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BALANCED INTEGRAL TREES

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ABSTRACT. A graph G is called integral if all the zeros of the characteristic polynomial $P(G;\lambda)$ are integers. In the present paper we investigate the question which trees are integral. Some positive and negative results are presented. Among others, we prove that there are infinitely many balanced integral trees of diameter 8, but there is none of diameter $4k+1$ for $k \geq 1$. A tree T is called balanced if the vertices at the same distance from the centre of T have the same degree. The problem of the existence of balanced integral trees of arbitrarily large diameter remains open.

1. Introduction

A graph G is called integral if it has an integral spectrum, i.e. if all the zeros of the characteristic polynomial $P(G;\lambda)$ are integers. In general, the problem of characterizing integral graphs seems to be difficult. Thus it makes sense to restrict our investigations to some interesting families of graphs. For instance it is known (see [1], [11]) that there are exactly 13 integral cubic graphs. Trees present another important family of graphs for which the problem has been considered ([4], [6], [8], [9], [12], [13]). In contrast to cubic graphs there are infinitely many integral trees, and it turns out that even the problem of determining integral trees seems to be not at hand. There are many unanswered questions related with this problem. For instance, all the integral trees constructed so far have diameter at most 6, which suggests the following problem:

(P1) *Are there integral trees of arbitrarily large diameter?*

It is well known that the centre $Z(T)$ of a tree T consists either of a central vertex, or of a central edge depending on whether the diameter of T is even, or odd, respectively. If all the vertices at the same distance from the centre

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$Z(T)$ are of the same degree, then the tree T will be called *balanced*. Clearly, the structure of a balanced tree (without vertices of degree 2) is determined by the parity of its diameter and the sequence $(n_k, n_{k-1}, \dots, n_1)$, where k is the radius of T and n_j $(1 \le j \le k)$ denotes the number of successors of a vertex at distance $k - j$ from the centre $Z(T)$. In what follows, n_i $(i = 1, 2, \dots)$ always stands for an integer ≥ 2. The balanced trees of diameter $2k$ will be encoded by the sequence $(n_k, \dots, n_1)$, while those with diameter $2k + 1$ by $(1; n_k, \dots, n_1)$. Sequences $(n_k, \dots, n_1)$ and $(1; n_k, \dots, n_1)$ will be called *integral* if the corresponding balanced trees are integral.

The main concern of this paper is to investigate balanced integral trees. Here is a brief survey of the related known results.

Harary and Schwenk [4] showed that (n_1) is integral if and only if n_1 is a square. Moreover, they gave the following examples of integral sequences: $(1;2)$, $(1;6)$, $(3,1)$. Later, Schwenk and Watanabe [12] proved that the sequence (n_2, n_1) is integral if and only if both n_1 and $n_1 + n_2$ are squares. Further, they showed that the sequence $(1; n_1)$ is integral if and only if $n_1 = r(r+1)$ for some $r \in \mathbb{N}$. In [12] they also proved that there is no integral sequence of shape $(n_k, 1, \dots, 1)$ for $k \ge 3$. Godsil (see [12]) found a family of integral sequences of length 3.

We extend the above by establishing the following results:

(1) If $(n_k, \dots, n_1)$ is integral, then $(n_j, \dots, n_1)$ is integral for $1 \le j \le k-1$ (Corollary 3.4);
(2) If $(1; n_k, \dots, n_1)$ is integral then $(n_j, \dots, n_1)$ is integral for $1 \le j \le k-1$ (Corollary 4.3);
(3) If $(n_k, \dots, n_1)$ is integral, then $(q^2 n_k, \dots, q^2 n_1)$ is integral for every $q \in \mathbb{N}$ (Theorem 3.5);
(4) There is no balanced integral tree of diameter $4k + 1$ (Theorem 4.5).

On the other hand, one can verify that the sequence (616, 225, 672, 4) is integral. It follows from (3) that for every integer q the sequence $(616q^2, 225q^2, 672q^2, 4q^2)$ is integral, too. Thus we have infinitely many integral sequences of length 4 implying that there are infinitely many balanced integral trees of diameter 8. Unfortunately, even using a computer, we have not been able to construct an integral sequence of length greater than 4.

The following problem remains open

(P2) *Are there integral sequences $(n_k, \dots, n_1)$ of arbitrary large length?*

Clearly, the positive solution of (P2) would imply the positive solution of (P1).

The current knowledge about the existence of balanced integral trees is summarized in the following table:

diameter:	existence:	authors:
2	yes	Harary, Schwenk [4]
3	yes	Watanabe, Schwenk [12]
4	yes	Harary, Schwenk [4]
5	no	Theorem 4.5
6	yes	Godsil (see [12])
7	no	Theorem 4.6
8	yes	Table 1
9	no	Theorem 4.5
4k+1	no	Theorem 4.5
diam ≥ 10 and diam $\neq 4k+1$	?	?

Hence the first unsolved case occurs for the diameter 10. An equivalent formulation of the problem reads as follows:

(P3) *Is there an integral sequence* $(n_5, n_4, n_3, n_2, n_1)$ *?*

It should be noted that the negative solution of (P3) together with the hereditary properties (1), (2) would imply that there is no balanced integral tree of diameter ≥ 12.

Let us note that there exist infinitely many integral trees of diameter 5 (see [9]), but there is no balanced integral tree of diameter 5 (see Theorem 4.5).

2. Preliminaries and general results

In the rest of this paper, a *graph* means a directed graph with multiple edges and loops. The main reason for dealing with directed graphs rather than undirected graphs is technical convenience. Most notions and definitions introduced in the paper for (directed) graphs will be also applied to undirected graphs in the following sense: Denote by $\overrightarrow{G}$ the directed graph which arises from an undirected graph G by replacing each edge of G by a pair of oppositely directed arcs. Since both G and $\overrightarrow{G}$ have identical adjacency matrices, to investigate the properties of the adjacency matrix of G we may consider the graph $\overrightarrow{G}$ instead of G and vice versa.

Let G be a graph. The *characteristic polynomial* $P(G;\lambda)$ of a graph G is defined to be the characteristic polynomial of the adjacency matrix of G. We say that G has an *integral spectrum* or that G is *integral* if all the zeros of $P(G;\lambda)$ are integers.

A partition of the vertex set $V(G) = \bigcup_{i=1}^{n} V_i$ of a graph G is called an *equitable partition* if there exists a square matrix $\mathbf{M} = (d_{ij})$ of order n such that for every $i, j \in \{1, 2, \dots, n\}$ and for every vertex $x \in V_i$ there are exactly d_{ij} arcs joining x to vertices in V_j. The graph D with the adjacency matrix $\mathbf{M}$ is called a *front-divisor* of G. The fact that D is a front-divisor of G will be denoted by $D \mid G$. Obviously, the vertices of D correspond to the classes V_i of the equitable partition. Two classes V_i, V_j of the equitable partition of G are called adjacent classes if $d_{ij} > 0$.

The most important property of a front-divisor D of a graph G is that the characteristic polynomial $P(D;\lambda)$ divides the characteristic polynomial of G ([2; Theorem 4.5]), i.e. there exists a polynomial $P(C;\lambda)$ such that $P(G;\lambda) = P(D;\lambda)P(C;\lambda)$. Unfortunately, in general, the polynomial $P(C;\lambda)$ need not be a characteristic polynomial of a graph. However, it is proved in [2] that it is always a characteristic polynomial of an integer matrix $\mathbf{M}_C$. The combinatorial object C corresponding to $\mathbf{M}_C$ is a graph whose arcs are valued by plus or minus one. Then C is called a *codivisor* of G and $P(C;\lambda)$ is the characteristic polynomial of C.

In [6] the properties of front-divisors of trees are discussed.

There, the following Lemma, which shows that each front-divisor of a tree has a tree-like structure, is proved.

LEMMA. (see [6; Proposition 3.4]) *Let D be a front-divisor of a tree T. Then the following statements hold:*

(1) *Let $u \in Z(T)$ and U be a class of equitable partition with $u \in U$. Then $|U| = 1$ or $|U| = 2$ and D has a directed u-based loop.*
(2) *D has exactly one loop, or it has none.*
(3) *Let u, v be two vertices of D and U, V be corresponding classes of the equitable partition. Let $y \in V$ and $x \in U$ be such vertices, that y is successor of x in any radial path of T. Then $|U| \le |V|$ and there exist exactly one $v - u$ path in D. (A $w - t$ path in T will be called a radial if $w \in Z(T)$).*
(4) *D has no cycle of length greater than two.*
(5) *Let $U = \{u \in T : u \in V_i,\ |V_i| = 1\}$, where V_i be a class of the equitable partition corresponding to D. Then $Z(T) \subseteq U$ and U is a connected subgraph of T.*

It follows from the above lemma that it is always possible to choose vertices representing the classes V_i of the equitable partition of T in such a way, that they induce a connected subtree of T. For this it is sufficient to take $v_0 \in Z(T) \subseteq V_0$ and if V_0, V_1 are two adjacent classes we choose the vertex $v_1 \in V_1$ such that $(v_0, v_1) \in E(T)$. Now, let $v_0, v_1, \dots, v_i$ be vertices representing the classes $V_0, V_1, \dots, V_i$ of the equitable partition such that they induce a connected subtree of T and V_{i+1} be a class adjacent to the some of the classes $V_0, V_1, \dots, V_i$, for example V_i. Then we choose the vertex $v_{i+1} \in V_{i+1}$ such that $(v_i, v_{i+1}) \in E(T)$.

In the following theorem we show that the characteristic polynomial $P(C; \lambda)$ of a codivisor of a tree T may be expressed in terms of proper subtrees of T.

THEOREM 2.1. *Let D be a front-divisor of a tree T and C be the corresponding codivisor. Then*

$$P(C; \lambda) = \prod_{T_i \subseteq T - V(D)} P(T_i; \lambda),$$

where T_i are connected components of $T - V(D)$ and in the case, that D has the v_1-based loop, T_1 is the connected component of $T - V(D)$ with the u_1-based loop valued by -1. ($u_1 \in Z(T)$).

P r o o f . Let $R = \{v_1, \dots, v_m\}$ be a selection of representatives of the vertex set of D such that the induced subgraph is connected and $v_1 \in Z(T)$. Every strongly connected component $T' \subseteq T - R$ is a rooted tree with the root u' where $u' \in T'$ is the end-vertex of the edge joining T' to R. Let $T_1, T_2, \dots, T_r$ be the strongly connected components of the graph $T - R$, and $u_1, u_2, \dots, u_r$ be the roots of $T_1, T_2, \dots, T_r$ where $u_i \in T_i$ be the end-vertex of the edge joining T_i to R. It follows from above Lemma that T_i can be put in order $T_1, T_2, \dots, T_r$ such that for each T_i, T_j, $i < j$ we have $d(v_1, u_i) \le d(v_1, u_j)$. Now, using the algorithm in [2; p. 126], we see that the adjacency matrix $\mathsf{M}(C)$ of the codivisor C has the following form

$$\begin{pmatrix} \mathsf{M}(T_1) & & & \\ & \mathsf{M}(T_2) & & -1 \text{ or } 0 \\ & & \ddots & \\ & 0 & & \\ & & & \mathsf{M}(T_r) \end{pmatrix}$$

where $\mathsf{M}(T_i)$ is the adjacency matrix of the tree T_i for $i = 1, 2, \dots, r$. In the case if D has a loop, then by the algorithm in [2; p. 126] $\mathsf{M}(T_1)$ is the adjacency matrix of the component T_1 with a loop denote by -1 which is u_1-based,

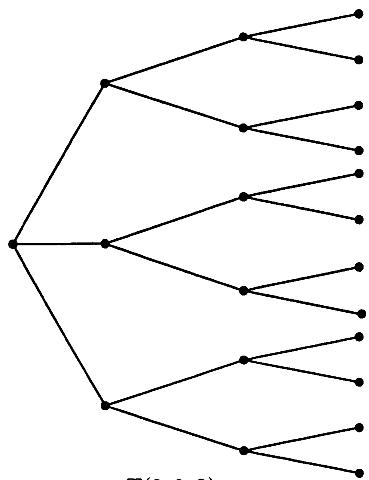
$u_1 \in Z(T)$. Since $\mathbf{M}(C)$ is a triangle matrix,

$$P(C;\lambda) = |\lambda\mathbf{I} - \mathbf{M}(C)| = \prod_{i=1}^{r} P(T_i;\lambda)\,.$$

□

3. Balanced rooted trees

A rooted (undirected) tree T will be called a *balanced rooted tree* if the distance partition Π of $V(T)$ with respect to the root of T is an equitable partition. It follows that vertices of the same class of Π have the same degrees. Thus, if T is a balanced rooted tree with a root w of the excentricity $k \geq 1$, then there exist integers $n_1, n_2, \dots, n_k$ such that a vertex v of T at distance i, $0 \leq i \leq k-1$ from w has exactly n_{k-i} successors. Hence, every nontrivial balanced rooted tree T being uniquely determined by the sequence $(n_k, n_{k-1}, \dots, n_1)$, T will be denoted by $T(n_k, n_{k-1}, \dots, n_1)$. Note that a balanced rooted tree $T(n_k, \dots, n_1)$ is a balanced tree of even diameter $2k$ if and only if $n_k \geq 2$.



$T(3,2,2)$

$D(3,2,2)$

FIGURE 1.

The front-divisor of $T = T(n_k, n_{k-1}, \dots, n_1)$ determined by the distance partition Π of T is a directed graph $D(n_k, n_{k-1}, \dots, n_1)$ with vertices $v_0, v_1, \dots, v_k$; a vertex v_i is joined by n_{k-i} parallel arcs to v_{i+1} and v_{i+1} is joined by one arc to v_i, for $0 \le i \le k-1$. The tree $T(3,2,2)$ and its divisor $D(3,2,2)$ are depicted in Fig. 1. The divisor $D(n_k, n_{k-1}, \dots, n_1)$ will be called a *rooted canonical divisor*. It is proved in [6] that for every tree T there exists a divisor D^* such that $D^* \mid D$ for every front-divisor D of T. The front-divisor D^* is called the *canonical divisor* of T. Note, that if T is a balanced rooted tree, then the rooted canonical divisor is the canonical divisor of T with the exception that T is a starlike tree (see [12]) and the root of T is not the central vertex of T.

In order to cover also the trivial case we denote by $T(\emptyset)$, $D(\emptyset)$ the graph consisting of one isolated vertex.

THEOREM 3.1. *Let $T(n_k, n_{k-1}, \dots, n_1)$ be a balanced rooted tree and $D(n_k, n_{k-1}, \dots, n_1)$ be the corresponding canonical rooted divisor. Then*

$$\begin{aligned} &P\big(D(n_k, n_{k-1}, \dots, n_1); \lambda\big) \\ &= \lambda P\big(D(n_{k-1}, n_{k-2}, \dots, n_1); \lambda\big) - n_k P\big(D(n_{k-2}, n_{k-3}, \dots, n_1); \lambda\big) \qquad (k \ge 2)\,. \end{aligned}$$

Proof. Let $\mathbf{M}$ be the adjacency matrix of $D(n_k, n_{k-1}, \dots, n_1)$. Then

$$P\big(D(n_k, n_{k-1}, \dots, n_1); \lambda\big) = |\lambda \mathbf{I} - \mathbf{M}|$$
$$= \begin{vmatrix} \lambda & -n_k & 0 & \dots & 0 & 0 \\ -1 & \lambda & -n_{k-1} & \dots & 0 & 0 \\ 0 & -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -n_1 \\ 0 & 0 & 0 & \dots & -1 & \lambda \end{vmatrix}.$$

Now, the expansion by cofactors of the first column yields the statement. □

THEOREM 3.2. *Let $D = D(n_k, n_{k-1}, \dots, n_1)$ be a canonical rooted divisor of a balanced rooted tree $T(n_k, n_{k-1}, \dots, n_1)$ and*

$$P(D; \lambda) = \lambda^{k+1} + \sum_{i=1}^{k+1} a_i \lambda^{k+1-i}$$

be its characteristic polynomial, then

(i) $a_{2i+1} = 0$, $i = 1, 2, \dots$;

(ii) $a_{2i} = (-1)^i \sum\limits_L \prod\limits_{j=1}^{i} n_{s_j}$, *where the summation is done over all subsets* $L = \{s_1, s_2, \dots, s_i\} \subset \{1, 2, \dots, k\}$, $s_r \ne s_q$, $|s_r - s_q| \ne 1$ *for every* $r, q \in \{1, 2, \dots, i\}$, $r \ne q$.

P r o o f . First observe that $D(n_k, n_{k-1}, \dots, n_1)$ contains only directed cycles of length 2. Further, every directed linear subgraph F of D with exactly $2i$ vertices contains exactly i directed cycles of length two. It is easy to verify that the number of such linear subgraphs of D is given by (ii). Now, the statement follows from [2; Theorem 1.2]. □

THEOREM 3.3. *Let $T = T(n_k, n_{k-1}, \dots, n_1)$ be a balanced rooted tree with the canonical rooted divisor $D = D(n_k, n_{k-1}, \dots, n_1)$ and let C denote the distinguished codivisor of D. Then the characteristic polynomial of C is*

$$P(C;\lambda) = \prod_{i=1}^{k} \left[P\bigl(T(n_{k-i}, n_{k-i-1}, \dots, n_1); \lambda\bigr)\right]^{n_{k-i+1}-1}.$$

P r o o f . Because for every $i = 1, \dots, k$ there are exactly $(n_{k-i+1} - 1)$ strongly connected components $T(n_{k-i}, n_{k-i-1}, \dots, n_1)$ in $T - V(D)$ the proof follows from Theorem 2.1. □

Remark. By Theorem 3.3

$$P\bigl(T(n_k, \dots, n_1); \lambda\bigr) = P\bigl(D(n_k, n_{k-1}, \dots, n_1); \lambda\bigr) \prod_{i=1}^{k} \left[P\bigl(T(n_{k-i}, n_{k-i-1}, \dots, n_1); \lambda\bigr)\right]^{n_{k-i+1}-1}.$$

Iterating this formula, $P\bigl(T(n_k, \dots, n_1); \lambda\bigr)$ is expressed as a product of powers of $P\bigl(T(\emptyset); \lambda\bigr) = \lambda$ and $P\bigl(D(n_{k-i}, \dots, n_1); \lambda\bigr)$ where the exponents can explicitly be calculated. Thus $(n_k, \dots, n_1)$ is integral if and only if all the zeros of all the $P\bigl(D(n_{k-i}, \dots, n_1); \lambda\bigr)$ are integral.

The above theorem has the following interesting corollary.

COROLLARY 3.4. *A sequence $(n_k, n_{k-1}, \dots, n_1)$ is integral if and only if all zeros of the polynomial $P\bigl(D(n_k, n_{k-1}, \dots, n_1); \lambda\bigr)$ are integral and $(n_{k-1}, n_{k-2}, \dots, n_1)$ is integral. In particular, if $(n_k, \dots, n_1)$ is integral then $(n_j, n_{j-1}, \dots, n_1$ is integral for every $1 \le j \le k-1$.*

THEOREM 3.5. *A sequence $(n_k, n_{k-1}, \dots, n_1)$ of positive integers is integral if and only if for every $q \in \mathbb{N}$ the sequence $(q^2 n_k, q^2 n_{k-1}, \dots, q^2 n_1)$ is integral.*

P r o o f . Let $q \in \mathbb{N}$. By Corollary 3.4 it is sufficient to prove that $D(n_k, n_{k-1}, \dots, n_1)$ has integral spectrum if and only if $D(q^2 n_k, q^2 n_{k-1}, \dots, q^2 n_1)$ has integral one.

Let $P_k(\lambda) = \lambda^{k+1} + \sum\limits_{i=1}^{k+1} a_i \lambda^{k+1-i}$ and $P'_k(\lambda) = \lambda^{k+1} + \sum\limits_{i=1}^{k+1} b_i \lambda^{k+1-i}$ be the characteristic polynomials of $D(n_k, n_{k-1}, \dots, n_1)$ and $D(q^2 n_k, q^2 n_{k-1}, \dots, q^2 n_1$, respectively. By Theorem 3.2 for every $i = 1, 2, \dots$, $b_{2i+1} = a_{2i+1} = 0$ and $b_{2i} = q^{2i} a_{2i}$. Now, $x_1, x_2, \dots, x_{k+1}$ are zeros of $P_k(\lambda)$ if and only if for every $i = 1, 2, \dots$

$$a_{2i} = \sum x_{j_1} x_{j_2} \dots x_{j_{2i}}\,; \qquad j_s \in \{1, 2, \dots, k+1\}.$$

Multiplying this equation by q^{2i} we get

$$q^{2i} a_{2i} = q^{2i} \sum x_{j_1} x_{j_2} \dots x_{j_{2i}}$$

and further

$$b_{2i} = \sum (q x_{j_1})(q x_{j_2}) \dots (q x_{j_{2i}}).$$

But, the last equation is equivalent to the statement $qx_1, qx_2, \dots, qx_{k+1}$ are zeros of polynomial $P'_k(\lambda)$. $\square$

COROLLARY 3.6. *If there exists an integral sequence $(n_k, n_{k-1}, \dots, n_1)$ of length k, then there are infinitely many integral sequences of length k.*

An integral sequence $(n_k, n_{k-1}, \dots, n_1)$ such that the $g \cdot c \cdot d \cdot (n_k, \dots, n_1)$ is square-free will be called a *primitive integral sequence.*

The following theorems contain results about integral sequences of length ≤ 4.

THEOREM 3.7. ([12; Theorem 1 and Theorem 2])

(a) *The balanced rooted tree $T(n_1)$ is integral if and only if n_1 is a square.*
(b) *The balanced rooted tree $T(n_2, n_1)$ is integral if and only if both n_1 and $(n_2 + n_1)$ are squares.*

Combining Theorem 3.7 with Corollary 3.4 we may obtain characterization theorems for some subfamilies of balanced integral trees. In [12; Theorem 1] the authors presented a characterization of starlike integral trees, i.e. the balanced trees $T(n_k, 1, \dots, 1)$. It follows from Theorem 3.7 that the sequence $(1, 1)$ is not integral. By Corollary 3.4 $k \leq 2$. Now the characterization follows from Theorem 3.7. Namely, we have that the only integral starlike trees are $T(m^2)$ and $T(m^2 - 1, 1)$ for $m \geq 2$. Analogously we obtain that the only r-uniform balanced integral tree $T(r, \dots, r)$ is the star $T(r)$, where $r = m^2$ and $m \geq 2$.

The following theorem gives us a characterization of integral sequences of length 3.

THEOREM 3.8. *A sequence (n_3, n_2, n_1) is integral if and only if $n_1 = k^2$, $n_2 = n^2 + 2nk$, $n_3 = \frac{a^2 b^2}{k^2}$ where a, b, k, n are positive integers satisfying*

$$(k^2 - b^2)(a^2 - k^2) = k^2(n^2 + 2nk), \qquad b < k < a. \tag{3.1}$$

Proof.

(a) Let $T(n_3, n_2, n_1)$ be an integral tree. Theorem 3.7 and Corollary 3.4 imply that there exist integers k, n such that $n_1 = k^2$, $n_2 = n^2 + 2nk$, and the polynomial $\lambda^4 - (n_1 + n_2 + n_3)\lambda^2 + n_1 n_2 = 0$ has only integral roots. But, this polynomial has only integral roots if and only if it may be factored into $(\lambda^2 - a^2)(\lambda^2 - b^2)$ with a, b positive; that is

$$\lambda^4 - (n_1 + n_2 + n_3)\lambda^2 + n_1 n_3 = (\lambda^2 - a^2)(\lambda^2 - b^2)\,. \tag{3.2}$$

From the above formulas we have

$$a^2 + b^2 = (k+n)^2 + n_3\,, \tag{3.3}$$

$$a^2 b^2 = k^2 n_3\,. \tag{3.4}$$

Excluding n_3 from (3.3) and (3.4) we get

$$a^2 b^2 = k^2\big(a^2 + b^2 - (n+k)^2\big)\,. \tag{3.5}$$

Now a simple modification of (3.5) gives us the required equation.

(b) Let $n_1 = k^2$, $n_2 = n^2 + 2nk$, $n_3 = \frac{a^2b^2}{k^2}$ and $a, b, k, n \in \mathbb{N}$ satisfying (3.1). From formula (3.1) we have

$$\frac{a^2 b^2}{k^2} = a^2 + b^2 - (n+k)^2\,. \tag{3.6}$$

Then

$$\begin{aligned} P\big(D(n_3, n_2, n_1); \lambda\big) &= \lambda^4 - \left(k^2 + n^2 + 2nk + \frac{a^2b^2}{k^2}\right)\lambda^2 + a^2b^2 \\ &= \lambda^4 - (a^2 + b^2)\lambda^2 + a^2b^2 \\ &= (\lambda^2 - a^2)(\lambda^2 - b^2) \\ &= (\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b)\,. \end{aligned}$$

Now, from Theorem 3.7 and Corollary 3.4 $T\big(\frac{a^2b^2}{k^2}, n^2 + 2nk, k^2\big)$ is integral. □

Theorem 3.8 implies that the problem of characterizing integral sequences of length 3 is equivalent with the problem of solving the diophantine equation (3.1). Let $d = g \cdot c \cdot d \cdot (b, k)$, $k = dx$, $b = dy$. Then (3.1) is equivalent with

$$(x^2 - y^2)(a^2 - d^2x^2) = x^2(n^2 + 2ndx)\,, \tag{3.7}$$

where $g \cdot c \cdot d \cdot (x, y) = 1$. Since $g \cdot c \cdot d \cdot (x, y) = 1$ we have $a = xc$ for some c. Hence (3.7) is equivalent with

$$(x^2 - y^2)(c^2 - d^2) = n(n + 2dx)\,. \tag{3.8}$$

Assuming $n = x^2 - y^2$ or $n = c^2 - d^2$ (3.8) can be reduced to

$$c^2 + y^2 = (x+d)^2\,, \tag{3.9}$$

or to

$$c^2 + y^2 = (x-d)^2\,, \tag{3.10}$$

respectively, the solutions being Pythagorian triples $(c, y, x+d)$, or $(c, y, x-d)$, respectively. One class of such solutions obtained in this way is presented in Corollary 3.9. Unfortunately, the solutions of (3.1) derived from the solutions of (3.9) and (3.10) do not cover all the solutions of (3.1). These formulas allow infinitely many primitive integral sequences to be generated.

We note that Godsil (see [12]) also constructed a class of integral sequences of length 3.

COROLLARY 3.9. *For every $u \in \mathbb{N}$, $u > 1$, the tree $T\big(4u^2(u^2-1)^2, 8u^4 - 6u^2 + 1, u^4\big)$ is a balanced integral rooted tree. In particular, there exist infinitely many primitive integral sequences of length 3.*

Proof. Let $a = 2u^3$, $b = u^2 - 1$, $n = 2u^2 - 1$ and $k = u^2$. By substituting these a, b, n and k in (3.1) one can check that they present a solution of (3.1). Further, it follows from $g\cdot c\cdot d\cdot(u^4, 8u^4 - 6u^2 + 1) = g\cdot c\cdot d\cdot(u^4, 6u^2 - 1) = 1$ that the sequence $\big(4u^2(u^2-1)^2, 8u^4 - 6u^2 + 1, u^4\big)$ is a primitive integral sequence for every $u \in \mathbb{N}$. $\square$

THEOREM 3.10. *A sequence (n_4, n_3, n_2, n_1) is integral if and only if $n_1 = k^2$, $n_2 - n^2 + 2nk$, $n_3 = \frac{a^2b^2}{k^2}$, $n_4 = \frac{c^2d^2 - a^2b^2}{(n+k)^2}$, where a, b, c, d, k, n are positive integers satisfying (3.1) and*

$$(c^2 + d^2)(n+k)^2k^2 = (n+k)^4k^2 + a^2b^2(n^2 + 2nk) + c^2d^2k^2\,, \qquad a^2b^2 < c^2d^2\,. \tag{3.11}$$

Proof. Let $T(n_4, n_3, n_2, n_1)$ be a balanced integral rooted tree. Using Theorems 3.7, 3.8 and Corollary 3.4 we have $n_1 = k^2$, $n_2 = n^2 + 2nk$, $n_3 = \frac{a^2b^2}{k^2}$, $(a^2 - k^2)(k^2 - b^2) = k^2(n^2 + 2nk)$; $b < k < a$, and all the zeros of

$$P\big(D(n_4, n_3, n_2, n_1); \lambda\big) = \lambda^5 - (n_1 + n_2 + n_3 + n_4)\lambda^3 + (n_1n_3 + n_2n_4 + n_1n_4)\lambda$$

are integral. Then there must exist $c, d \in \mathbb{N}$ such that

$$\lambda^4 - (n_1 + n_2 + n_3 + n_4)\lambda^2 + (n_1n_3 + n_2n_4 + n_1n_4) = (\lambda^2 - c^2)(\lambda^2 - d^2)\,. \tag{3.12}$$

From the above formulas we have

$$(n+k)^2 + \frac{a^2b^2}{k^2} + n_4 = c^2 + d^2\,, \tag{3.13}$$

$$a^2b^2 + (n+k)^2 n_4 = c^2d^2\,. \tag{3.14}$$

(3.14) implies $n_4 = \frac{c^2d^2-a^2b^2}{(n+k)^2}$, which inserted into (3.13) yields (3.11). Let $n_1 = k^2$, $n_2 = n^2 + 2nk$, $n_3 = \frac{a^2b^2}{k^2}$, $n_4 = \frac{c^2d^2-a^2b^2}{(n+k)^2}$, where $a, b, c, d, k, n \in \mathbb{N}$, satisfy (3.1) and (3.11). It follows from Theorems 3.7, 3.8 and Corollary 3.4 that it is sufficient to show that the polynomial

$$P\big(D(n_4, n_3, n_2, n_1); \lambda\big) = \lambda^5 - \left(\frac{c^2d^2 - a^2b^2}{(n+k)^2} + \frac{a^2b^2}{k^2} + (n+k)^2\right)\lambda^3 + c^2d^2\lambda \tag{3.15}$$

has only integral zeros. From (3.11) we have

$$c^2 + d^2 = (n+k)^2 + \frac{a^2b^2}{k^2} + \frac{c^2d^2 - a^2b^2}{(n+k)^2}. \tag{3.16}$$

Substituting this into (3.15) we get

$$P\big(D(n_4, n_3, n_2, n_1); \lambda\big) = \lambda^5 - (c^2+d^2)\lambda^3 + c^2d^2\lambda = \lambda(\lambda-c)(\lambda+c)(\lambda-d)(\lambda+d). \tag{3.17}$$

□

Using a computer we have found 182 "small" solutions of (3.1) and (3.11). A sample of them are given in the following table.

n_1	n_2	n_3	n_4	a	b	c	d
4	672	225	616	30	1	19	34
9	9792	1225	6336	105	1	71	111
9	9792	1225	38784	105	1	97	201
16	105	144	676	16	3	10	29
16	560	729	360	36	3	12	39
16	560	729	2736	36	3	21	60
36	693	1600	1209	48	5	17	57

It turns out that for every k we may find a system (S_k) of diophantine equations such that every solution of (S_k) gives an integral sequence $(n_k, n_{k-1}, \dots, n_1)$, and vice versa. Unfortunately, we have not been able to find any solution of (S_k) for $k \geq 5$. Thus the problem of the existence of integral sequences of length ≥ 5 remains open. In fact, we do not know any example of an integral tree of diameter ≥ 9. The above results imply that there e i infinitely many balanced integral rooted trees of diam t r 8.

Balanced rooted trees can be used also for deriving g n r l r ults on p r of trees. For instance we have the following theorem A *bran h* of a tr e subtree $T' \subseteq T$ such that every endvertex of T' 's an ndv rt of T.

THEOREM 3.11. *Let T be an integral tree. If $T(2, n_k, \dots, n_1) \subseteq T$ is a branch of T, then $T(n_k, \dots, n_1)$ is integral.*

Proof. Since $T(2, n_k, \dots, n_1)$ contains two copies of $T(n_k, \dots, n_1)$ there is a front-divisor D of T such that the corresponding codivisor C contains $T(n_k, \dots, n_1)$ as a connectivity component. By Theorem 2.1 $T(n_k, \dots, n_1)$ is integral. □

For instance, it follows from Theorem 3.11 and Theorem 3.7 that an integral tree cannot contain a branch isomorphic with the tree which is depicted in Fig. 2.

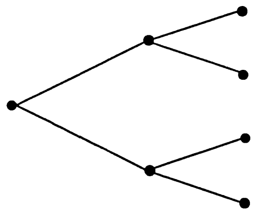
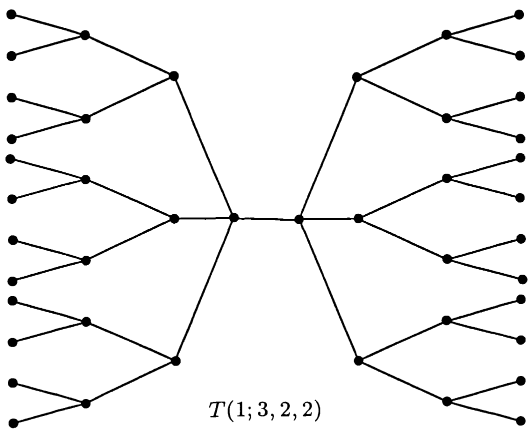
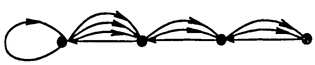


FIGURE 2.

4. Balanced trees of odd diameter



$T(1; 3, 2, 2)$



$D(1; 3, 2, 2)$

FIGURE 3.

Let $T = T(n_k, n_{k-1}, \dots, n_1)$ be a balanced rooted tree. We may form the tree $T(1; n_k, n_{k-1}, \dots, n_1)$ by taking two copies of T and joining their roots by an edge. Note that the tree $T(1; n_k, n_{k-1}, \dots, n_1)$ is the *balanced tree* of diameter $2k+1$ corresponding to the sequence $(1; n_k, \dots, n_1)$. The graph $D = D(1; n_k, n_{k-1}, \dots, n_1)$ formed from $D' = D(n_k, n_{k-1}, \dots, n_1)$ by joining a directed loop to the root of D' is clearly the canonical divisor of $T(1; n_k, n_{k-1}, \dots, n_1)$. Later we shall use also the graph $\overline{D} = D(-1; n_k, n_{k-1}, \dots, n_1)$ which is formed from D' by joining a loop valued -1 to the root of D'. Note that the adjacency matrix of $D(-1; n_k, n_{k-1}, \dots, n_1)$ arises from the adjacency matrix of $D(1; n_k, n_{k-1}, \dots, n_1)$ by changing 1 on the main diagonal into -1. The balanced tree $T(1; 3, 2, 2)$ and its canonical divisor $D(1; 3, 2, 2$ are depicted in Fig. 3.

THEOREM 4.1. *Let $T(1; n_k, n_{k-1}, \dots, n_1)$ be a balanced tree of diameter $2k+1$ and $D(1; n_k, n_{k-1}, \dots, n_1)$ be the corresponding canonical divisor. Then*

$$P\big(D(\pm 1; n_k, \dots, n_1); \lambda\big) = P\big(D(n_k, \dots, n_1); \lambda\big) \mp P\big(D(n_{k-1}, \dots, n_1); \lambda\big).$$

Proof. For the proof it is sufficient to expand the corresponding determinants $|\lambda I - D|$ and $|\lambda I - \overline{D}|$ by cofactors of the first column. □

Using the algorithm [2; p. 126] and Theorem 2.1 we get:

THEOREM 4.2. *Let $D = D(1; n_k, n_{k-1}, \dots, n_1)$ be the canonical divisor of the balanced tree $T(1; n_k, n_{k-1}, \dots, n_1)$ and C be the corresponding codivisor. Then*

$$P(C; \lambda) = P(\overline{D}; \lambda) \prod_{i=1}^{k} \big[P\big(T(n_{k-i}, n_{k-i-1}, \dots, n_1); \lambda\big)\big]^{2(n_{k-i+1}-1)}.$$

COROLLARY 4.3. *A balanced tree $T(1; n_k, n_{k-1}, \dots, n_1)$ is integral if and only if the following conditions hold:*

(i) *$D(1; n_k, n_{k-1}, \dots, n_1)$ is integral,*
(ii) *$D(-1; n_k, n_{k-1}, \dots, n_1)$ is integral,*
(iii) *$T(n_r, \dots, n_1)$ is integral for every r with $1 \le r \le k-1$.*

THEOREM 4.4. (see [12; Theorem 2]) *A balanced tree $T(1; n_1)$ is integral if and only if $n_1 = r(r+1)$, $r \in \mathbb{N}$.*

Note. If a tree $T(1; n_k, n_{k-1}, \dots, n_1)$ is integral, then the tree $T(1; n_{k-1}, \dots, n_1)$ need not be integral. For example, assume that both $T(1; n_2, n_1)$ and $T(1; n_1)$ are integral then by Corollary 4.3 and Theorem 3.7 n_1 is a square but by Theorem 4.4 $n_1 = r(r+1)$, $r \in \mathbb{N}$, a contradiction.

As concerns the balanced integral trees of diameter $4k+1$ we have the following theorem.

THEOREM 4.5. *There is no balanced integral tree of diameter $4k+1$.*

Proof. Let $T = T(1; n_{2k}, n_{2k-1}, \dots, n_1)$ be a balanced tree of diameter $4k+1$, and $D(1; n_{2k}, n_{2k-1}, \dots, n_1)$ be its rooted canonical divisor. Combining Theorem 4.1 and Theorem 3.1 we get

$$\begin{aligned} &P\big(D(1; n_{2k}, \dots, n_1); \lambda\big) \\ &\quad = (\lambda - 1)P\big(D(n_{2k-1}, \dots, n_1); \lambda\big) - n_{2k}P\big(D(n_{2k-2}, \dots, n_1); \lambda\big). \end{aligned} \tag{4.1}$$

If T is integral then there exist $\eta_0, \eta_1, \dots, \eta_{2k} \in \mathbb{Z}$ such that $P\big(D(1; n_{2k}, \dots, n_1); \eta_i\big) = 0$, for each $i = 1, 2, \dots, 2k$. Hence η_i are solutions of

$$(\lambda - 1)P\big(D(n_{2k-1}, \dots, n_1); \lambda\big) - n_{2k}P\big(D(n_{2k-2}, \dots, n_1); \lambda\big) = 0. \tag{4.2}$$

Now, put

$$f(\lambda) = \frac{(\lambda - 1)P\big(D(n_{2k-1}, \dots, n_1); \lambda\big)}{P\big(D(n_{2k-2}, \dots, n_1); \lambda\big)}. \tag{4.3}$$

Clearly, every solution of the equation $f(\lambda) = n_{2k}$ is a solution of the equation (4.2). By Theorem 3.2 we see that the rational function (4.3) has the following form:

$$\begin{aligned} &f(\lambda) = \\ &= \frac{(\lambda - 1)\left[\lambda^{2k} - (n_1 + n_2 + \dots + n_{2k-1})\lambda^{2k-2} + \dots + (-1)^k n_1 n_3 \dots n_{2k-1}\right]}{\lambda\left[\lambda^{2k-2} - (n_1 + \dots + n_{2k-2})\lambda^{2k-4} + \dots + (-1)^{k-1}(n_1 \dots n_{2k-3} + n_2 \dots n_{2k-2})\right]} \end{aligned} \tag{4.4}$$

It is easy to check that the following conditions hold:

(a) $f(\lambda)$ is continuous for $\lambda \in (0,1)$ (see Corollary 3.4),

(b) $\lim\limits_{\lambda \to 0+} f(\lambda) = \infty$,

(c) $f(1) = 0$ (see [7; Corollary 1]).

Hence, the equation $f(\lambda) = n_{2k}$ has a solution $\eta \in (0,1)$. Since every solution of $f(\lambda) = n_{2k}$ is also a solution (4.2), then $\eta \in \{\eta_0, \eta_1, \dots, \eta_{2k}\}$. However, this contradicts the fact that η_i is an integer for $i = 0, 1, \dots, 2k$. □

Now we shall discuss the balanced integral trees of diameter $4k-1$. It follows from [12; Theorem 4.4] that there exist infinitely many balanced integral trees of diameter 3. In contrast we prove that there is no balanced integral tree of diameter 7. The problem of the existence of balanced integral trees of diameter $4k-1$ for $k \geq 3$ remains open.

THEOREM 4.6. *There is no balanced integral tree of diameter 7.*

P r o o f . Let $T(1; n_3, n_2, n_1)$ be such a tree. According to Theorems 4.1, 4.2 and Corollaries 4.3, 3.4 we get

$$P\big(D(1; n_3, n_2, n_1); \lambda\big) = \lambda^4 - \lambda^3 - (n+k)^2\lambda^2 + (n+k)^2\lambda - n_3(\lambda^2 - k^2)\,, \tag{4.5}$$

$$P\big(D(-1; n_3, n_2, n_1); \lambda\big) = \lambda^4 + \lambda^3 - (n+k)^2\lambda^2 - (n+k)^2\lambda - n_3(\lambda^2 - k^2)\,. \tag{4.6}$$

Clearly, η is a zero of equation (4.5) if and only if $-\eta$ is a zero of equation (4.6). Now, we consider the functions

$$f(\lambda) = \frac{\lambda(\lambda-1)\big(\lambda^2 - (n+k)^2\big)}{\lambda^2 - k^2}\,,$$

$$g(\lambda) = \frac{\lambda(\lambda+1)\big(\lambda^2 - (n+k)^2\big)}{\lambda^2 - k^2}\,.$$

It is easy to see that every solution of equations $f(\lambda) = n_3$ or $g(\lambda) = n_3$ is also a solution of (4.5) or (4.6), respectively. We distinguish two cases:

Case 1. $k = 1$.

Then the functions $f(\lambda)$ and $g(\lambda)$ have the following properties:

(a) $\lim\limits_{\lambda\to -1+} f(\lambda) = \lim\limits_{\lambda\to 1-} g(\lambda) = \infty$,

(b) $f(0) = g(0) = 0$,

(c) $f(\lambda)$, $g(\lambda)$ are continuous in $(-1, 1)$.

Then there exist $\eta_0 \in (-1, 0)$ and $\eta_0' \in (0, 1)$ for which $f(\eta_0) = n_3$ and $g(\eta_0') = n_3$ but this contradicts the integral nature of zeros of (4.5), (4.6).

Case 2. $k > 1$.

Then the function $f(\lambda)$ and $g(\lambda)$ have the following properties:

(a') $\lim\limits_{\lambda\to k^-} f(\lambda) = \lim\limits_{\lambda\to k^-} g(\lambda) = \infty$,

(b') $g(\lambda) > f(\lambda)$ for every $\lambda \in (1, k)$,

(c') $f(\lambda)$, $g(\lambda)$ are continuous in $(0, k)$.

Clearly, there exist $r, s \in \mathbb{N}$; $1 < r < r + s < k$ and $g(r) = f(r+s) = n_3$, thus

$$\frac{(r+1)r\big(r^2 - (n+k)^2\big)}{r^2 - k^2} = \frac{(r+s)(r+s-1)\big((r+s)^2 - (n+k)^2\big)}{(r+s)^2 - k^2}\,.$$

This equation can be modified to the form:

$$\frac{(r+1)r\big(r^2 - (n+k)^2\big)}{r^2 - k^2} = \frac{\big[r(r+1) + (2r+s)(s-1)\big]\big[(r+s)^2 - (n+k)^2\big]}{(r^2 - k^2) + s(2r+s)}$$

and finally we obtain the equation

$$r(r+1)s\big[k^2 - (n+k)^2\big] = (r^2 - k^2)(2r+s)\big[(r+s)^2 - (n+k)^2\big](s-1)\,.$$

The left side of the last equation is < 0 and the right one is ≥ 0, except for $s < 1$. But, this is a contradiction with the fact $s \in \mathbb{N}$. □

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