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ON (WEAK) ZERO-FIXING ISOMETRIES IN DUALLY RESIDUATED LATTICE ORDERED SEMIGROUPS

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ABSTRACT. The group of (weak) zero-fixing isometries of a dually residuated lattice ordered semigroup is isomorphic to the group of zero-fixing isometries of an abelian lattice ordered group.

An algebra $A = (A; 0; +; -; \wedge; \vee)$ of type $\langle 0; 2; 2; 2; 2\rangle$ is a *dually residuated lattice ordered semigroup* (abbreviated, a *DRℓ-semigroup*) if the following holds ([5; Definition 1], [3; Corollary 3] and [4; Corollary 3]):

(i) $(A; 0; +)$ is an abelian monoid,

(ii) $(A; \wedge; \vee)$ is a lattice (the induced order is denoted by $\leq$),

(iii) $(x \vee y) + z = (x + z) \vee (y + z)$ for all $x, y, z \in A$,

(iv) $(x - y) + y \geq x$ and if $z + y \geq x$, then $z \geq x - y$ for all $x, y, z \in A$,

(v) $\big[(x - y) \vee 0\big] + y \leq x \vee y$ for all $x, y \in A$.

In a DRℓ-semigroup A, a metric operation ϱ is introduced (cf. [5; Theorem 9]):

$$\varrho(x; y) = (x - y) \vee (y - x)\,.$$

A *weak zero-fixing isometry* of A is a mapping $f \colon A \to A$ such that

$$f(0) = 0 \qquad \text{and} \qquad \varrho(x; y) = \varrho\big(f(x); f(y)\big) \quad \text{for all } x, y \in A$$

(cf. [1; Preliminaries]).

A surjective weak zero-fixing isometry is a *zero-fixing isometry.*

In what follows, A stands for a DRℓ-semigroup, $\mathrm{In}(A)$ stands for the lattice ordered group of all invertible elements of A (cf. [6; Theorem 1.1]), $\mathrm{Si}(A)$ stands for the DRℓ-semigroup of all singular elements of A (cf. [2; Definition 2, Theorem 8]) and f denotes a zero-fixing isometry of A.

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1. THEOREM. (cf. [2; Theorem 12]) $A = \mathrm{In}(A) \times \mathrm{Si}(A)$.

2. THEOREM. *In A, the notions of a zero-fixing isometry and a weak zero-fixing isometry coincide.*

Proof. Due to [1; Theorem 3.12], any weak zero-fixing isometry is a semigroup automorphism and hence a bijection. □

3. COROLLARY. *Weak zero-fixing isometries of A form a group (under the composition of mappings).*

4. LEMMA. *The following holds:*

(i) *if $x \in \mathrm{In}(A)$, then $f(x) \in \mathrm{In}(A)$,*
(ii) *if $y \in \mathrm{Si}(A)$, then $f(y) = y$,*
(iii) *if $x \in \mathrm{In}(A)$ and $y \in \mathrm{Si}(A)$, then $f(x+y) = f(x) + y$.*

Proof.

(i) Assume $x \in \mathrm{In}(A)$. Due to [1; Theorem 3.12] we have $0 = f(0) = f\big(x + (-x)\big) = f(x) + f(-x)$ and therefore $f(x) \in \mathrm{In}(A)$.

(ii) Assume $y \in \mathrm{Si}(A)$. Due to [2; Definition 2], [2; Lemma 4] and [5; Lemma 1] we have $y = y \vee 0 = (y - 0) \vee (0 - y) = \varrho(y; 0) = \varrho\big(f(y); f(0)\big) = \varrho\big(f(y); 0\big) = \big(f(y) - 0\big) \vee \big(0 - f(y)\big) = f(y) \vee \big(0 - f(y)\big)$ and hence $0 - f(y) \leq y$. In view of [2; Theorem 7] in conjunction with [2; Lemma 10] we obtain $0 - f(y) \leq 0$ and [5; Lemma 7] implies $f(y) \geq 0$. Consequently, $y = f(y) \vee \big(0 - f(y)\big) = f(y)$.

(iii) It follows from (ii) and [1; Theorem 3.12]. □

5. THEOREM. *The group of zero-fixing isometries of a dually residuated lattice ordered semigroup A is isomorphic to the group of zero-fixing isometries of an abelian lattice ordered group $\mathrm{In}(A)$.*

Proof. It follows directly from Theorem 1 and Lemma 4. □

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