)$$

is relatively fuzzy continuous for each $b \in Y$. Hence

$$\begin{aligned}\mu_{R_a^{-1}(U)}(x) &= \mu_V\big(\alpha(x) * \alpha(a)\big) = \mu_V\big(R_{\alpha(a)}(\alpha(x))\big)\\ &= \mu_{R_{\alpha(a)}^{-1}(V)}\big(\alpha(x)\big) = \mu_{\alpha^{-1}(R_{\alpha(a)}^{-1}(V))}(x),\end{aligned}$$

which implies that $R_a^{-1}(U) = \alpha^{-1}\big(R_{\alpha(a)}^{-1}(V)\big)$ so that

$$R_a^{-1}(U) \cap \alpha^{-1}(G) = \alpha^{-1}\big(R_{\alpha(a)}^{-1}(V)\big) \cap \alpha^{-1}(G)$$

is open in the induced fuzzy topology on $\alpha^{-1}(G)$. This completes the proof. □

The membership function μ_G of a fuzzy d-algebra G of a d-algebra X is said to be α*-invariant* ([12]) if, for all $x, y \in X$, $\alpha(x) = \alpha(y)$ implies $\mu_G(x) = \mu_G(y)$.

THEOREM 3.9. *Given d-algebras X, Y and a d-homomorphism α of X onto Y, let $\mathcal{T}$ be the fuzzy topology on X and let $\mathcal{U}$ be the fuzzy topology on Y such that $\alpha(\mathcal{T}) = \mathcal{U}$. Let D be a fuzzy topological d-algebra of X. If the membership function μ_D of D is α-invariant, then $\alpha(D)$ is a fuzzy topological d-algebra of Y.*

Proof. It is sufficient to show that the mapping

$$R_b\colon \big(\alpha(D), \mathcal{U}_{\alpha(D)}\big) \longrightarrow \big(\alpha(D), \mathcal{U}_{\alpha(D)}\big)$$

is relatively fuzzy continuous for each $b \in Y$. Note that α is relatively fuzzy open; for if $U' \in \mathcal{T}_D$, there exists $U \in \mathcal{T}$ such that $U' = U \cap D$, and by the α-invariance of μ_F,

$$\alpha(U') = \alpha(U) \cap \alpha(D) \in \mathcal{U}_{\alpha(D)}.$$

Let V' be an open fuzzy set in $\mathcal{U}_{\alpha(D)}$. Since α is onto, for each $b \in Y$ there exists $a \in X$ such that $b = \alpha(a)$. Hence

$$\begin{aligned}\mu_{\alpha^{-1}(R_b^{-1}(V'))}(x) &= \mu_{\alpha^{-1}(R_{\alpha(a)}^{-1}(V'))}(x) = \mu_{R_{\alpha(a)}^{-1}(V')}\big(\alpha(x)\big)\\ &= \mu_{V'}\big(R_{\alpha(a)}(\alpha(x))\big) = \mu_{V'}\big(\alpha(x) * \alpha(a)\big)\\ &= \mu_{V'}\big(\alpha(x * a)\big) = \mu_{\alpha^{-1}(V')}(x * a)\\ &= \mu_{\alpha^{-1}(V')}\big(R_a(x)\big) = \mu_{R_a^{-1}(\alpha^{-1}(V'))}(x),\end{aligned}$$

which implies that $\alpha^{-1}\big(R_b^{-1}(V')\big) = R_a^{-1}\big(\alpha^{-1}(V')\big)$. By hypothesis, R_a is a relatively fuzzy continuous mapping from $(D, \mathcal{T}_D)$ to $(D, \mathcal{T}_D)$ and α is a relatively fuzzy continuous mapping from $(D, \mathcal{T}_D)$ to $\big(\alpha(D), \mathcal{U}_{\alpha(D)}\big)$. Hence

$$\alpha^{-1}\big(R_b^{-1}(V')\big) \cap G = R_a^{-1}\big(\alpha^{-1}(V')\big) \cap D$$

is open in $\mathcal{T}_D$. Since α is relatively fuzzy open,

$$\alpha\big(\alpha^{-1}\big(R_b^{-1}(V')\big) \cap D\big) = R_b^{-1}(V') \cap \alpha(D)$$

is open in $\mathcal{U}_{\alpha(D)}$. This completes the proof. □

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* *Department of Mathematics Education*
Gyeongsang National University
Chinju 660-701
KOREA
E-mail: ybjun@nongae.gsnu.ac.kr

** *Department of Mathematics*
Hanyang University
Seoul 133-791
KOREA
E-mail: heekim@email.hanyang.ac.kr