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THE BOX-COUNTING DIMENSION OF THE SINE-CURVE

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ABSTRACT. We show that the box-counting dimension of the sine-curve is $\frac{3}{2}$.

It is well known that the "dimension" (box-counting, Hausdorff-Besicovitch, divider dimension etc.) of the graph of a real valued continuous function on an interval can take values strictly greater than one ([1], [2]). The examples studied of this kind are mostly of a very ill-behaved nature (being nowhere differentiable). We show that the celebrated sine-curve provides a simple example of a smooth function on an interval whose graph has box-counting dimension exceeding one.

PROPOSITION 1. *The box-counting dimension of the graph of the function*

$$f\colon (0,1] \to \mathbb{R}, \qquad f(x) = \sin\frac{1}{x}$$

exists and is equal to $\frac{3}{2}$.

Proof. Let G denote the graph of f:

$$G = \left\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1],\ y = \sin\tfrac{1}{x}\right\}.$$

It suffices to show that the box-counting dimension of the closure $\bar{G} = G \cup \{0\} \times [-1,1]$ of G in $\mathbb{R}^2$ exists and is equal to $\frac{3}{2}$. We compute the box dimension of $\bar{G}$ with mesh-counting. Consider a mesh of size $\varepsilon_k = \frac{1}{2k\pi} - \frac{1}{(2k+1)\pi}$ where $k \in \mathbb{N}$, and let $N(\bar{G}, \varepsilon_k)$ denote the number of mesh-squares intersecting $\bar{G}$. It is not difficult to see that $\frac{k}{\varepsilon_k}$ can be taken as a lower bound for $N(\bar{G}, \varepsilon_k)$ (compare the figure and consider the vertical middle-segments of the "wave-hills"). To find an upper bound, we define

$$G_1 = \left\{(x,y) \in \bar{G} \mid x \le \tfrac{1}{(2k+1)\pi}\right\} \quad \text{and} \quad \bar{G}_2 = \left\{(x,y) \in \bar{G} \mid x \ge \tfrac{1}{(2k+1)\pi}\right\}.$$

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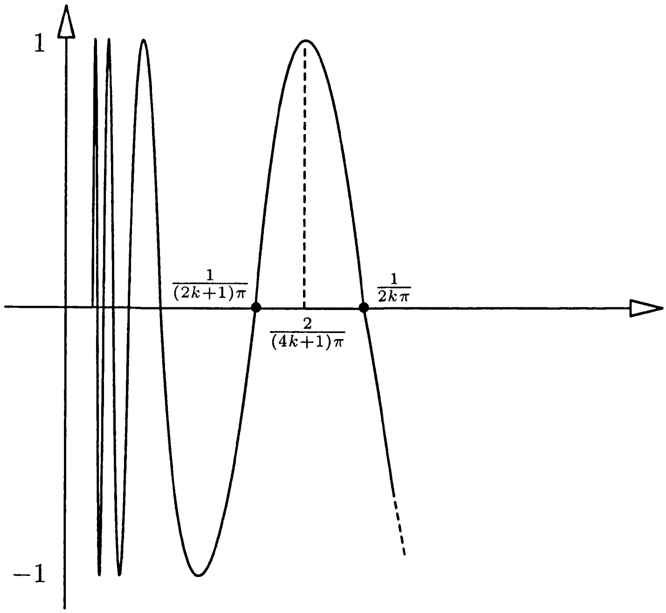

FIGURE 1.

Obviously

$$N(\bar{G},\varepsilon_k) \le N(\bar{G}_1,\varepsilon_k) + N(\bar{G}_2,\varepsilon_k)\,.$$

Covering the rectangle $\left[0, \frac{1}{(2k+1)\pi}\right] \times [-1,1]$ we get

$$N(\bar{G}_1,\varepsilon_k) \le (2k+2)\left(\frac{2}{\varepsilon_k}+2\right) \le 2^7k^3 \qquad \text{(for } k \text{ sufficiently large).}$$

To find an estimate for $N(\bar{G}_2,\varepsilon_k)$, we apply Proposition 11.1 of [2] to a number of subintervals of $\left[\frac{1}{(2k+1)\pi}, 1\right]$, on which the function is monotone. This proposition can be formulated for a monotone function $f\colon [a,b] \to \mathbb{R}$ as follows:

Let a δ-mesh for $\mathbb{R}^2$ be given. Then the number of squares intersecting the graph of f can be bounded from above by

$$2\frac{b-a}{\delta} + 4 + \frac{1}{\delta}|f(b)-f(a)|\,.$$

Applying this upper bound to the $4k+2$ intervals

$$\left[\frac{1}{(2k+1)\pi}, \frac{2}{(4k+1)\pi}\right],\ \left[\frac{2}{(4k+1)\pi}, \frac{1}{2k\pi}\right], \dots,\ \left[\frac{2}{3\pi}, \frac{1}{\pi}\right],\ \left[\frac{1}{\pi}, \frac{2}{\pi}\right],\ \left[\frac{2}{\pi}, 1\right]$$

and adding them up, we obtain:

$$N(\bar{G}_2, \varepsilon_k) \le \frac{2}{\varepsilon_k} + 4(4k+2) + \frac{4k+2}{\varepsilon_k} \le 2^7 k^3 \qquad \text{(for } k \text{ sufficiently large).}$$

Hence

$$\begin{aligned} \frac{k}{\varepsilon_k} &\le N(\bar{G}, \varepsilon_k) \le 2^8 k^3\,, \\ 2k^2(2k+1)\pi &\le N(\bar{G}, \varepsilon_k) \le 2^8 k^3\,, \\ k^3 &\le N(\bar{G}, \varepsilon_k) \le 2^8 k^3\,. \end{aligned}$$

Now we can pass to the dimension calculation using the bounds

$$\begin{aligned} \frac{\log k^3}{\log \frac{1}{\varepsilon_k}} &\le \frac{\log N(\bar{G}, \varepsilon_k)}{\log \frac{1}{\varepsilon_k}} \le \frac{\log 2^8 k^3}{\log \frac{1}{\varepsilon_k}}\,, \\ \frac{\log k^3}{\log 2k(2k+1)\pi} &\le \frac{\log N(\bar{G}, \varepsilon_k)}{\log \frac{1}{\varepsilon_k}} \le \frac{\log 2^8 k^3}{\log 2k(2k+1)\pi}\,, \\ \frac{\log k^3}{\log 2^5 k^2} &\le \frac{\log N(\bar{G}, \varepsilon_k)}{\log \frac{1}{\varepsilon_k}} \le \frac{\log 2^8 k^3}{\log k^2} \qquad \text{(for sufficiently large } k\text{)}\,, \\ \frac{3\log k}{5\log 2 + 2\log k} &\le \frac{\log N(\bar{G}, \varepsilon_k)}{\log \frac{1}{\varepsilon_k}} \le \frac{8\log 2 + 3\log k}{2\log k}\,. \end{aligned}$$

Hence $\lim_{k\to\infty} \frac{\log N(\bar{G},\varepsilon_k)}{\log \frac{1}{\varepsilon_k}} = \frac{3}{2}$. □

Remark. To see the existence of the limit $\lim_{\varepsilon\to 0} \frac{\log N(\bar{G},\varepsilon)}{\log \frac{1}{\varepsilon}}$ rigorously, one must consider a continuous approach $\varepsilon \to 0$, or a geometric-sequential approach $\varepsilon_k = r^k \to 0$ ([3]). But the test of Barnsley can easily be improved as follows:

Assume there is a monotone decreasing null-sequence ε_k such that

1. there are numbers $0 < c_1 < c_2$ and $0 < r < 1$ with $c_1 < \frac{\varepsilon_k}{r^k} < c_2$ for all $k \in \mathbb{N}$,
2. $\lim_{k\to\infty} \frac{\log N(X,\varepsilon_k)}{\log \frac{1}{\varepsilon_k}}$ exists ($X \subset \mathbb{R}^n$ compact).

Then $\lim_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{\log \frac{1}{\varepsilon}}$ exists.

In our case, a subsequence of $\varepsilon_k = \frac{1}{2k(2k+1)\pi}$ satisfying the first condition also can easily be chosen.

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