Štefan Černák

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CANTOR EXTENSION OF A HALF LATTICE ORDERED GROUP

Štefan Černák

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ABSTRACT. In this note the Cantor extension of a half lattice ordered group with an Abelian increasing part is constructively described and studied.

C. J. Everett [2] has defined and studied the notion of the Cantor extension of an abelian lattice ordered group (cf. also L. Fuchs [3], F. Papangelou [5] and F. Dashiell, A. Hager, M. Henriksen [1]).

M. Giraudet and F. Lucas [4] have introduced and investigated the notion of a half lattice ordered group as a generalization of a lattice ordered group. Every half lattice ordered group is a subgroup of monotonic permutations of a chain.

In this note the Cantor extension of a half lattice ordered group with an abelian increasing part is studied.

1. Preliminaries

In this section the basic definitions concerning the Cantor extension of an Abelian lattice ordered group are given and the fundamental results of Everett [2] and Papangelou [5] (which will be applied in Section 2) are recalled. Further, we recall some definitions and results concerning half lattice ordered groups that are due to Giraudet and Lucas [4].

Let H be an Abelian lattice ordered group (l-group) and let $\mathbb{N}$ be the set of all positive integers. We say that (x_n) is a sequence in H if $x_n \in H$ for each $n \in \mathbb{N}$. Assume that (t_n) is a sequence in H such that $t_n \geq t_{n+1}$ for each $n \in \mathbb{N}$ and that there exists $\bigwedge t_n = t$ in H. Then we write $t_n \downarrow t$ in H. We say that

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a sequence (x_n) in H *o-converges* to $x \in H$ (or x is an *o-limit* of (x_n)) in H and we write $x_n \to x$ if there exists a sequence (t_n) in H such that $t_n \downarrow 0$ in H and

$$-t_n \le x_n - x \le t_n \qquad \text{for each} \quad n \in \mathbb{N}.$$

It is easy to verify that if $x_n \ge x_{n+1}$ for each $n \in \mathbb{N}$, then $x_n \to x$ if and only if $\bigwedge x_n = x$.

By a *zero sequence* we understand a sequence (x_n) with $x_n \to 0$.

A sequence (x_n) is called *fundamental* in H if there exists a sequence (t_n) in H such that $t_n \downarrow 0$ in H and

$$-t_n \le x_n - x_m \le t_n \qquad \text{for each} \quad n \in \mathbb{N}, \quad m \in \mathbb{N}, \quad m \ge n.$$

Every o-convergent sequence is fundamental. If the converse holds then H is called *o-complete.*

The set of all fundamental (zero) sequences in H will be denoted by F (E). F is an Abelian group under the operation $(x_n) + (y_n) = (x_n + y_n)$ and E is a subgroup of F. We can form the factor group $F/E = C(H)$. If $(x_n), (y_n) \in F$ then also $(x_n \vee y_n) \in F$. A coset of $C(H)$ containing a sequence $(x_n) \in F$ is denoted by $(x_n)^*$. For $(x_n)^*, (y_n)^* \in C(H)$ we put $(x_n)^* \le (y_n)^*$ if and only if $(x_n \vee y_n)^* = (y_n)^*$ or equivalently $x_n \le y_n + t_n$ for each $n \in \mathbb{N}$ and for some $t_n \downarrow 0$ in H (see [2]). Then $C(H)$ is an Abelian l-group which is called the *Cantor extension* of H.

For $(x_n) \in F$ and for $n \in \mathbb{N}$ we denote $X_n = (x_n, x_n, \dots)^*$ and $X = (x_n)^*$. Then we have (cf. [2; Theorem 4]):

(A) If $(x_n) \in F$, then $X_n \to X$ in $C(H)$.

Let φ be a mapping from H into $C(H)$ defined by the rule

$$\varphi(x) = (x, x, \dots)^*$$

for each $x \in H$.

In [2; Theorem 4] there is derived the following result:

(B) φ is an injection and φ preserves the group operation, order on H, all joins and intersections existing in H.

If x and $\varphi(x)$ are identified for each $x \in H$ then the following assertions are true (cf. [2; Theorem 4], [5; Corollary 4.5]):

(α) $C(H)$ is o-complete.
(β) H is an l-subgroup of $C(H)$.
(γ) Every element of $C(H)$ is an o-limit in $C(H)$ of a fundamental sequence in H.

1.1. THEOREM. (cf. [5; Theorem 4.6]) *Let H_1 and H_2 be Abelian l-groups satisfying (α)–(γ) (H_1 and H_2 instead of $C(H)$). Then there exists an isomorphism ψ of a lattice ordered group H_1 onto H_2 such that $\psi(x) = x$ for each $x \in H$.*

We recall the definition of a half lattice ordered group (cf. [4; Section 1]).

Let G be a group and, at the same time, a partially ordered set. We denote by $G\uparrow$ and $G\downarrow$ the set of all elements $x \in G$ such that whenever $y, z \in G$, $y \leq z$, then $x + y \leq x + z$ or $x + y \geq x + z$, respectively. $G\uparrow$ ($G\downarrow$) is called an *increasing* (*decreasing*) part of G.

G is said to be a *half lattice ordered group* if the following conditions are satisfied:

(I) $\leq$ is a non-trivial partial order on G,
(II) if $x, y, z \in G$ and $y \leq z$, then $y + x \leq z + x$,
(III) $G = G\uparrow \cup G\downarrow$,
(IV) $G\uparrow$ is a lattice.

From the definition it follows that $G\uparrow$ is a lattice ordered group. We shall apply (I)–(IV) without special references.

1.2. PROPOSITION. (cf. [4; Proposition I.1.3]) *Let G be a half lattice ordered group such that $G\downarrow \neq \emptyset$. Then*

(i) *$G\uparrow$ is a subgroup of G having the index 2,*
(ii) *the partially ordered sets $G\uparrow$ and $G\downarrow$ are isomorphic and also dually isomorphic,*
(iii) *if $x \in G\uparrow$ and $y \in G\downarrow$, then x and y are incomparable.*

1.3. PROPOSITION. (cf. [4; Proposition I.3.1]) *Let G be a half lattice ordered group such that $G\downarrow \neq \emptyset$. Then $A = \{a \in G : a \neq 0 \text{ and } 2a = 0\} \neq \emptyset$.*

Evidently, $A \subseteq G\downarrow$.

2. Cantor extension of a half lattice ordered group

In what follows we assume that G is a half lattice ordered group such that $G\uparrow$ is an Abelian lattice ordered group and that $G\downarrow \neq \emptyset$. Therefore G is neither Abelian nor a lattice ordered group.

Let G' be a half lattice ordered group such that

(i) the group G is a subgroup of the group G',
(ii) $G\uparrow$ is a sublattice of $G'\uparrow$ and $G\downarrow$ is a sublattice of $G'\downarrow$.

Then we say that G is an *hl-subgroup* of G'.

We shall use the notations $G\uparrow = H$ and $G\downarrow = K$.

Let (x_n) be a sequence in G. We say that (x_n) *o-converges* to $x \in G$ (or x is an *o-limit* of (x_n)) in G and we write $x_n \to x$ if there are sequences (t_n) and (u_n) in G such that $t_n \downarrow 0$, $u_n \downarrow 0$ in G and

$$-t_n \le x_n - x \le t_n, \quad -u_n \le -x + x_n \le u_n \qquad \text{for each} \quad n \in \mathbb{N}.$$

A sequence (x_n) in G is said to be *fundamental* in G if there are sequences (t_n) and (u_n) in G such that $t_n \downarrow 0$, $u_n \downarrow 0$ in G and

$$-t_n \le x_n - x_m \le t_n, \qquad -u_n \le -x_m + x_n \le u_n$$
$$\text{for each} \quad n \in \mathbb{N}, \quad m \in \mathbb{N}, \quad m \ge n.$$

Every o-convergent sequence is fundamental. If every fundamental sequence in G is o-convergent in G then G is said to be *o-complete.*

In view of the above mentioned properties (α)–(γ) of the Cantor extension of a lattice ordered group we introduce the following definition.

DEFINITION. A half lattice ordered group G' is said to be a *Cantor extension* of G if the following conditions are satisfied:

(a) G' is o-complete.

(b) G is an hl-subgroup of G'.

(c) Every element of G' is an o-limit in G' of a fundamental sequence in G.

In this section we prove that a Cantor extension of G exists and that it is uniquely determined up to isomorphisms leaving all elements of G fixed.

We need some auxiliary results.

The set of all fundamental sequences in G (H) will be denoted by F_G (F_H).

Let $x_i \in H$ $(i \in I)$. By using 1.2.(iii) we get that there exists $\bigwedge_{i \in I} x_i$ in H if and only if there exists $\bigwedge_{i \in I} x_i$ in G and $\bigwedge_{i \in I} x_i$ in H is equal to $\bigwedge_{i \in I} x_i$ in G. An analogous result holds for K. Further from 1.2.(iii) it follows:

2.1. LEMMA.

(i) *(t_n) is a sequence in G and $t_n \downarrow 0$ in G if and only if (t_n) is a sequence in H and $t_n \downarrow 0$ in H.*

(ii) *Let $x \in G$ and let (x_n) be a sequence in G such that $x_n \to x$ in G. Then either (x_n) is a sequence in H and $x \in H$ or (x_n) is a sequence in K and $x \in K$.*

(iii) *Let $(x_n) \in F_G$. Then (x_n) is a sequence either in H or in K.*

(iv) *Let (x_n) be a sequence in H. Then $(x_n) \in F_H$ if and only if $(x_n) \in F_G$.*

(v) *Let $x \in H$ and let (x_n) be a sequence in H. Then $x_n \to x$ in H if and only if $x_n \to x$ in G.*

Since $K \neq \emptyset$, with respect to 1.3 there exists an element $a \in A$.

The mapping $\alpha\colon x \mapsto a + x$ $(x \in H)$ is a dual isomorphism of the partially ordered set H onto K.

2.2. Lemma. *Let (x_n) be a sequence in H and $x \in H$. Then*

(i) $(x_n) \in F_H$ *if and only if* $(a + x_n) \in F_G$,
(ii) $x_n \to x$ *in* H *if and only if* $a + x_n \to a + x$ *in* G,
(iii) $x_n \to x$ *in* H *if and only if* $a + x_n + a \to a + x + a$ *in* H,
(iv) $(x_n) \in F_H$ *if and only if* $(a + x_n + a) \in F_H$.

Proof.

(i) Assume that $(x_n) \in F_H$. There exists $t_n \downarrow 0$ in H with $-t_n \le x_n - x_m \le t_n$ for each $n \in \mathbb{N}$, $m \in \mathbb{N}$, $m \ge n$. By applying $a \in K$ we get $a + t_n + a \le a + x_n - x_m + a = (a + x_n) - (a + x_m) \le a - t_n + a$. Since $a \in A$, we obtain $a + t_n + a = -(a - t_n + a)$. It can be verified that $a - t_n + a \downarrow 0$ in H and with respect to 2.1.(i) in G as well. Further we have $-(a + x_m) + (a + x_n) = -x_m + x_n = x_n - x_m$. We conclude that $(a + x_n) \in F_G$.

To prove the converse and (ii)–(iv), analogous steps can be applied. □

2.3. Lemma. *G is o-complete if and only if H is o-complete.*

Proof. Assume that G is o-complete and let $(x_n) \in F_H$. According to 2.2.(i) we have $(a + x_n) \in F_G$. The hypothesis yields that $(a + x_n)$ is an o-convergent sequence in G. Therefore $a + x_n \to a + x$ in G where x is an element of H. By 2.2.(ii) we get $x_n \to x$ in H.

Assume that H is o-complete and let $(z_n) \in F_G$. Then in view of 2.1.(iii) (z_n) is a sequence either in H or in K. If (z_n) is a sequence in H then 2.1.(iv) yields that $(z_n) \in F_H$. Thus (z_n) is o-convergent in H and by 2.1.(v) in G as well. If (z_n) is a sequence in K then $z_n = a + x_n$ $(n \in \mathbb{N})$ for some $x_n \in H$. By using of 2.2.(i) we get that $(x_n) \in F_H$. This implies that there is $x \in H$ with $x_n \to x$ in H. Then by 2.2.(ii) $a + x_n \to a + x$ in G. Therefore G is o-complete. □

Let us form the sets

$$a + C(H) = \{a + (x_n)^* : \ (x_n)^* \in C(H)\}$$

and

$$C_h(G) = C(H) \cup (a + C(H)). \tag{$*$}$$

We intend to define a group operation $+$ and a partial order $\le$ on $C_h(G)$ in such a way that $C_h(G)$ turns out to be a half lattice ordered group.

Let $(x_n)^*, (y_n)^* \in C(H)$. Since $(x_n) \in F_H$, according to 2.2.(iv) we obtain that $(a + x_n + a) \in F_H$ as well.

First, the operation $(x_n)^* + (y_n)^*$ was already defined in $C(H)$; we apply the same definition in $C_h(G)$, i.e.,

$$(x_n)^* + (y_n)^* = (x_n + y_n)^* .$$

In the remaining cases for pairs of elements of $C_h(G)$ we put

$$\begin{aligned}\big(a + (x_n)^*\big) + \big(a + (y_n)^*\big) &= (a + x_n + a + y_n)^* ,\\ (x_n)^* + \big(a + (y_n)^*\big) &= a + (a + x_n + a + y_n)^* ,\\ \big(a + (x_n)^*\big) + (y_n)^* &= a + (x_n + y_n)^* .\end{aligned}$$

Further we put $a+(x_n)^* \le a+(y_n)^*$ if and only if $(y_n)^* \le (x_n)^*$; we consider $a + (x_n)^*$ and $(y_n)^*$ as incomparable.

2.4. Lemma. $\big(C_h(G),+\big)$ *is a group.*

Proof. At first we show that the operation $+$ on $C_h(G)$ is associative. Only two cases will be investigated. Proofs of the remaining cases are similar.

$$\big(\big(a + (x_n)^*\big) + \big(a + (y_n)^*\big)\big) + \big(a + (z_n)^*\big) = (a + x_n + a + y_n)^* + \big(a + (z_n)^*\big)$$
$$= a + (a + a + x_n + a + y_n + a + z_n)^* = a + (x_n + a + y_n + a + z_n)^* ;$$

$$\big(a + (x_n)^*\big) + \big(\big(a + (y_n)^*\big) + \big(a + (z_n)^*\big)\big) = \big(a + (x_n)^*\big) + (a + y_n + a + z_n)^*$$
$$= a + (x_n + a + y_n + a + z_n)^* .$$

Hence

$$\big(\big(a+(x_n)^*\big)+\big(a+(y_n)^*\big)\big)+\big(a+(z_n)^*\big) = \big(a+(x_n)^*\big)+\big(\big(a+(y_n)^*\big)+\big(a+(z_n)^*\big)\big) .$$

Now we show that $\big((x_n)^*+(y_n)^*\big)+\big(a+(z_n)^*\big) = (x_n)^*+\big((y_n)^*+\big(a+(z_n)^*\big)\big)$.

$$\big((x_n)^* + (y_n)^*\big) + \big(a + (z_n)^*\big) = (x_n + y_n)^* + \big(a + (z_n)^*\big)$$
$$= a + (a + x_n + y_n + a + z_n)^* ;$$

$$(x_n)^* + \big((y_n)^* + \big(a + (z_n)^*\big)\big) = (x_n)^* + \big(a + (a + y_n + a + z_n)^*\big)$$
$$= a + (a + x_n + a + a + y_n + a + z_n)^* = a + (a + x_n + y_n + a + z_n)^* .$$

Every element of $C_h(G)$ has an inverse in $C_h(G)$. It is evident that it suffices to consider elements of $a+C(H)$. Let $a+(x_n)^* \in a+C(H)$. Then the element $a + (a - x_n + a)^* \in a + C(H)$ and it is an inverse to $a + (x_n)^*$.

Therefore $\big(C_h(G),+\big)$ is a group. □

It is obvious that $\big(C_h(G),\le\big)$ is a partially ordered set.

2.5. Lemma. *Let $(x_n)^*, (y_n)^* \in C(H)$. Then $(x_n)^* \le (y_n)^*$ if and only if $(a + x_n + a)^* \ge (a + y_n + a)^*$.*

Proof. Suppose that $(x_n)^* \le (y_n)^*$. Then there is a sequence $t_n \downarrow 0$ in H such that $x_n \le y_n + t_n$ for each $n \in \mathbb{N}$. Hence $x_n - t_n \le y_n$. Therefore $(a+x_n+a)+(a-t_n+a) \ge a+y_n+a$. According to 2.2.(iv) we have $(a+x_n+a)$, $(a + y_n + a) \in F_H$. Since $a - t_n + a \downarrow 0$ in H, $(a + y_n + a)^* \le (a + x_n + a)^*$. The converse is similar. □

2.6. Lemma. *Let $(x_n)^*, (y_n)^*, (z_n)^* \in C(H)$.*

(i) *If $(x_n)^* \le (y_n)^*$, then $(x_n)^* + (z_n)^* \le (y_n)^* + (z_n)^*$ and $(x_n)^* + (a + (z_n)^*) \le (y_n)^* + (a + (z_n)^*)$.*

(ii) *If $a + (x_n)^* \le a + (y_n)^*$ then $(a + (x_n)^*) + (z_n)^* \le (a + (y_n)^*) + (z_n)^*$ and $(a + (x_n)^*) + (a + (z_n)^*) \le (a + (y_n)^*) + (a + (z_n)^*)$.*

Proof.

(i) Since $C(H)$ is an l-group, $(x_n)^* \le (y_n)^*$ implies that $(x_n)^* + (z_n)^* \le (y_n)^* + (z_n)^*$.

Let $(x_n)^* \le (y_n)^*$. According to 2.5 we obtain that $(a+x_n+a)^* \ge (a+y_n+a)^*$. Then $(a + x_n + a)^* + (z_n)^* \ge (a + y_n + a)^* + (z_n)^*$ and so $(a + x_n + a + z_n)^* \ge (a + y_n + a + z_n)^*$. Then $a + (a + x_n + a + z_n)^* \le a + (a + y_n + a + z_n)^*$. It means that $(x_n)^* + (a + (z_n)^*) \le (y_n)^* + (a + (z_n)^*)$.

(ii) can be proved in a similar way. □

2.7. Lemma. *$C_h(G)\uparrow = C(H)$ and $C_h(G)\downarrow = a + C(H)$.*

Proof. We have to prove the validity of the following assertions:

(i_1) if $(x_n)^* \le (y_n)^*$ then $(z_n)^* + (x_n)^* \le (z_n)^* + (y_n)^*$,

(ii_1) if $a + (x_n)^* \le a + (y_n)^*$ then $(z_n)^* + (a + (x_n)^*) \le (z_n)^* + (a + (y_n)^*)$

and

(i_2) if $(x_n)^* \le (y_n)^*$ then $(a + (z_n)^*) + (x_n)^* \ge (a + (z_n)^*) + (y_n)^*$,

(ii_2) if $a + (x_n)^* \le a + (y_n)^*$ then $(a + (z_n)^*) + (a + (x_n)^*) \ge (a + (z_n)^*) + (a + (y_n)^*)$.

(i_1) holds because of the fact that $C(H)$ is an l-group.

(ii_1) Let $a+(x_n)^* \le a+(y_n)^*$. Then $(x_n)^* \ge (y_n)^*$ and we get $(a+z_n+a)^* + (x_n)^* \ge (a + z_n + a)^* + (y_n)^*$, $(a + z_n + a + x_n)^* \ge (a + z_n + a + y_n)^*$. Hence $a + (a + z_n + a + x_n)^* \le a + (a + z_n + a + y_n)^*$ and so $(z_n)^* + (a + (x_n)^*) \le (z_n)^* + (a + (y_n)^*)$.

(ii_2) Assume that $a + (x_n)^* \le a + (y_n)^*$. Hence $(x_n)^* \ge (y_n)^*$ and $(a + z_n + a)^* + (x_n)^* \ge (a + z_n + a)^* + (y_n)^*$. Thus $(a + (z_n)^*) + (a + (x_n)^*) \ge (a + (z_n)^*) + (a + (y_n)^*)$.

The proof of (i_2) is analogous. □

The partial order $\leq$ is not trivial on G. This yields that $\leq$ is also a non-trivial partial order on $C_h(G)$. Then by applying 2.6, 2.7 and $(*)$ we conclude that $C_h(G)$ is a half lattice ordered group.

Let f be a mapping from G into $C_h(G)$ defined as follows:

$f(x) = (x, x, \dots)^*$ and $f(a+x) = a + f(x)$ for each $x \in H$.

2.8. Lemma. *The mapping f is an injection and preserves the group operation, partial order, all joins and intersections existing in G.*

P r o o f. Since f restricted to H is equal to φ and $C(H) \cap (a + CH) = \emptyset$, from (B) it follows that f is an injection.

Let $x, y \in H$. We have

$f\big((a+x)+(a+y)\big) = f(a+x+a+y) = (a+x+a+y,\, a+x+a+y, \dots)^* = \big(a+(x,x,\dots)^*\big)+\big(a+(y,y,\dots)^*\big) = \big(a+f(x)\big)+\big(a+f(y)\big) = f(a+x)+f(a+y)$.

$f\big(x+(a+y)\big) = f\big(a+(a+x+a+y)\big) = a+f(a+x+a+y) = a+(a+x+a+y,\, a+x+a+y, \dots)^* = (x,x,\dots)^* + \big(a+(y,y,\dots)^*\big) = f(x)+\big(a+f(y)\big) = f(x)+f(a+y)$.

$f\big((a+x)+y\big) = f\big(a+(x+y)\big) = a+f(x+y) = a+(x+y,\, x+y, \dots)^* = \big(a+(x,x,\dots)^*\big)+(y,y,\dots)^* = \big(a+f(x)\big)+f(y) = f(a+x)+f(y)$.

From this and from (B) we infer that f preserves the group operation on G.

Let $x, y \in H$, $a+x \leq a+y$. Then $x \geq y$. According to (B) we obtain $f(x) \geq f(y)$. Hence $a+f(x) \leq a+f(y)$, $f(a+x) \leq f(a+y)$. Therefore f preserves the partial order on G.

Now we prove that f preserves also all joins and intersections existing in G.

Assume that $x_i \in H$ $(i \in I)$ and that there exists $\bigwedge_{i\in I}(a+x_i)$ in G. We shall prove that then there exist $\bigvee_{i\in I}(a+x_i)$ in G, $\bigwedge_{i\in I} f(a+x_i)$, $\bigvee_{i\in I} f(a+x_i)$ in $C_h(G)$ and that

(1) $f\Big(\bigwedge_{i\in I}(a+x_i)\Big) = \bigwedge_{i\in I} f(a+x_i)$,

(2) $f\Big(\bigvee_{i\in I}(a+x_i)\Big) = \bigvee_{i\in I} f(a+x_i)$

are valid.

At first we prove that there exists $\bigvee_{i\in I} x_i$ in G and that

(3) $\bigvee_{i\in I} x_i = a + \bigwedge_{i\in I}(a+x_i)$

holds.

Denote $z = \bigwedge_{i\in I}(a+x_i)$. We have $z \leq a+x_i$, $a+z \geq x_i$ $(i \in I)$. Assume that $z' \in G$, $z' \geq x_i$ $(i \in I)$. Then $a+z' \leq a+x_i$ $(i \in I)$ and thus $a+z' \leq z$, $z' \geq a+z$. From this it follows that (3) holds.

Then from (B) we infer that $\bigvee_{i\in I} f(x_i)$ does exist in $C(H)$ and $\bigvee_{i\in I} f(x_i) = f\left(\bigvee_{i\in I} x_i\right)$.

Since f preserves the partial order, $\bigwedge_{i\in I}(a + x_i) \le a + x_i$ $(i \in I)$ implies that $f\left(\bigwedge_{i\in I}(a + x_i)\right) \le f(a + x_i)$ $(i \in I)$. Let $z \in C_h(G)$, $z \le f(a + x_i)$ $(i \in I)$. Then $z \in a + C(H)$, $z = a + (x_n)^*$ where $(x_n)^*$ is an element of $C(H)$. We have $a + (x_n)^* \le f(a + x_i) = a + f(x_i)$, $(x_n)^* \ge f(x_i)$ $(i \in I)$. Hence $(x_n)^* \ge \bigvee_{i\in I} f(x_i) = f\left(\bigvee_{i\in I} x_i\right)$. Thus $a + (x_n)^* \le a + f\left(\bigvee_{i\in I} x_i\right) = f\left(a + \bigvee_{i\in I} x_i\right)$. According to (3) we get $z \le f\left(\bigwedge_{i\in I}(a + x_i)\right)$. Therefore (1) is satisfied.

Since $\bigvee_{i\in I} x_i$ does exist in H, there exists also $\bigwedge_{i\in I} x_i$ in H. In an analogous way as above we prove that there exists $\bigvee_{i\in I}(a+x_i)$ in G, $\bigwedge_{i\in I} x_i = a + \bigvee_{i\in I}(a+x_i)$ and that (2) is valid. □

2.9. LEMMA. *Let $(x_n) \in F_H$. Then $a + X_n \to a + X$ in $C_h(G)$.*

P r o o f. By (A) we have $X_n \to X$ in $C(H)$. Then there exists $T_n \downarrow E$ in $C(H)$ with $-T_n \le X_n - X \le T_n$ for each $n \in \mathbb{N}$. Therefore $a + T_n + a \le a + X_n - X + a = (a + X_n) - (a + X) \le a - T_n + a$, $a - T_n + a \downarrow E$ in $C(H)$, $a + T_n + a = -(a - T_n + a)$. Further we have $-(a + X) + (a + X_n) = -X + X_n = X_n - X$. We conclude that $a + X_n \to a + X$ in $C_h(G)$. □

From 2.1.(iii), 2.9 and (A) it follows that every fundamental sequence in $f(G)$ has an o-limit in $C_h(G)$.

Moreover, with respect to $(*)$, 2.7 and (A) we have shown in 2.9 that every element of $C_h(G)$ is an o-limit of a fundamental sequence in $f(G)$.

According to 2.3 a half lattice ordered group is o-complete if and only if its increasing part is o-complete. Then from (α) and 2.7 it follows that $C_h(G)$ is o-complete.

By summarizing the above results, we infer from 2.7 and 2.8 that the following theorem is valid (x and $f(x)$ are identified for each $x \in G$).

2.10. THEOREM. *$C_h(G)$ is a half lattice ordered group with the following properties:*

(a) *$C_h(G)$ is o-complete.*
(b) *G is an hl-subgroup of $C_h(G)$.*
(c) *Every element of $C_h(G)$ is an o-limit in $C_h(G)$ of a fundamental sequence in G.*

2.11. COROLLARY. *$C_h(G)$ is a Cantor extension of G.*

By using 2.3 it is easy to verify that the following assertion is valid.

2.12. LEMMA. *Let G_1 be a half lattice ordered group such that $G_1\uparrow$ is Abelian and $K \subseteq G_1\downarrow$. Then G_1 is a Cantor extension of G if and only if $G_1\uparrow$ fulfils (α)-(γ) ($G_1\uparrow$ instead of $C(H)$).*

2.13. THEOREM. *Let G_1 and G_2 be Cantor extensions of G. Then there exists an isomorphism ϕ of a half lattice ordered group G_1 onto G_2 which restricts to the identity on G.*

Proof. With respect to (b) G is an hl-subgroup of G_1 and G_2. According to 2.12 $G_1\uparrow$ and $G_2\uparrow$ satisfy (α)-(γ) ($G_1\uparrow$ and $G_2\uparrow$ instead of $C(H)$). An arbitrary element of $G_1\downarrow$ has the form $a + x^1$ where x^1 is an element of $G_1\uparrow$ and a is as above. With respect to (γ) there is a sequence $(x_n) \in F_H$ with $x_n \to x^1$ in G_1. The condition (α) implies that there exists $x^2 \in G_2$ with $x_n \to x^2$ in G_2. We have $a + x_n \in K$ for each $n \in \mathbb{N}$ and by 2.2.(i) $(a + x_n) \in F_G$. Therefore $a + x_n \to a + x^1$ in G_1 and $a + x_n \to a + x^2$ in G_2.

We put $\phi(x^1) = x^2$, $\phi(a + x^1) = a + \phi(x^1)$ for each $x^1 \in G_1\uparrow$. Then ϕ is a mapping from G_1 into G_2. It is easy to verify that ϕ is correctly defined and that ϕ is an isomorphism of a half lattice ordered group G_1 onto G_2 with the desired property. □

Assume that $a' \in A$, $a' \neq a$. We can construct a half lattice ordered group $C'_h(G)$ (a' instead of a) in the same way as $C_h(G)$ above. Hence $C'_h(G)$ is a Cantor extension of G. Then, under the notation from 2.13 it follows:

2.14. COROLLARY. *Half lattice ordered groups $C_h(G)$ and $C'_h(G)$ are isomorphic.*

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Department of Mathematics
Faculty of Civil Engineering
Technical University
ul. Vysokoškolská 4
SK–042 02 Košice
SLOVAKIA
E-mail: cernak@ocsun.tuke.sk