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THE BOCHNER AND THE MONOTONE INTEGRALS WITH RESPECT TO A NUCLEAR-VALUED FINITELY ADDITIVE MEASURE

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ABSTRACT. The comparison obtained in [Isidori, M. C.—Martellotti, A.—Sambucini, A. R.: *Integration with respect to orthogonally scattered measures,* Math. Slovaca. **48** (1998), 253–269] is extended to the case of nuclear spaces making use of their representation as a projective limit of a family of Hilbert spaces.

1. Introduction

In many applications of mathematical analysis locally convex topological vector spaces are too general tools: a very useful specialization is represented by nuclear spaces; on the other hand, most of the examples of locally convex topological vector spaces, which are important for the applications, are nuclear spaces.

One consideration for which the concept of nuclear space has great importance is that the mere hypothesis of completeness allows to represent them as projective limit of Hilbert spaces, thus permitting the use of the inner product in each projection. Different kinds of integrations with respect to a Hilbert-valued finitely additive measure m have been already studied in [3], and compared with the integral with respect to a particular orthogonally scattered dilation of m.

The main idea of this paper is to use the projective limit structure of a complete nuclear space E, in order to define and compare different concepts of integrability with respect to a finitely additive measure m ranging on E.

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Lavoro svolto nell' ambito dello G.N.A.F.A. del C.N.R.

In fact we show that, in the countably additive case, the Bochner integrability with respect to m coincides with the Bochner integrability with respect to each projection of m, and the same is true for the monotone integral (Theorems 3.8 and 3.10).

Hence the comparison between these two concepts obtained in [3] for Hilbert-valued measures transfers to the case of an E-valued measure.

We then introduce the orthogonally scattered dilation of m, and again apply the results in [3] to obtain their analogous in E. Finally we consider the case of a finitely additive E-valued measure.

2. Preliminaries

Throughout the sequel $m \colon \Sigma \to E$ will denote a bounded countably additive measure and E a complete nuclear space.

THEOREM 2.1. ([5]) *Every complete locally convex space E is isomorphic to a projective limit of a family of Banach spaces; this family can be chosen so that its cardinality equals the cardinality of a given 0-neighbourhood base in E.*

We remember that in [5] if $\{U_\alpha : \alpha \in I\}$ is a basis of convex and circled neighbourhoods of 0 in E and if p_α is the gauge of U_α, we can form the projective limit $E = \mathrm{pjl}(E_\alpha, g_{\alpha,\beta})$ where $E_\alpha = E_{U_\alpha}$ is a complete Banach space and $g_{\alpha,\beta}$ is a continuous linear map of E_β into E_α defined by $g_{\alpha,\beta}([x]_\beta) = [x]_\alpha$, for every $\alpha \leq \beta$, where $[x]_\alpha$ denotes the equivalence class of the element x with respect to $\ker p_\alpha$. In the particular case when E is nuclear, the family U_α can be chosen so that each E_α is Hilbert.

For other notations and results concerning nuclear spaces we refer to [4]. We denote by $m_\alpha \colon \Sigma \to E_\alpha$ the bounded countably additive measure defined by $m_\alpha(B) = \big[m(B)\big]_\alpha$ for every $\alpha \in I$ and for every $B \in \Sigma$.

We refer to [3] for the notations and definitions relative to each m_α. In particular, $\|m_\alpha\|$ denotes the usual semivariation.

Note that, since m is bounded, each $\|m_\alpha\|$ is bounded as well, and for $\alpha < \beta$ one easily obtains that $\|m_\alpha\| \leq \|m_\beta\|$.

Our aim is to compare the m-integrability, the $(\widehat{\ })$-integrability with respect to m and the $(\widetilde{\ })$-integrability with respect to m with the same kinds of integrability with respect to each m_α.

3. The m-integral and the monotone integral

DEFINITION 3.1. Let $f\colon \Omega \to \mathbb{R}$ be a measurable function. Then f is *m-integrable* if and only if there exists a sequence of simple functions $(f_n)_n$ which $\|m_\alpha\|$-converges to f, for every $\alpha \in I$ and such that $\left(\int_F f_n \, \mathrm{d}m\right)_n$ is Cauchy in E for every $F \in \Sigma$. In this case we set

$$\int_{\bullet} f \, \mathrm{d}m = \lim_{n\to\infty} \int_{\bullet} f_n \, \mathrm{d}m .$$

We denote by $L^1(m)$ the space of m-integrable functions.

In order to define the $(\hat{\ })$-integral, we shall need a definition by seminorm which is an extension of that studied in [1], to σ-finite measure spaces.

DEFINITION 3.2. Let $(Y, \mathcal{B}, \mu)$ be a complete σ-finite measure space with respect to positive measure μ. A function $\varphi\colon \Omega \to E$ is said to be *measurable by seminorm* if and only if for every $\alpha \in I$ there exist a μ-null set $\Omega_0^\alpha \subset \Omega$ and a sequence of simple functions $(\varphi_n^\alpha)_n$ which satisfies $\lim\limits_{n\to\infty} p_\alpha\big([\varphi(x)]_\alpha - \varphi_n^\alpha(x)\big) = 0$ for every $x \in \Omega - \Omega_0^\alpha$.

DEFINITION 3.3. A function φ measurable by seminorm is said to be *integrable by seminorm* if and only if

1) for every $\alpha \in I$ there exists a sequence of simple Bochner μ-integrable functions $(\varphi_n^\alpha)_n$ such that $p_\alpha\big(\varphi_n^\alpha - [\varphi]_\alpha\big)$ is μ-integrable and
$$\lim_{n\to\infty} \int_\Omega p_\alpha\big(\varphi_n^\alpha - [\varphi]_\alpha\big) \, \mathrm{d}\mu = 0 ;$$

2) for every $F \in \mathcal{B}$ with $\mu(F) < \infty$ there exists $y_F \in E$ such that for every $\alpha \in I$
$$\lim_{n\to\infty} p_\alpha\left(\int_F \varphi_n^\alpha \, \mathrm{d}\mu - [y_F]_\alpha\right) = 0 .$$
In this case we set
$$\int_F \varphi \, \mathrm{d}\mu =: y_F .$$

PROPOSITION 3.4. *Let $f\colon \Omega \to [0, +\infty)$ be a measurable function. Then the function $\varphi\colon [0, +\infty) \to E$ defined by $\varphi(t) = m(f > t)$ is measurable by seminorm, with respect to the ordinary Lebesgue measure μ on $[0, +\infty)$.*

Proof. For every $\alpha \in I$ the function $t \mapsto m_\alpha(f > t)$ is measurable ([2; p. 291]) and so, for every $y \in E$, $\langle [y]_\alpha \mid \varphi_\alpha \rangle$ is measurable. The function φ_α takes values in E_α which is a separable Hilbert space and thus by [1; Theorem 2.2] it is totally measurable. $\square$

DEFINITION 3.5. A measurable function $f \colon \Omega \to \mathbb{R}_0^+$ is said to be $(\hat{\ })$*-integrable* with respect to m if and only if the following conditions hold:

1) for every $B \in \Sigma$ the function $\varphi^B(t) = m(f \cdot 1_B > t)$ is integrable by seminorm in $([0,+\infty), \mathcal{B}, \mu)$, where μ is the Lebesgue measure;
2) for every $\alpha \in I$ there exists $F_\alpha \in L^1([0,+\infty), \mathcal{B}, \mu)$ such that $p_\alpha(\varphi_\alpha^B(t)) \leq F_\alpha(t)$ for every $B \in \Sigma$ and for every $t \in [0,+\infty)$, where $\varphi_\alpha^B(t) = m_\alpha(f \cdot 1_B > t)$.

$f \colon \Omega \to \mathbb{R}$ is $(\hat{\ })$*-integrable* with respect to m if and only if f^+ and f^- are $(\hat{\ })$-integrable in the above sense.

We denote by $\widehat{L}^1(m)$ the space of $(\hat{\ })$-integrable functions.

DEFINITION 3.6. A function $f \colon \Omega \to \mathbb{R}$ is said to be $(\tilde{\ })$*-integrable* with respect to m if and only if $\|m_\alpha\|^2(f > t)$ is Lebesgue integrable for every $\alpha \in I$.

We denote by $\widetilde{L}^1(m)$ the space of $(\tilde{\ })$-integrable functions.

Note that by definition the $(\tilde{\ })$-integrability with respect to m is equivalent to the $(\tilde{\ })$-integrability with respect to m_α for every $\alpha \in I$.

DEFINITION 3.7. For every $\alpha \in I$, $\Sigma_{*,\alpha}$ is the σ-algebra generated by Σ and all $\|m_\alpha\|$-null sets.

THEOREM 3.8. *Let $f \colon \Omega \to \mathbb{R}$ be a measurable function. Then f is m-integrable if and only if f is m_α-integrable for every $\alpha \in I$.*

Proof. Assume first that f is m-integrable. Let $\alpha \in I$ and let $V_\varepsilon^\alpha = \{x : p_\alpha(x) \leq \varepsilon\} \in \mathcal{U}_0^E$ be fixed. Let $(f_n)_n$ be a defining sequence for f. Since $\left(\int_F f_n \, \mathrm{d}m\right)_n$ is Cauchy, for every $V \in \mathcal{U}_0^E$ there exists $\overline{n} \in \mathbb{N}$ such that for every $r, s > \overline{n}$

$$\left(\int_F f_r \, \mathrm{d}m - \int_F f_s \, \mathrm{d}m\right) \in V;$$

hence, for r, s suitable large

$$p_\alpha\left(\int_F f_r \, \mathrm{d}m - \int_F f_s \, \mathrm{d}m\right) \leq \varepsilon.$$

Reducing f_r and f_s to the same decomposition we obtain

$$p_\alpha\left(\sum_{k=1}^{q}(\xi_k^r - \xi_k^s)m(A_k)\right) \leq \varepsilon .$$

Thus we have

$$\left\|\int_F f_r \, \mathrm{d}m_\alpha - \int_F f_s \, \mathrm{d}m_\alpha\right\|_{E_\alpha} = \left\|\sum_{k=1}^{q}(\xi_k^r - \xi_k^s)m_\alpha(A_k)\right\|_{E_\alpha}$$

$$= p_\alpha\left(\sum_{k=1}^{q}(\xi_k^r - \xi_k^s)m_\alpha(A_k)\right) \leq \varepsilon .$$

We now prove the converse. Assume f is m_α-integrable for every $\alpha \in I$. Then for every $\alpha \in I$ there exists a sequence of simple functions $(f_n^\alpha)_n$ converging in m_α-measure to f, such that $\left(\int_\bullet f_n \, \mathrm{d}m_\alpha\right)_n$ is Cauchy in E_α. We first prove that $\left(\int_\bullet f \, \mathrm{d}m_\alpha\right)_\alpha$ is in E, namely

$$g_{\alpha,\beta}\left(\int_\bullet f \, \mathrm{d}m_\beta\right) = \int_\bullet f \, \mathrm{d}m_\alpha$$

for every $\alpha < \beta$. Obviously $\int_\bullet f \, \mathrm{d}m_\beta = \lim_{n\to\infty} \int_\bullet f_n^\beta \, \mathrm{d}m_\beta$. Since f_n^β $\|m_\beta\|$-converges to f, f is $\Sigma_{*,\beta}$-measurable.

So we have $(|f - f_n^\beta| > t) \in \Sigma_{*,\beta} \subset \Sigma_{*,\alpha}$ and thus $\|m_\alpha\|(|f - f_n^\beta| > t) \leq \|m_\beta\|(|f - f_n^\beta| > t)$. Therefore f_n^β $\|m_\alpha\|$-converges to f. Since $\left(\int_\bullet f_n^\beta \, \mathrm{d}m_\beta\right)_n$ is Cauchy in E_β, for every $\varepsilon > 0$ there exists $\overline{n} \in \mathbb{N}$ such that for every $r, s > \overline{n}$

$$p_\beta\left(\int_\bullet (f_r^\beta - f_s^\beta) \, \mathrm{d}m_\beta\right) < \varepsilon ,$$

and so $\left(\int_\bullet f_n^\beta \, \mathrm{d}m_\alpha\right)_n$ is Cauchy in E_α. Thus f is m_α-integrable and we obtain

$$g_{\alpha,\beta}\left(\int_\bullet f_n^\beta \, \mathrm{d}m_\beta\right) = \int_\bullet f_n^\beta \, \mathrm{d}m_\alpha .$$

The sequence on the left hand side converges to $\int_\bullet f \, \mathrm{d}m_\alpha$. Therefore we have

$$g_{\alpha,\beta}\left(\int_\bullet f \, \mathrm{d}m_\beta\right) = g_{\alpha,\beta}\left(\lim_{n\to\infty}\int_\bullet f_n^\beta \, \mathrm{d}m_\beta\right) = \lim_{n\to\infty} g_{\alpha,\beta}\left(\int_\bullet f_n^\beta \, \mathrm{d}m_\beta\right)$$

$$= \lim_{n\to\infty}\int_\bullet f_n^\beta \, \mathrm{d}m_\alpha = \int_\bullet f \, \mathrm{d}m_\alpha .$$

Thus $\left(\int_{\bullet} f \, \mathrm{d}m_\alpha\right)_\alpha \in E$.

We now prove that f is m-integrable. Since f is Σ-measurable, there exists a sequence of simple functions $(f_n)_n$ converging to f almost everywhere and such that for every $\beta \in I$, $\int_{\bullet} f_n \, \mathrm{d}m_\beta$ converges to $\int_{\bullet} f \, \mathrm{d}m_\beta$.

Therefore we have $\lim_{n\to\infty} \int_{\bullet} f_n \, \mathrm{d}m_\beta = \lim_{n\to\infty} \int_{\bullet} f_n^\beta \, \mathrm{d}m_\beta = \int_{\bullet} f \, \mathrm{d}m_\beta$. Since, as one easily can see, $\left(\int_F f_n \, \mathrm{d}m\right)_n$ is Cauchy in E for every $F \in \Sigma$, the assertion follows. □

Remark 3.9. Observe that the "if" implication is valid even when m is only finitely additive, while the "only if" part is true only when m is countably additive.

THEOREM 3.10. *Let $f: \Omega \to \mathbb{R}$ be a measurable function. Then f is $(\widehat{\ })$-integrable with respect to m if and only if f is $(\widehat{\ })$-integrable with respect to m_α for every $\alpha \in I$.*

Proof. We first prove the assertion for non negative f. Assume first that f is $(\widehat{\ })$-integrable with respect to m. The function φ_α^B is measurable for every $\alpha \in I$ and for every $B \in \Sigma$. Fix $\alpha \in I$ and $B \in \Sigma$. By hypothesis there exists a sequence $(\varphi_{\alpha,B}^n)_n$ of simple Bochner integrable functions such that $p_\alpha[\varphi_{\alpha,B}^n - \varphi_\alpha^B]$ converges to 0 and $\lim_{n\to\infty} \int_0^\infty p_\alpha[\varphi_{\alpha,B}^n - \varphi_\alpha^B] \, \mathrm{d}\mu = 0$. Therefore $p_\alpha[\varphi_\alpha^B]$ is Lebesgue integrable in $[0, +\infty]$. Furthermore we have

$$\|m_\alpha\|(f > t) = \sup_{B \subset (f>t)} \|m_\alpha(B)\| = \sup_{B \subset (f>t)} p_\alpha[m_\alpha(f \cdot 1_B > t)] \leq F_\alpha(t)$$

for every $t \in [0, +\infty)$ and so $t \mapsto \|m_\alpha\|(f > t)$ is integrable in $[0, \infty)$ for every $\alpha \in I$.

We now prove the converse.

Assume f is $(\widehat{\ })$-integrable with respect to m_α for every $\alpha \in I$. By hypothesis the function $t \mapsto \|m_\alpha\|(f > t)$ is Lebesgue integrable in $[0, +\infty]$ for every $\alpha \in I$. So we have

$$p_\alpha[\varphi_\alpha^B(t)] = \|m_\alpha(f \cdot 1_B > t)\|_{E_\alpha} \leq \|m_\alpha\|(f > t).$$

Thus we set $F_\alpha(t) = \|m_\alpha\|(f > t)$. As φ_α is measurable for every $\alpha \in I$, then φ is measurable by seminorm.

Fix $B \in \Sigma$. We obtain $\int_0^\infty p_\alpha[\varphi_\alpha^B] \, \mathrm{d}\mu \leq \int_0^\infty \|m_\alpha\|(f > t) \, \mathrm{d}t$. Since φ_α is Bochner integrable, $p_\alpha[\varphi_\alpha^B]$ is Lebesgue integrable. So φ_α^B is Bochner integrable and therefore there exists a sequence of simple functions $(\varphi_{\alpha,B}^n)$ such that $p_\alpha\big[\varphi^n_{\alpha,B} - \varphi^B_\alpha\big]$ converges to 0 μ-a.e. and $\lim\limits_{n\to\infty} \int\limits_0^\infty p_\alpha\big[\varphi^n_{\alpha,B} - \varphi^B_\alpha\big]\ \mathrm{d}\mu = 0$. Thus $\varphi^n_{\alpha,B}$ is Bochner integrable. We have proved 1) of Definition 3.3 for φ^B. To prove 2) of Definition 3.3 we must divide our proof into steps. Fix $B \in \Sigma$.

a) First case: $F = (a,b)$.
Let $A = \big\{\omega \in \Omega :\ a < f(\omega) < b\big\}$. It is

$$
\begin{aligned}
m_\alpha(1_B \cdot f \cdot 1_A > t) &= m_\alpha\big(B \cap (f \cdot 1_A > t)\big) \\
&= \begin{cases} m_\alpha(A \cap B) & t \le a \\ m_\alpha\big(B \cap (t < f < b)\big) & a < t < b \\ 0 & t \ge b \end{cases} \\
&= \begin{cases} m_\alpha(A \cap B) & t \le a \\ m_\alpha\big[\big(B \cap (f > t)\big) - \big(B \cap (f \ge b)\big)\big] & a < t < b \\ 0 & t \ge b\,. \end{cases}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\int_F \varphi^B_\alpha\ \mathrm{d}\mu &= \int_a^b m_\alpha(f \cdot 1_B > t)\ \mathrm{d}\mu = \int_a^b m_\alpha\big[B \cap (f > t)\big]\ \mathrm{d}\mu \\
&= \int_a^b m_\alpha\big[B \cap (1_A \cdot f > t)\big]\ \mathrm{d}\mu + \int_a^b m_\alpha\big[B \cap (f \ge b)\big]\ \mathrm{d}\mu \\
&= \int_0^a m_\alpha(A \cap B)\ \mathrm{d}\mu + \int_a^b m_\alpha\big[B \cap (1_A \cdot f > t)\big]\ \mathrm{d}\mu \\
&\quad + \int_a^b m_\alpha\big[B \cap (f \ge b)\big]\ \mathrm{d}\mu - \int_0^a m_\alpha(A \cap B)\ \mathrm{d}\mu \\
&= \int_0^{+\infty} m_\alpha\big[B \cap (1_A \cdot f > t)\big]\ \mathrm{d}\mu \\
&\quad + m_\alpha\big[B \cap (f \ge b)\big] \cdot (b-a) - m_\alpha(A \cap B) \cdot a\,.
\end{aligned}
$$

We set $\varphi^+_{\alpha,B}(b) = m_\alpha\big[B \cap (f \ge b)\big]$. Then we have

$$
\int_F \varphi^B_\alpha\ \mathrm{d}\mu = \int_{A \cap B} f\ \mathrm{d}m_\alpha + \varphi^+_{\alpha,B}(b) \cdot (b-a) - m_\alpha(A \cap B) \cdot a\,.
$$

By the boundedness of f and [2; Theorem 3.2] $\widehat{\int\limits_{A\cap B}} f \, \mathrm{d}m_\alpha = \int\limits_{A\cap B} f \, \mathrm{d}m_\alpha$. If we set $\varphi^+_B(t) = \left(\varphi^+_{\alpha,B}(t)\right)_\alpha$, then $\varphi^+_B(t) \in E$; putting

$$y_F = \left(\widehat{\int\limits_{A\cap B}} f \, \mathrm{d}m_\alpha \right)_\alpha + \varphi^+_B(b) \cdot (b-a) - m(A \cap B) \cdot a$$

we have proved that $\int\limits_F \varphi^B_\alpha \, \mathrm{d}\mu = [y_F]_\alpha$.

b) Second case: $F = \dot{\bigcup\limits_n}\,]a_n, b_n[$ is an open set of finite measure.

Set $A_n = \{\omega \in \Omega : \; a_n < f(\omega) < b_n\}$. Then, from a), it follows that

$$\int\limits_{a_n}^{b_n} \varphi^B_\alpha \, \mathrm{d}\mu = \int\limits_{A_n \cap B} f \, \mathrm{d}m_\alpha + \varphi^+_{\alpha,B}(b_n) \cdot (b_n - a_n) - \int\limits_0^{a_n} m_\alpha(A_n \cap B) \, \mathrm{d}\mu .$$

We want to prove that $\sum\limits_n \int\limits_{a_n}^{b_n} \varphi^B_\alpha \, \mathrm{d}\mu$ converges in E_α.

Since $\int\limits_{A_n\cap B} f \, \mathrm{d}m_\alpha$ is a countable additive measure the series $\sum\limits_n \int\limits_{A_n\cap B} f \, \mathrm{d}m_\alpha$ converges in E_α. Moreover it holds

$$\begin{aligned} \int\limits_{A_n\cap B} f \, \mathrm{d}m_\alpha &= \int\limits_0^{+\infty} \varphi^{A_n \cap B}_\alpha(t) \, \mathrm{d}\mu \\ &= \int\limits_0^{a_n} m_\alpha(A_n \cap B) \, \mathrm{d}\mu + \int\limits_{a_n}^{b_n} m_\alpha[B \cap (f \cdot 1_{A_n} > t)] \, \mathrm{d}\mu , \end{aligned}$$

and so we have

$$\int\limits_0^{a_n} m_\alpha(A_n \cap B) \, \mathrm{d}\mu = - \int\limits_{a_n}^{b_n} m_\alpha[B \cap (f \cdot 1_{A_n} > t)] \, \mathrm{d}\mu + \int\limits_{A_n\cap B} f \, \mathrm{d}m_\alpha . \qquad (1)$$

Let us prove that $\sum\limits_n \left[\int\limits_{a_n}^{b_n} m_\alpha[B \cap (f \cdot 1_{A_n} > t)] \, \mathrm{d}\mu \right]$ is absolutely convergent. In fact we have

$$\left\| \int\limits_{a_n}^{b_n} m_\alpha[B \cap (f \cdot 1_{A_n} > t)] \, \mathrm{d}\mu \right\| \leq \|m_\alpha\|(\Omega) \cdot (b_n - a_n)$$

and so we obtain

$$\sum_n \left\| \int_{a_n}^{b_n} m_\alpha[B \cap (f \cdot 1_{A_n} > t)] \, \mathrm{d}\mu \right\| \leq \|m_\alpha\|(\Omega) \cdot \mu(F) < +\infty .$$

Therefore from (1), the series $\sum_n \int_0^{a_n} m_\alpha(A_n \cap B) \, \mathrm{d}\mu$ converges in E_α. It remains to prove the convergence of the series $\sum_n \varphi^+_{\alpha,B}(b_n) \cdot (b_n - a_n)$. Indeed

$\|\varphi^+_{\alpha,B}(b_n) \cdot (b_n - a_n)\| \leq (b_n - a_n) \cdot \|m_\alpha[B \cap (f \geq b_n)]\| \leq \|m_\alpha\|(\Omega) \cdot (b_n - a_n)$,

and so we have

$$\begin{aligned}\sum_n \|\varphi^+_{\alpha,B}(b_n) \cdot (b_n - a_n)\| &\leq (b_n - a_n) \cdot \|m_\alpha[B \cap (f \geq b_n)]\| \\ &\leq \|m_\alpha\|(\Omega) \cdot \mu(F) < +\infty .\end{aligned}$$

Therefore the series $\sum_n \int_{a_n}^{b_n} \varphi^B_\alpha \, \mathrm{d}\mu$ converges in E_α.

If we set

$$x^B_n = \left(\int_{A_n \cap B} f \, \mathrm{d}m_\alpha \right)_\alpha + \varphi^+_B(b_n) \cdot (b_n - a_n) - m(A_n \cap B) \cdot a_n ,$$

we obtain $\int_{a_n}^{b_n} \varphi^B_\alpha \, \mathrm{d}\mu = [x^B_n]_\alpha$. Now we want to prove that the series $\sum_n x^B_n$ is absolutely convergent in E, namely that the series $\sum_n p_\alpha(x^B_n)$ is convergent for every $\alpha \in I$. In fact it is

$$p_\alpha(x^B_n) = p_\alpha([x^B_n]_\alpha) = p_\alpha\left(\int_{a_n}^{b_n} \varphi^B_\alpha(t) \, \mathrm{d}\mu \right) \leq \int_{a_n}^{b_n} \|m_\alpha\|[B \cap (f > t)] \, \mathrm{d}\mu$$

and so we obtain

$$\begin{aligned}\sum_n p_\alpha(x^B_n) &\leq \sum_n \int_{a_n}^{b_n} \|m_\alpha\|[B \cap (f > t)] \, \mathrm{d}\mu \\ &= \int_F \|m_\alpha\|[B \cap (f > t)] \, \mathrm{d}\mu \\ &\leq \int_0^{+\infty} \|m_\alpha\|(B \cap (f > t)) \, \mathrm{d}\mu < +\infty .\end{aligned}$$

Let x_F^B be the sum of the series $\sum\limits_n x_n^B$. Then we have $\int\limits_F \varphi_\alpha^B(t)\,\mathrm{d}\mu = [x_F^B]_\alpha$ for every $\alpha \in I$. Note that if $\alpha < \beta$ we obtain

$$g_{\alpha,\beta}\left(\int\limits_F \varphi_\beta^B\,\mathrm{d}\mu\right) = g_{\alpha,\beta}([x_F^B]_\beta) = \int\limits_F \varphi_\alpha^B\,\mathrm{d}\mu\,.$$

c) Third case: F is a Borel set of finite measure.
Fix $\alpha < \beta$ in I. By hypothesis for every $\varepsilon > 0$ there exists $\delta_\alpha(\varepsilon) > 0$ such that if $\mu(G) < \delta_\alpha$ then

$$\int\limits_G \|m_\alpha\|[B \cap (f > t)]\,\mathrm{d}\mu < \varepsilon\,. \tag{2}$$

Since μ is a regular measure there exists an open set A, with $F \subset A$ such that

$$\mu(A - F) < \inf\left\{\delta_\alpha\left(\frac{\varepsilon}{2}\right), \delta_\beta\left(\frac{\varepsilon}{2 \cdot \|g_{\alpha,\beta}\|}\right)\right\}.$$

Note that A is of finite measure. By (2) we obtain

$$\int\limits_{A-F} \|m_\alpha\|[B \cap (f > t)]\,\mathrm{d}\mu < \frac{\varepsilon}{2}\,, \qquad \int\limits_{A-F} \|m_\beta\|[B \cap (f > t)]\,\mathrm{d}\mu < \frac{\varepsilon}{2 \cdot \|g_{\alpha,\beta}\|}\,.$$

Therefore applying step b) to the open set A, it follows that

$$\begin{aligned}
&p_\alpha\left[\int\limits_F \varphi_\alpha^B\,\mathrm{d}\mu - g_{\alpha,\beta}\left(\int\limits_F \varphi_\beta^B\,\mathrm{d}\mu\right)\right] \\
\leq\, & p_\alpha\left[\int\limits_F \varphi_\alpha^B\,\mathrm{d}\mu - \int\limits_A \varphi_\alpha^B\,\mathrm{d}\mu\right] + p_\alpha\left[\int\limits_A \varphi_\alpha^B\,\mathrm{d}\mu - g_{\alpha,\beta}\left(\int\limits_F \varphi_\beta^B\,\mathrm{d}\mu\right)\right] \\
=\, & p_\alpha\left[\int\limits_{A-F} \varphi_\alpha^B\,\mathrm{d}\mu\right] + p_\alpha\left[g_{\alpha,\beta}\left(\int\limits_A \varphi_\beta^B\,\mathrm{d}\mu\right) - g_{\alpha,\beta}\left(\int\limits_F \varphi_\beta^B\,\mathrm{d}\mu\right)\right] \\
\leq\, & \int\limits_{A-F} p_\alpha(\varphi_\alpha^B)\,\mathrm{d}\mu + p_\alpha\left[g_{\alpha,\beta}\left(\int\limits_{A-F} \varphi_\beta^B\,\mathrm{d}\mu\right)\right] \\
\leq\, & \int\limits_{A-F} \|m_\alpha\|[B \cap (f > t)]\,\mathrm{d}\mu + \|g_{\alpha,\beta}\| \cdot p_\beta\left[\int\limits_{A-F} \varphi_\beta^B\,\mathrm{d}\mu\right] \\
\leq\, & \frac{\varepsilon}{2} + \|g_{\alpha,\beta}\| \cdot \frac{\varepsilon}{2 \cdot \|g_{\alpha,\beta}\|} = \varepsilon\,.
\end{aligned}$$

In all three cases we set $y_F = \left(\int\limits_F \varphi_\alpha^B \, \mathrm{d}\mu\right)_\alpha$; we have proven that $y_F \in E$. Furthermore

$$\begin{aligned}\lim_{n\to\infty} p_\alpha\left[\int\limits_F \varphi_{n,\alpha}^B \, \mathrm{d}\mu - [y_F]_\alpha\right] &= \lim_{n\to\infty} p_\alpha\left[\int\limits_F \varphi_{n,\alpha}^B \, \mathrm{d}\mu - \int\limits_F \varphi_\alpha^B \, \mathrm{d}\mu\right] \\ &\le \lim_{n\to\infty} \int\limits_F p_\alpha[\varphi_{n,\alpha}^B - \varphi_\alpha^B] \, \mathrm{d}\mu \\ &\le \lim_{n\to\infty} \int\limits_0^{+\infty} p_\alpha[\varphi_{n,\alpha}^B - \varphi_\alpha^B] \, \mathrm{d}\mu = 0 .\end{aligned}$$

So we have proved 2) of Definition 3.3 for φ^B and therefore f is $(\widehat{\ })$-integrable with respect to m. □

COROLLARY 3.11. *Let $m\colon \Sigma \to E$ be an s-bounded finitely additive measure. Let $f\colon \Omega \to \mathbb{R}$ be a measurable function. Then f is $(\widehat{\ })$-integrable with respect to m if and only if f is $(\widehat{\ })$-integrable with respect to m_α for every $\alpha \in I$.*

Proof. The proof substantially goes along the same lines as that of Theorem 3.10. The only difference lies in the proof of the convergence of the series $\sum\limits_n \int\limits_{A_n \cap B} f \, \mathrm{d}m_\alpha$. Since $f \in \widehat{L}^1(m_\alpha)$, by [2; Theorem 3.9] $f \in L^1(m_\alpha)$ and by [3; Proposition 3.4], for every $A \in \Sigma$,

$$\int\limits_A f \, \mathrm{d}m_\alpha = \int\limits_{h(A)} \overline{f} \, \mathrm{d}\overline{m}_\alpha .$$

If $A_n \cap A_m = \emptyset$ for $n \neq m$ then $h(A_n) \cap h(A_m) = \emptyset$ then

$$\sum_n \int\limits_{h(A_n \cap B)} \overline{f} \, \mathrm{d}\overline{m}_\alpha = \sum_n \int\limits_{A_n \cap B} f \, \mathrm{d}m_\alpha$$

converges in E_α because $\int\limits_\bullet \overline{f} \, \mathrm{d}\overline{m}_\alpha$ is countably additive. □

4. Orthogonally scattered dilations of a nuclear valued finitely additive measure and applications to comparison of integrals

DEFINITION 4.1. Let $m\colon \Sigma \to E$ be an s-bounded finitely additive measure. $\widetilde{m}\colon \Sigma \to E \times \prod_{\alpha\in I} H_\alpha$ is an *orthogonally scattered dilation* of m if, for every $\alpha \in I$, $V_\alpha[Pr_\alpha(\widetilde{m}_\alpha)] = m_\alpha$, where Pr_α is an orthogonal projection of $E_\alpha \oplus H_\alpha$ onto a closed linear manifold M of $E_\alpha \oplus H_\alpha$ and $V_\alpha\colon M \to E_\alpha$ is a unitary isomorphism (surjective). The relationship between m_α, $\widetilde{m_\alpha}$, E_α, $E_\alpha \oplus H_\alpha$ can be clearly seen trough the following commutative diagram:

$$\begin{tikzcd} \Sigma \arrow[r, "m_\alpha"] \arrow[d, "\widetilde{m}_\alpha"'] & E_\alpha \arrow[d, leftrightarrow, "V_\alpha"] \\ E_\alpha \oplus H_\alpha \arrow[r, "Pr_\alpha"] & M \end{tikzcd}$$

THEOREM 4.2. *Let $f\colon \Omega \to \mathbb{R}$ be a measurable function, let $m\colon \Sigma \to E$ be a bounded countably additive measure and let $\widetilde{m}\colon \Sigma \to E \times \prod_{\alpha\in I} H_\alpha$ be an orthogonally scattered dilation of m. Then f is $\widetilde{m}$-integrable if and only if f is $\widetilde{m}_\alpha$-integrable, for every $\alpha \in I$.*

Proof. Let $P_\alpha = p_\alpha + \rho_\alpha$ defined by $P_\alpha(x,y) = p_\alpha(x) + \rho_\alpha(y)$ be the norm of $E_\alpha \times H_\alpha$, where p_α and ρ_α are the norms of E_α and H_α respectively. Let $\mathcal{F}(I) = \{\pi \subset I : \pi \text{ finite}\}$. We define the seminorm $P_\pi\colon \prod_{\alpha\in I}(E_\alpha \times H_\alpha) \to \mathbb{R}_0^+$ by

$$P_\pi\big((x_\alpha, y_\alpha)_{\alpha\in I}\big) = \sum_{\alpha\in\pi} P_\alpha(x_\alpha, y_\alpha) = \sum_{\alpha\in\pi} [p_\alpha(x_\alpha) + \rho_\alpha(y_\alpha)]_{\alpha\in I}\,.$$

The product topology of $\prod_{\alpha\in I}(E_\alpha \times H_\alpha)$ is generated by the family of seminorms P_π, $\pi \in \mathcal{F}(I)$. So, by Theorem 2.1, $\prod_{\alpha\in I}(E_\alpha \times H_\alpha)$ is the projective limit of Banach spaces.

The set $\mathcal{F}(I)$ is ordered by inclusion and directed, namely $\pi \le \tau \iff \pi \subseteq \tau$, $\pi \vee \pi_1 = \pi \cup \pi_1$. So $\prod_{\alpha\in I}(E_\alpha \times H_\alpha)|_{P_\pi} = [(x_\alpha, y_\alpha)]_{P_\pi}$ is a Banach space and we have

$$[(x_\alpha, y_\alpha)]_{P_\pi} = \{(\xi_\alpha, \eta_\alpha) : \; P_\pi((\xi_\alpha, \eta_\alpha) - (x_\alpha, y_\alpha)) = 0\}\,.$$

Then $[(x_\alpha, y_\alpha)]_{P_\pi} = \{(x_\alpha, y_\alpha)_\alpha + (\xi_\alpha, \eta_\alpha)_\alpha : \; \forall \lambda \in \pi \quad p_\lambda(\xi_\alpha) = 0,\; \rho_\lambda(\eta_\alpha) = 0\}$.
So we define $\widetilde{m}_\pi$ as follows:

$$\begin{aligned} \widetilde{m}_\pi(A) &= [\widetilde{m_\alpha}(A)]_{P_\pi} \\ &= \big\{\big(m_\alpha(A), \nu_\alpha(A)\big)_\alpha + (\xi_\alpha, \eta_\alpha)_\alpha : \; \forall \lambda \in \pi \quad p_\lambda(\xi_\alpha) = 0\,,\; \rho_\lambda(\eta_\alpha) = 0\big\}\,. \end{aligned}$$

By Theorem 3.8, we obtain $L^1(\tilde{m}) = \bigcap_{\pi\in\mathcal{F}(I)} L^1(\tilde{m}_\pi)$. We have now to show that $L^1(\tilde{m}) = \bigcap_{\alpha\in I} L^1(\tilde{m}_\alpha)$.

Observe that if $\mathcal{F}_1(I) = \{\{\alpha\} : \alpha \in I\}$, then $\mathcal{F}_1(I) \subset \mathcal{F}(I)$ and hence $\bigcap_{\pi\in\mathcal{F}(I)} L^1(\tilde{m}_\pi) \subset \bigcap_{\pi\in\mathcal{F}_1(I)} L^1(\tilde{m}_\pi)$. It is obvious that if $\pi \in \mathcal{F}_1(I)$, say $\pi = \{\alpha\}$, then $L^1(\tilde{m}_\pi) = L^1(\tilde{m}_\alpha)$, and therefore the inclusion $L^1(\tilde{m}) \subset \bigcap_{\alpha\in I} L^1(\tilde{m}_\alpha)$ holds.

Conversely, if $f \in \bigcap_{\alpha\in I} L^1(\widetilde{m_\alpha})$ then for every $\pi \subset I$, π finite, it is $f \in \bigcap_{\pi\in\mathcal{F}(I)} L^1(\widetilde{m_\pi}) = L^1(\tilde{m})$. □

Remark 4.3. Note that, by Remark 3.9, the inclusion $L^1(\tilde{m}) \subset \bigcap_{\alpha\in I} L^1(\tilde{m}_\alpha)$ is valid also when m is an s-bounded finitely additive measure.

THEOREM 4.4. *Let $m\colon \Sigma \to E$ be a countably additive bounded measure. If for every $\alpha \in I$ there exists $y_\alpha \in E$ such that*

1) $|\langle [y_\alpha]_\alpha \mid m_\alpha\rangle|$ *is a control for* m_α,
2) $\dfrac{\mathrm{d}m_\alpha}{\mathrm{d}|\langle [y_\alpha]_\alpha \mid m_\alpha\rangle|}$ *is bounded,*

then the following implications hold:

$$f \in L^1(\tilde{m}) \implies f \in L^1(m) \iff f \in \widehat{L}^1(m) \implies f \in \widetilde{L}^1(\tilde{m}).$$

P r o o f. It suffices to apply [3; Corollary 3.10], Theorem 3.8, Theorem 3.10, Theorem 4.2 and Definition 3.6. □

We now shall extend the last result to the case of E-valued finitely additive measures.

Analogously to the proof of Proposition 4.2 of [3], the following lemma can be proved.

LEMMA 4.5. *Let $m\colon \Sigma \to E$ be an s-bounded finitely additive measure, let $\tilde{m}$ be its orthogonally scattered dilation and let $f\colon \Omega \to \mathbb{R}$ be a measurable function. Then $f \in L^1(\tilde{m}) \implies f \in L^1(m)$.*

THEOREM 4.6. *Let $m\colon \Sigma \to E$ be an s-bounded finitely additive measure. If for every $\alpha \in I$ there exists $y_\alpha \in E$ such that*

1) $|\langle [y_\alpha]_\alpha \mid \overline{m}_\alpha\rangle|$ *is a control for* $\overline{m}_\alpha$,
2) $\dfrac{\mathrm{d}m_\alpha}{\mathrm{d}| < [y_\alpha]_\alpha \mid m_\alpha > |}$ *is bounded,*

then the following two chains of implications hold:

4.6.A
$$f \in L^1(\tilde{m}) \implies f \in L^1(m) \implies f \in \bigcap_{\alpha \in I} L^1(m_\alpha)$$
$$\iff f \in \bigcap_{\alpha \in I} \widehat{L}^1(m_\alpha) = \widehat{L}^1(m) \implies f \in \bigcap_{\alpha \in I} \widetilde{L}^1(\widetilde{m_\alpha}) \iff f \in \widetilde{L}^1(\tilde{m}).$$

4.6.B
$$f \in L^1(\tilde{m}) \implies f \in \bigcap_{\alpha \in I} L^1(\widetilde{m_\alpha}) \implies f \in \bigcap_{\alpha \in I} L^1(m_\alpha)$$
$$\iff f \in \bigcap_{\alpha \in I} \widehat{L}^1(m_\alpha) = \widehat{L}^1(m) \implies f \in \bigcap_{\alpha \in I} \widetilde{L}^1(\widetilde{m_\alpha}) \iff f \in \widetilde{L}^1(\tilde{m}).$$

Proof. To prove 4.6.A it suffices to apply Lemma 4.5, Remark 3.9, [3; Theorem 3.9], Corollary 3.11, [3; Theorem 5.1] and Definition 3.6.

4.6.B follows immediately from Remark 4.3, [3; Proposition 4.2], [3; Theorem 3.9], Corollary 3.11, [3; Theorem 5.1] and Definition 3.6. □

REFERENCES

[1] BLONDIA, C.: *Integration in locally convex spaces*, Simon Stevin **55** (1981), 81 102.

[2] BROOKS, J. K.—MARTELLOTTI, A.: *On the De Giorgi-Letta integral in infinite dimension*, Atti Sem. Mat. Fis. Univ. Modena **XL** (1992), 285–302.

[3] ISIDORI, M. C.—MARTELLOTTI, A.—SAMBUCINI, A. R.: *Integration with respect to orthogonally scattered measures*, Math. Slovaca. **48** (1998), 253–269.

[4] PIETSCH, A.: *Nuclear Locally Convex Spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[5] SCHAEFER, H. H.: *Topological Vector Spaces*, Springer-Verlag, New York-Heidelberg-Berlin, 1971.

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