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λ-STATISTICAL CONVERGENCE

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ABSTRACT. In this paper, we use the notion of (V,λ)-summability to generalize the concept of statistical convergence. We call this new method a *λ-statistical convergence* and denote by S_λ the set of sequences which are λ-statistically convergent. We find its relation to statistical convergence, $(C,1)$-summability and strong (V,λ)-summability.

1. Introduction

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \le \lambda_n + 1\,, \qquad \lambda_1 = 1\,.$$

The generalized de la Valée-Pousin mean is defined by

$$t_n(x) := \frac{1}{\lambda_n} \sum_{k\in I_n} x_k\,,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V,λ)-*summable* to a number L (see [8]) if

$$t_n(x) \to L \qquad \text{as} \quad n \to \infty\,.$$

If $\lambda_n = n$, then (V,λ)-summability reduces to $(C,1)$-summability.

We write

$$[C,1] := \left\{x = (x_n):\ \exists\, L \in \mathbb{R}, \quad \lim_{n\to\infty} \tfrac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0\right\}$$

and

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$$[V,\lambda] := \Big\{ x = (x_n) : \ \exists L \in \mathbb{R}, \quad \lim_{n\to\infty} \tfrac{1}{\lambda_n} \sum_{k\in I_n} |x_k - L| = 0 \Big\}$$

for the sets of sequences $x = (x_k)$ which are *strongly Cesàro summable* and *strongly* (V,λ)*-summable* to L, i.e. $x_k \to L \ [C,1]$ and $x_k \to L \ [V,\lambda]$ respectively.

The idea of statistical convergence was introduced by Fast [3] and studied by various authors (see [1], [5] and [9]).

A sequence $x = (x_k)$ is said to be *statistically convergent* to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \big| \{ k \le n : \ |x_k - L| \ge \varepsilon \} \big| = 0 ,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S\text{-}\lim x = L$ or $x_k \to L \ (S)$ and S denotes the set of all statistically convergent sequences.

In this paper, we introduce and study the concept of λ-statistical convergence and determine how it is related to $[V,\lambda]$ and S.

Definition. A sequence $x = (x_n)$ is said to be λ*-statistically convergent* or S_λ*-convergent* to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \big| \{ k \in I_n : \ |x_k - L| \ge \varepsilon \} \big| = 0 .$$

In this case we write $S_\lambda\text{-}\lim x = L$ or $x_k \to L \ (S_\lambda)$, and

$$S_\lambda := \{ x : \ \exists L \in \mathbb{R}, \quad S_\lambda\text{-}\lim x = L \} .$$

Remark.

(i) If $\lambda_n = n$, then S_λ is the same as S.

(ii) λ-statistical convergence is a special case of $\mathbf{A}$-statistical convergence (see [2], [7]) if the matrix $\mathbf{A} = (a_{nk})$ is taken as

$$a_{nk} = \begin{cases} \frac{1}{\lambda_n} & \text{if } k \in I_n , \\ 0 & \text{if } k \notin I_n . \end{cases}$$

2.

In this section, we find the relationship of S_λ with $[V,\lambda]$ and $(C,1)$ methods.

Let Λ denote the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n$ and $\lambda_1 = 1$. The following theorem is the analogue of [6; Theorem 1].

THEOREM 2.1. *Let $\lambda \in \Lambda$, then*

(i) $x_k \to L\ [V,\lambda] \implies x_k \to L\ (S_\lambda)$ *and the inclusion* $[V,\lambda] \subseteqq S_\lambda$ *is proper,*

(ii) *if* $x \in \ell_\infty$ *and* $x_k \to L\ (S_\lambda)$, *then* $x_k \to L\ [V,\lambda]$ *and hence* $x_k \to L(C,1)$ *provided* $x = (x_k)$ *is not eventually constant,*

(iii) $S_\lambda \cap \ell_\infty = [V,\lambda] \cap \ell_\infty$,

where ℓ_∞ *denotes the set of bounded sequences.*

Proof.

(i) Let $\varepsilon > 0$ and $x_k \to L\ [V,L]$. We have

$$\sum_{k\in I_n} |x_k - L| \ge \sum_{\substack{k\in I_n \\ |x_k-L|\ge\varepsilon}} |x_k - L| \ge \varepsilon \big|\{k \in I_n :\ |x_k - L| \ge \varepsilon\}\big| .$$

Therefore $x_k \to L\ [V,\lambda] \implies x_k \to L\ (S_\lambda)$.

The following example shows that $S_\lambda \subsetneqq [V,\lambda]$.

Define $x = (x_k)$ by

$$x_k = \begin{cases} k & \text{for } n - \big[\sqrt{\lambda_n}\,\big] + 1 \le k \le n\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \notin \ell_\infty$ and for every ε $(0 < \varepsilon \le 1)$

$$\frac{1}{\lambda_n}\big|\{k \in I_n :\ |x_k - 0| \ge \varepsilon\}\big| = \frac{[\sqrt{\lambda_n}\,]}{\lambda_n} \to 0 \qquad \text{as} \quad n \to \infty\,,$$

i.e. $x_k \to 0\ (S_\lambda)$. On the other hand,

$$\frac{1}{\lambda_n}\sum_{k\in I_n} |x_k - 0| \to \infty \qquad (n \to \infty)\,,$$

i.e. $x_k \not\to 0\ [V,\lambda]$.

(ii) Suppose that $x_k \to L\ (S_\lambda)$ and $x \in \ell_\infty$, say $|x_k - L| \le M$ for all k. Given $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{\lambda_n}\sum_{k\in I_n} |x_k - L| &= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |x_k-L|\ge\varepsilon}} |x_k - L| + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |x_k-L|<\varepsilon}} |x_k - L| \\ &\le \frac{M}{\lambda_n}\big|\{k \in I_n :\ |x_k - L| \ge \varepsilon\}\big| + \varepsilon\,, \end{aligned}$$

which implies that $x_k \to L\ [V,\lambda]$.

Further, we have

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - L) = \frac{1}{n}\sum_{k=1}^{n-\lambda_n}(x_k - L) + \frac{1}{n}\sum_{k\in I_n}(x_k - L)$$

$$\leq \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}|x_k - L| + \frac{1}{\lambda_n}\sum_{k\in I_n}|x_k - L|$$

$$\leq \frac{2}{\lambda_n}\sum_{k\in I_n}|x_k - L|\,.$$

Hence $x_k \to L\ (C,1)$, since $x_k \to L\ [V,\lambda]$.

(iii) This immediately follows from (i) and (ii). □

3.

It is easily seen that $S_\lambda \subseteqq S$ for all λ, since λ_n/n is bounded by 1. In this section, we prove the following relation.

THEOREM 3.1. $S \subseteq S_\lambda$ *if and only if*

$$\liminf_{n\to\infty}\frac{\lambda_n}{n} > 0\,. \tag{3.1.1}$$

P r o o f. For given $\varepsilon > 0$ we have

$$\{k \leq n:\ |x_k - L| \geq \varepsilon\} \supset \{k \in I_n:\ |x_k - L| \geq \varepsilon\}\,.$$

Therefore

$$\frac{1}{n}\big|\{k \leq n:\ |x_k - L| \geq \varepsilon\}\big| \geq \frac{1}{n}\big|\{k \in I_n:\ |x_k - L| \geq \varepsilon\}\big|$$

$$\geq \frac{\lambda_n}{n}\cdot\frac{1}{\lambda_n}\big|\{k \in I_n:\ |x_k - L| \geq \varepsilon\}\big|\,.$$

Taking the limit as $n \to \infty$ and using (3.1.1), we get

$$x_k \to L\ (S) \implies x_k \to L\ (S_\lambda)\,.$$

Conversely, suppose that $\liminf_{n\to\infty}\frac{\lambda_n}{n} = 0$. As in [4; p. 510], we can choose a subsequence $\big(n(j)\big)_{j-1}^{\infty}$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$. Define a sequence $x = (x_i)$ by

$$x_i = \begin{cases} 1 & \text{if } i \in I_{n(j)}\,,\ \ j = 1,2,\dots\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in [C,1]$, and hence, by [1; Theorem 2.1], $x \in S$. But on the other hand, $x \notin [V,\lambda]$ and Theorem 2.1(ii) implies that $x \notin S_\lambda$. Hence (3.1.1) is necessary. □

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