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FACE SIZE AND THE MAXIMUM GENUS OF A GRAPH. PART 2: NONSIMPLE GRAPHS

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ABSTRACT. It is proved that a loopless graph which can be cellularly embedded on some closed surface in such a way that the size of each face does not exceed 5 is upper embeddable. This settles the first of two conjectures posed by Nedela and Škoviera in [NEDELA, R.—ŠKOVIERA, M.: *On* graphs embeddable with short faces. In: Topics in Combinatorics and Graph Theory (R. Bodendiek, R. Henn, eds.), Physica Verlag, Heidelberg, 1990, pp. 519–529]. The second conjecture is established in [HUANG, Y.—LIU, Y.: *Face* size and the maximum genus of a graph. Part 1: Simple graphs, J. Combin. Theory Ser. B **80** (2000), 356–370].

1. Introduction

This paper is a sequel to our paper "Face size and the maximum genus of a graph. Part 1: Simple graphs" [3]. There we started our investigation of the relationship between embeddings with short faces and the maximum genus of a graph, the largest integer k such that the graph has a cellular embedding on an orientable surface of genus k (i.e., closed surface with k handles).

In contrast to Part 1, here we deal with the maximum genus of graphs that may contain multiple edges, but not loops. If not stated otherwise, however, our graphs may, in general, contain loops.

All the definitions necessary for this paper can be found in Part 1. Nevertheless, in order to keep this paper as self-contained as possible, we repeat, at the corresponding places, all the non-standard definitions.

In 1990, Nedela and Škoviera [4] proved that a loopless graph which has a cellular embedding on a closed surface such that the size of each face does not exceed 4 is upper embeddable. This means that its maximum genus is

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$\lfloor \beta/2 \rfloor$, where β is the Betti number (cycle rank) of the graph in question, which is the general upper bound for the maximum genus of any graph. Moreover, they made two conjectures. First, they conjectured that every loopless graph admitting a cellular embedding on a closed surface with maximum face size at most 5 is upper embeddable. Second, that restricting to simple graphs the conclusion remains true when the condition is relaxed to requiring the maximum face size not to exceed 7. The later conjecture has been proved in Part 1 of this series ([3]). In the present paper we give a confirmative answer to the former conjecture by proving the following result:

MAIN THEOREM. *Let G be a loopless graph. If G has a cellular embedding on a surface closed S (orientable or nonorientable) such that the size of each face does not exceed 5, then G is upper embeddable.*

A direct consequence of the above theorem is the following interesting fact.

COROLLARY. *Let G be a loopless graph. If G is not upper embeddable, then every cellular embedding of G on any surface (orientable or nonorientable) contains a face with size at least 6.*

A simple example given at the end of this paper shows that the condition of maximum face size not exceeding 5 in the above theorem is best possible.

We will prove Main Theorem in Section 4. The proof will be performed by induction on the genus of a surface. The two sections preceding the proof will be devoted to preparations. In the next section we study the maximum genus of a graph in a greater detail. Section 3 is devoted to the surgery on surfaces, our main technical device in the induction step.

2. Maximum genus

This section together with the next one will be devoted to preparations for the main proof. Here we will deal with the maximum genus of a graph in a greater detail.

Recall that every cellular embedding of a graph satisfies *Euler's formula*. Let G be a connected graph with p vertices and q edges embedded on a surface S with r faces. Then there is a number $g(S)$ depending only on the surface S such that

$$p - q + r = \begin{cases} 2 - 2g(S) & \text{if } S \text{ is orientable,} \\ 2 - g(S) & \text{if } S \text{ is nonorientable.} \end{cases}$$

This number $g(S)$ is called the *genus* of S.

The *maximum genus* $\gamma_M(G)$ of a connected graph G is the largest integer k with such that G admits a cellular embedding on the orientable surface S of genus k. It readily follows from the Euler formula that

$$\gamma_M(G) \le \lfloor \beta(G)/2 \rfloor ,$$

where $\beta(G) = q - p + 1$ is the *Betti number* (or *cycle rank*) of G.

A graph connected G is said to be *upper embeddable* if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$. The quantity $\beta(G) - 2\gamma_M(G)$ is called the *deficiency* of G (or *Betti deficiency* of G) and is denoted by $\xi(G)$. Thus a graph is upper embeddable if and only if $\xi(G) \le 1$. A graph G with $\xi(G) \ge 2$ will be called to be a *deficient* graph.

Note that a disconnected graph does not admit a cellular embedding on any surface, therefore the concepts of maximum genus, upper embeddability, and deficiency are meaningful only for connected graphs. For further information about the maximum genus and the upper embeddability of graphs the reader is referred to Gross and Tucker [2] (or the survey article by Ringeisen [7]).

Let G be a graph and $A \subseteq E(G)$. Let $c(G-A)$ and $b(G-A)$ denote the number of components of $G-A$ and the number of components of $G-A$ with odd Betti number, respectively. For two subgraphs F and K of G denote by $E_G(F,K)$ the set of all edges whose two end-vertices are respectively in F and K.

The following lemma is due to Nebeský [5].

LEMMA 2.1. ([5]) *Let G be a graph. Then:*

(1) $\xi(G) = \max\limits_{A \subseteq E(G)} \{c(G-A) + b(G-A) - |A| - 1\}$;

(2) *G is upper embeddable if and only if $c(G-A) + b(G-A) - |A| \le 2$ for any subset $A \subseteq E(G)$.*

We call a subset $A \subseteq E(G)$ a *Nebeský set* in G if $\xi(G) = c(G-A) + b(G-A) - |A| - 1$. A *minimal Nebeský set* is a Nebeský set that is minimal under inclusion.

Based on the results of Nebeský [6], and Fu and Tsai [1], the following lemma provides structural information about a deficient graph G via minimal Nebeský sets. Before stating it let $E_G(F,K)$ denote the set of all edges of G whose two end-vertices are respectively in the subgraphs F and K of G. For the proof, see [3; Part 1].

Lemma 2.2. *Let G be a deficient graph, and let $A \subseteq E(G)$ be a minimal Nebeský set in G. Then:*

(i) *$c(G-A) \geq 2$, and each component F of $G-A$ has an odd Betti number, that is, $\beta(F) = 1 \pmod 2$;*
(ii) *each component F of $G-A$ is a vertex-induced subgraph of G;*
(iii) *$|A| \leq 2c(G-A)-3$;*
(iv) *for any two distinct components F and K of $G-A$ one has $|E_G(F,K)| \leq 1$.*

Let G be a connected graph and let F be a connected vertex-induced subgraph of G. Let $E(G,F)$ denote the set of all edges of G which do not belong to $E(F)$ but are incident with vertices in $V(F)$. We now form a graph, denoted by $\bar{G}$, as follows:

(1) we remove $V(F)$ and $E(F)$ from G but do not remove $E(G,F)$;
(2) we take a family $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$, $n \geq 1$, of pairwise disjoint circuits with arbitrary lengths ≥ 2;
(3) finally, we put each edge of $E(G,F)$ formerly incident with a vertex of F to be incident with an arbitrary vertex on some C_j $(1 \leq j \leq n)$.

In other words, the graph $\bar{G}$ is obtained from G by replacing F with a collection $\mathcal{C}$ of pairwise disjoint circuits. Note that $\bar{G}$ may happen to be disconnected. Let G_1, G_2, ..., G_m $(m \geq 1)$ be the components of $\bar{G}$. Then $\{G_1, G_2, \dots, G_m\}$ is called an *F-resolution* of G by the family of cycles $\mathcal{C}$, or simply an F-resolution of G.

It is clear from the definition that the number and the lengths of the chosen circuits as well as their incidence with the edges of $E(G,F)$ is not important. Therefore, the resulting graph $\bar{G}$ and thus also the F-resolution of G are not uniquely determined.

We now have the following lemmas which have been proved in [3; Part 1].

Lemma 2.3. *Let G be a connected graph and let $\{G_1, G_2, \dots, G_n\}$ be an F-resolution of G by a family of circuits $\mathcal{C}$. Then each G_i contains at least one of the circuits $C_j \in \mathcal{C}$ (it may occur that $G_i = C_j$).*

Lemma 2.4. *Let $\{G_1, G_2, \dots, G_m\}$ be an F-resolution of a deficient graph G, where F is a component of $G-A$ for some minimal Nebeský set $A \subseteq E(G)$. Then at least one of the graphs G_1, G_2, ..., G_m is a deficient graph.*

Lemma 2.5. *Let $\{G_1, G_2, \dots, G_n\}$ be an F-resolution of a graph G by a family circuits $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$, and assume that for each i the graph G'_i is obtained from G_i by contracting some edges of circuits in $\mathcal{C}$, $1 \leq i \leq m$. Then $\{G'_1, G'_2, \dots, G'_m\}$ is also an F-resolution of G.*

3. Surgery

A 2-cell embedding of a graphs will be called a *short-face* embedding if the size of each face (the number of edges on its boundary, repeated edges being counted twice) is at most five. Moreover, any face of an embedded graph will be said to be *short* provided that the size of the face does not exceed five.

In this section we develop a surgical technique to obtain an F-resolution $\{G_1, G_2, \ldots, G_m\}$ of a graph G short-face embedded on some surface such that each G_i has an induced embedding that is either short-face or has a few exceptional longer faces.

Let $j\colon G \to S$ be a cellular embedding of a graph G on a surface S, and let F be a connected subgraph of G. For any positive and arbitrarily small real number ε, let $N(F,\varepsilon)$ denote the open collaring of $j(F)$ in S in which the distance of all points from $j(F)$ is smaller than ε. Analogously, for each vertex $v \in V(F)$, let $N(v,\varepsilon)$ denote the open ε-neighborhood of $j(v)$ in S, that is, the distance of all points of $N(v,\varepsilon)$ from $j(v)$ is less than ε. Notice that ε can be chosen arbitrarily small, and therefore we can assume its complement $S - N(F,\varepsilon)$ on S to be a bordered surface, possibly disconnected. For any connected component M of $S - N(F,\varepsilon)$, we take a closed disk and identify each boundary circuit of M with that of a closed disk, thereby obtaining an new surface denoted by $S(M)$. Observe that the orientability of $S(M)$ may happen to be different from that of the original embedding surface S. We will call M the *connected component* and $S(M)$ the *surface* obtained by *removing* $N(F,\varepsilon)$ from S. If C is a circuit of G, then $S - N(C,\varepsilon)$ has clearly at most two components. We will call C a *contractible* circuit on S if $S - N(C,\varepsilon)$ has two precisely components and at least one homeomorphic to a closed disc; otherwise, C will be said to be a *noncontractible* circuit. Equivalently, a contractible circuit on a surface is one that can be continuously contracted to a point in the surface. (Purely combinatorial definitions of contractible and noncontractible circuits were given by Thomassen in [8].)

The following lemma can be found in [3].

Lemma 3.2. *Let F be a connected subgraph of a graph G that is cellularly embedded on a surface S, and let M be a connected component of $S - N(F,\varepsilon)$. Then:*

(1) *If each circuit in F is contractible on S, then either $g(S_M) = g(S)$, or $g(S_M) = 0$, and furthermore, M has exactly one boundary circuit.*
(2) *If F has at least one noncontractible circuit on S, then $g(S_M) < g(S)$.*

In the remaining part of this section we consider a deficient loopless graph G with short-face embedding on some a surface. Let $j\colon G \to S$ be a short-face embedding of G on S. Throughout, we will identify $j(G)$ with G.

Let F be a component of $G - A$, where A is a minimal Nebeský subset of $E(G)$. By Lemma 2.2(i), F is not a tree, and hence contains some cycles; by Lemma 2.2(ii), F is a vertex-induced subgraph of G. Intuitively, we want to cut the surface S along F and leave a copy of each edge on each side of the resulting (possibly disconnected) bordered surface. We perform this as follows. For a sufficiently small positive real number ε, let us take the open ε-collaring $L = N(F,\varepsilon)$ of F, that is, the set of all points on S whose distance from F in S is smaller than ε. Let us recall that, in general, an open collaring of F in the surface S is an open set $L \subseteq S$ such that $F \subseteq L$ and F is a *deformation retract* of L. This means that there exists a continuous mapping, called a *deformation retraction*, $\Phi\colon L \times [0,1] \to L$ such that Φ_0 is the identity mapping on L, $\Phi_1(L) = F$ and $\Phi_1|_F = \mathrm{id}_F$. In this particular case, where $L = N(F,\varepsilon)$, we clearly can and will adopt the following useful "regularity" assumption: for each $t > 0$ we let $\Phi_t\big(N(F,\varepsilon)\big) = N\big(F,\varepsilon(1-t)\big)$. Now we remove $L' = N(F,\varepsilon/2)$ from S, thereby getting a bordered, possibly disconnected, surface $S - N(F,\varepsilon/2)$. Observe that its border consists of all elements of $N(F,\varepsilon)$ whose distance from F equals $\varepsilon/2$. Let M_i be any component of $S - N(F,\varepsilon/2)$ $(i = 1,\dots,m)$ and let $L_i = L \cap M_i$. Define the graph F_i to be $\Phi_{\varepsilon/2}(L_i) \cup \big(j(G) \cap \mathrm{int}(M_i)\big)$.

It is obvious that F_i is embedded in M_i. Moreover, $B(F_i) = \Phi_{\varepsilon/2}(L_i)$ is the part of F_i which lies on the boundary of M_i whereas $j(G) \cap \mathrm{int}(M_i)$ lies in the interior of M_i.

We observe that for each edge e' of F_i lying on the boundary $B(F_i)$ of M_i, there exists the original edge e of F such that e' consists of all elements of $N(F,\varepsilon)$ whose distance from e equals $\varepsilon/2$. Analogously, for each vertex v' of F_i lying on $B(F_i)$, there exists the original vertex v of F such that the distance between v' and v equals $\varepsilon/2$. In this sense we can say that e' and v' correspond to e and v, respectively. We must note that for each edge e, there exist exactly two such edges e' and e'' (not necessarily belonging to the same $B(F_i)$) corresponding to e, while for each vertex v of F the number of such vertices corresponding to v depends on the number of corners on S formed by the edges of F incident with v. Let the boundary $B(F_i)$ of M_i be composed of n_i disjoint circuits of F_i, say C_i^1, C_i^2, …, $C_i^{n_i}$. We now cap each boundary circuit C_i^j $(1 \le j \le n_i)$ by a closed 2-cell D_i^j, and thus obtain a new surface $S(F_i)$, together with an embedding of the graph F_i.

Now we establish several useful properties of the graph F_i and its embedding on the surface $S(F_i)$ $(1 \le i \le m)$.

Let us denote by $f(C_i^j)$ the face of F_i in $S(F_i)$ which is bounded by C_i^j and obtained from D_i^j by removing its boundary. Clearly, the face $f(C_i^j)$ is an open 2-cellular face, and the size of $f(C_i^j)$ is the length of C_i^j.

Claim 1. *For each i $(1 \le i \le m)$, F_i is a connected loopless graph which is cellularly embedded in $S(F_i)$. Moreover, each face of the embedding of F_i in $S(F_i)$ is a short face except possibly the faces $f(C_i^j)$, $1 \le j \le n_i$.*

P r o o f . Since G is loopless, so must be F_i by the definitions of F_i and M_i. In order to prove that F_i is connected, it suffices to prove that F_i is cellularly embedded in $S(F_i)$. To do this, we analyse the possible position of each face of F_i with respect to $B(F_i)$. There are the following three possible cases (a), (b) and (c):

(a) f, together with its boundary, lies entirely in the interior of M_i, and thus f may be viewed as an original face of G on S.

(b) f lies entirely in the interior of M_i, however its boundary intersects the boundary of M_i at some vertices or edges of $B(F_i)$. In this case, the face f is homeomorphic to the original face f' of G on S, where the boundary of f' is formed from the boundary of f by replacing the vertices or the edges belonging to $B(F_i)$ with their corresponding vertices or edges of F, respectively. Therefore f is cellular and has the same size as f'.

(c) f is a face $f(C_i^j)$, $1 \le j \le n_i$.

From the above three cases we see that each face of F_i on $S(F_i)$ is an open 2-cell, implying that the embedding of F_i is cellular and thus F_i is connected. Moreover, since the original embedding of G in S is a short-face embedding, each face f of the embedding is a short face except possibly the faces $f(C_i^j)$, $1 \le j \le n_i$. □

We also observe that there are three kinds of edges in F_i. We will say that an edge e of F_i is of *type* k, $k = 0, 1, 2$, if k end-vertices of e belong to $B(F_i)$.

Claim 2. *An edge e of F_i is of type 2 if and only if e belongs to $B(F_i)$.*

P r o o f . If e belongs to $B(F_i)$, then by the definitions F_i and M_i the two end-vertices of e must belong to $B(F_i)$ as well. Thus e is of type 2. For the converse, let e be of type 2. Assume to the contrary that e does not belong to $B(F_i)$. Then it is easily seen that in G, the two end-vertices of e are in $V(F)$ but $e \in E(G) - E(F)$. This contradicts Lemma 2.2(ii) that F is a vertex-induced subgraph of G. Therefore e belongs to $B(F_i)$. □

Claim 3. *$\{F_1, F_2, \ldots, F_m\}$ is an F-resolution of G.*

P r o o f . See [3; Section 3]. □

Now we turn each of the circuits C_i^j into a 2-circuit by contracting some of its edges (if necessary, because some C_i^j may already be a 2-circuit) along the surface $S(F_i)$. We let F_i' and $S(F_i')$ be respectively the resulting graph and the resulting surface obtained by the edge contraction process. Obviously $S(F_i')$ is homeomorphic to $S(F_i)$.

The following Claims 4 and 5 describe the graph F_i' and its embeddings on $S(F_i')$.

CLAIM 4. *Each F_i' is a loopless connected graph ($1 \le i \le m$), and $\{F_1', F_2', \dots, F_m'\}$ is an F-resolution of G.*

Proof. Since F_i is connected (Claim 1), the connectivity of F_i' is immediate. Next we prove that F_i' has no loops. Let B_i^j, $1 \le j \le n_i$, be the 3-circuit obtained from C_i^j by the contraction process.

By way of contradiction, suppose that F_i' has a loop e. Then e must be incident with a vertex in some a B_i^j. In F_i, the edge e does not belong to C_i^j while its two end-vertices do. This implies that in F_i, e does not entirely lie in $B(F_i)$ but at the same time is of type 2, contradicting Claim 2. □

CLAIM 5.

(1) *If every circuit of F is contractible on S, then each F_i' ($1 \le i \le m$) is short-face embedded on the surface $S(F_i')$ with $g(S(F_i')) = g(S)$ or $g(S(F_i')) = 0$. Moreover, one has $|V(F_i')| + |E(F_i')| < |V(G)| + |E(G)|$ unless F is a 2-circuit bounding a face.*

(2) *If F contains a noncontractible circuit on S, then each F_i' ($1 \le i \le m$) is short-face embedded on the surface $S(F_i')$ with $g(S(F_i')) < g(S)$.*

Proof. We first prove the conclusion (1). Since each circuit of F is contractible on S, it follows directly from Lemma 3.2(1) that $g(S(F_i')) = g(S(F_i)) = g(S)$, or $g(S(F_i')) = g(S(F_i)) = 0$ for each $1 \le i \le m$. The fact that F_i' is short-face embedded in $S(F_i')$ is easily seen from Claim 1 as well as the definitions of F_i' and $S(F_i')$.

Now we prove the rest of the conclusion (1). By Lemma 3.2(1), each M_i has exactly one boundary component; hence $n_i = 1$. Let C_i denote the unique boundary circuit of M_i and let B_i be the circuit of F_i' obtained from C_i by the edge contracting process. Then B_i is a 2-circuit, and thus $|V(B_i)| = |E(B_i)| = 2$. Keeping in mind that F is loopless and contains circuits, we have to consider the following cases:

Case 1. $|V(F)| \ge 3$.
Obviously F is not a 2-circuit. Furthermore $|E(F)| \ge 3$ because F contains circuits. Thus, $|V(F_i')| + |E(F_i')| < |V(G)| + |E(G)|$ because we have $\big(|V(G)| + |E(G)|\big) - \big(|V(F_i')| + |E(F_i')|\big) \ge |V(F)| + |E(F)| - \big(|V(F_i')| + |E(F_i')|\big) > 0$.

Case 2. $|V(F)| = 2$ and F is not a 2-circuit.
We see that $|E(F)| \ge 3$ because F is loopless and contains circuits. Noting that $|V(F)| + |E(F)| > |V(F_i')| + |E(F_i')|$, similarly we have $|V(F_i')| + |E(F_i')| < |V(G)| + |E(G)|$.

Case 3. $|V(F)| = 2$, and
F is a 2-circuit but does not bound a face of G on S.
By the hypothesis, F is a contractible 2-circuit on S. Thus $S - N(F, \varepsilon/2)$ has exactly two connected components M_1 and M_2, that is, $m = 2$. Furthermore, the unique boundary circuit C_i of M_i ($i = 1, 2$) is a 2-circuit. Again, since F does not bound a face of G on S, both M_1 and M_2 must contain some vertices in $V(G) - V(F)$ and some edges in $E(G) - E(F)$, and thus so must both F_1 and F_2. Therefore, by the definition of F_i', $|V(F_i')| + |E(F_i')| < |V(G)| + |E(G)|$ ($1 \le i \le m = 2$).

The above three cases imply that $|V(F_i')| + |E(F_i')| < |V(G)| + |E(G)|$ unless F is a 2-circuit bounding a face. This completes the proof of (1).

If we apply Lemma 3.2(2), the proof of the conclusion (2) can be performed along the same lines as the proof of the conclusion (1) above. The difference in this case is in that the number of the boundary circuits of each M_i may be larger than one. However, here we need not consider the inequality on the number of vertices and edges of F_i' and G. We leave the details to the reader. □

The following lemma obtained by summarizing the above claims is crucial for the proof of the main theorem in the next section.

Lemma 3.3. *Let G be a deficient loopless graph that is short-face embedded in a surface S. Let F be a component of $G - A$, where A is a minimal Nebeský subset of $E(G)$. Then there exists an F-resolution $\{G_1, G_2, \dots, G_m\}$ of G with the following properties:*

(a) *If every circuit of F is contractible on S, then each G_i ($1 \le i \le m$) is loopless and has a short-face embedding on a surface S_i with $g(S_i) = g(S)$ or $g(S_i) = 0$. Moreover, one has $|V(G_i)| + |E(G_i| < |V(G)| + |E(G)|$ unless F is a 2-circuit bounding a face.*
(b) *If F contains a noncontractible circuit on S, then each G_i ($1 \le i \le m$) is loopless and has a short-face embedding on a surface S_i with $g(S_i) < g(S)$.*

4. Proof of Main Theorem

As we have already indicated, the proof of the Main Theorem will be performed by induction on the genus. The next lemma verifies the induction basis.

Lemma 4.1. *Let G be a loopless graph. If G has a short-face embedding on the 2-sphere, then G is upper embeddable.*

Proof. Assume to the contrary that the conclusion does not hold. Then there is a loopless planar deficient graph with a short-face embedding on the 2-sphere. From among these choose G such that the value $|V(G)| + |E(G)|$ is minimum. Fix a short-face embedding of G on the 2-sphere, and let A be a minimal Nebeský subset of $E(G)$. Since G is short-face embedded on the 2-sphere, for any component F of $G - A$ it follows that each circuit in F is contractible. So we can employ part (1) of Lemma 3.3 and distinguish the following two cases.

Case 1. There exists a component F of $G-A$ that is not a 2-circuit bounding a face.

By Lemma 3.3, there exists an F-resolution $\{G_1, G_2, \dots, G_m\}$ of G such that each G_i ($1 \le i \le m$) is loopless and has a short-face embedding on the 2-sphere. Furthermore, $|V(G_i)| + |E(G_i)| < |V(G)| + |E(G)|$. Therefore, each G_i is upper embeddable by the choice of G ($1 \le i \le m$), contradicting Lemma 2.4.

Case 2. Each component F of $G - A$ is a 2-circuit bounding a face of G on the 2-sphere.

Obviously, G has at least $c(G - A)$ faces with the size two. Moreover,

$$|V(G)| = \sum_F |V(F)| = 2c(G - A)\,, \tag{1}$$

$$|E(G)| = \sum_F |E(F)| + |A| = 2c(G - A) + |A|\,, \tag{2}$$

where the sum ranges over all the components F of $G - A$. Let $F(G)$ denote the set of faces of G. By (1), (2), and Euler's formula for the plane we have

$$|F(G)| = 2 + |E(G)| - |V(G)| = |A| + 2\,. \tag{3}$$

As G is short-face embedded on the 2-sphere with at least $c(G - A)$ faces of the size two, we have

$$2|E(G)| = \sum_{f \in F(G)} |f| \le 2c(G - A) + 5\big(|F(G)| - c(G - A)\big)\,. \tag{4}$$

Substituting (2) for $|E(G)|$ and (3) for $|F(G)|$ into (4), and simplifying, we obtain that $|A| \ge (7/3)c(G-A) - 10/3$. Since $c(G-A) > 1$ by Lemma 2.2(i), it follows that $|A| > 2(G - A) - 3$, contradicting the property (iv) of Lemma 2.2. The proof of this lemma is complete. □

Now we are ready to finish the proof of the Main Theorem.

Proof of Main Theorem. We employ induction on $g(S)$, the genus of the embedding surface S. By Lemma 4.1, we can assume that $g(S) \ge 1$ and that the conclusion holds for any surface with its genus $< g(S)$. Assume to the contrary that the conclusion does not hold for the surface S. We choose a graph G with the minimum value $|V(G)|+|E(G)|$ such that G satisfies the hypothesis of the theorem but G is not upper embeddable, that is, G is a deficient graph. Let A be a minimal Nebeský subset of $E(G)$. We now apply Lemma 3.3 and consider the following cases.

Case 1. For any component F of $G-A$, each circuit of F is contractible on S. In this case we shall distinguish two subcases according to Lemma 3.3(1).

Subcase 1.1. There exists a component F of $G-A$ that is not a 2-circuit bounding a face.

By Lemma 3.3(1), there exists an F-resolution $\{G_1, G_2, \ldots, G_m\}$ of G such that each G_i $(1 \le i \le m)$ is loopless and has a short-face embedding on a surface S_i with either $g(S_i) = g(S)$ or $g(S_i) = 0$, and furthermore $|V(G_i)| + |E(G_i)| < |V(G)| + |E(G_i)|$. Then we easily get that each G_i $(1 \le i \le m)$ is upper embeddable by either the basis of induction (Lemma 4.1) or the choice of G. This contradicts Lemma 2.4.

Subcase 1.2. Each component F of $G-A$ is a 2-circuit bounding a face of G on S.

In this subcase the proof can be done along the same line as the proof of Lemma 4.1 in Case 2 so as to induce a similar contradiction. The only difference here is in that we substitute the general Euler's formula (orientable or nonorientable) for the planar Euler's formula.

Case 2. There exists a component F of $G-A$ that contains a noncontractible circuit on S.

By Lemma 3.3(2) there exists an F-resolution $\{G_1, G_2, \ldots, G_m\}$ of G such that each G_i $(1 \le i \le m)$ is loopless and has a short-face embedding on a surface S_i with $g(S_i) < g(S)$. Therefore each G_i $(1 \le i \le m)$ is upper embeddable by the inductive hypothesis, a contradiction to Lemma 2.4 as well.

Thus, in each case we have arrived at a contradiction, thereby completing the induction step and the whole proof. □

EXAMPLE. To conclude, we give an example of graph which has an embedding with each face of size at most six and which is not upper-embeddable. This shows that the condition requiring a short-face embedding of a loopless graph to have faces of size at most five is best possible.

Let G be a graph formed as follows: let $C = v_1v_2v_3v_4v_5v_6v_1$ be a 6-circuit. Add the multiple edges v_1v_2, v_3v_4, and v_5v_6. Clearly, G is loopless, and G has a planar embedding with the size of each face ≤ 6.

However, for the subset $A = \{v_2v_3, v_4v_5, v_6v_1\} \subseteq E(G)$, we have $c(G-A) = b(G-A) = 3$ and $c(G-A) + b(G-A) - |A| \not\le 2$, implying (by Lemma 2.1(2)) that G is not upper embeddable.

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