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ON THE DICHROMATIC NUMBER IN KERNEL THEORY

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ABSTRACT. The dichromatic number of a digraph D is the minimum cardinal of a partition of $V(D)$ into acyclic classes. The dichromatic number gives a measure of the complexity of the cyclic structure of D.

A kernel N of a digraph D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D. When every induced subdigraph of D has a kernel, the digraph D is said to be kernel-perfect. We say that D is a critical kernel-imperfect digraph if D does not have a kernel but every proper induced subdigraph of D does have at least one.

In this paper we prove the existence of kernel-perfect digraphs with arbitrarily large dichromatic number whose underlying graph has no triangles and we prove the existence of critical kernel-imperfect digraphs with arbitrarily large dichromatic number and without directed cycles of length two or three. The earliest sufficient conditions for the existence of kernels in digraphs include only digraphs with dichromatic number at most two. Finally we state some open problems relating the dichromatic number and the kernel of a digraph.

1. Introduction

For general concepts we refer the reader to [1]. Let D be a digraph; $V(D)$ and $F(D)$ will denote the sets of vertices and arcs of D, respectively. An arc u_1u_2 of D will be called an S_1S_2-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of D induced by S_1. A digraph is called asymmetric if it does not contain a directed cycle of length two. A set $I \subset V(D)$ is independent if $FD[I] = \emptyset$. A *kernel* N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D (we say that N is absorbing). When every induced subdigraph of D has a kernel, the digraph D is said to be *kernel-perfect*. We say that D is a *critical kernel-imperfect digraph* if D does not have a kernel but every proper induced subdigraph of D does have at least

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one. We denote by $g(D)$ (resp. $\vec{g}(D)$) the *girth* of D, the minimum length of a cycle (resp. directed cycle) in the underlying graph of D (resp. in D).

The concept of dichromatic number $d_k(D)$ of a digraph D was defined in [6] and independently in [5] as the minimum number of colours required to colour the vertices of D in such a way that the chromatic classes induce acyclic subdigraphs of D. Clearly $d_k(D) \leq \chi(D)$ where $\chi(D)$ denotes the chromatic number of the underlying graph of D. The dichromatic number is a generalization of the chromatic number. In particular they coincide for symmetric digraphs.

The earliest sufficient conditions for the existence of kernels in digraphs ([7], [8], [9], [10]) include only digraphs with dichromatic number at most two. In [2] the authors found kernel-perfect and critical kernel-imperfect digraphs with arbitrarily large dichromatic number. However, their construction allows a lot of symmetric arcs.

For asymmetrical digraphs the situation is more complicated: the only tournament which is a critical kernel-imperfect digraph is the directed triangle; therefore a tournament T is a kernel-perfect digraph if and only if T is acyclic or equivalently, $d_k(T) = 1$. In [3] the existence of kernel-perfect and critical kernel-imperfect digraphs with arbitrarily large dichromatic number was proved, but the construction given there allows many directed triangles and symmetric arcs.

In this paper we prove that there exist kernel perfect asymmetrical digraphs with arbitrarily large dichromatic number whose underlying graphs have no triangles. We also prove the existence of critical kernel-imperfect digraphs with arbitrarily large dichromatic number and without directed cycles of length two or three.

We conclude this section with two open problems.

CONJECTURE 1. Given natural numbers n and k, there exists an asymmetric critical kernel-imperfect digraph with dichromatic number n and girth k.

An instance of Conjecture 1 is:

CONJECTURE 2. Given natural numbers n and k, there exists a critical kernel-imperfect digraph with dichromatic number n and directed girth k.

2. The dichromatic number in kernel theory

DEFINITION 1. Let D be a digraph, $\widetilde{\alpha} = \big((\alpha_v, J_v)\big)_{v\in V(D)}$ a family where the α_v are mutually disjoint digraphs and J_v is a non-empty subset of $V(\alpha_v)$ for $v \in V(D)$. We define the digraph $\sigma(D, \widetilde{\alpha})$ by the following conditions:

(i) $V\big(\sigma(D,\widetilde{\alpha})\big) = \bigcup_{v\in V(D)} V(\alpha_v)$,

(ii) $F(\sigma(D,\tilde{\alpha})) = \Big(\bigcup_{v\in V(D)} F(\alpha_v)\Big) \cup \{xy \mid x\in J_u,\ y\in J_v,\ uv\in F(D)\}$.

Notice that if $j_v \in J_v$ for $v\in V(D)$, then $\sigma(D,\tilde{\alpha})[\{j_v \mid v\in V(D)\}]$ is isomorphic to D. If $f_v\colon \alpha_v \to \alpha$ is an isomorphism such that $f_v(J_v) = J$ for every $v\in V(D)$, we will write $\sigma(D,(\alpha,J))$ instead of $\sigma(D,\tilde{\alpha})$.

Lemma 1. *If $g(D)\ge 4$, $g(\alpha_v)\ge 4$ and J_v is independent in α_v for $v\in V(D)$ then $g(\sigma(D,\tilde{\alpha}))\ge 4$.*

The proof of Lemma 1 is trivial.

Lemma 2. *Let D be a digraph such that $d_k(D) = n$. If the family*

$$\tilde{\alpha} = ((\alpha_v, J_v))_{v\in V(D)}$$

satisfies

(i) *$\alpha_v[J_v]$ is acyclic for every $v\in V(D)$,*

(ii) *for all $v\in V(D)$, $d_k(\alpha_v) = n$ and in every n-colouring of α_v where the chromatic classes are acyclic (acyclic n-colouring), J_v is not monochromatic,*

then $d_k(\sigma(D,\tilde{\alpha})) = n+1$.

Proof. Let φ be an acyclic n-colouring of D and f_v an acyclic $(n+1)$-colouring of α_v such that f_v has a constant value $\varphi(v)$ on J_v for every $v\in V(D)$ (since $d_k(\alpha_v) = n$ such a colouring indeed exists). Define an acyclic $(n+1)$-colouring of $\sigma(D,\tilde{\alpha})$ by the conditions $f/V(\alpha_v) = f_v$ for every $v\in V(D)$. Therefore $d_k(\sigma(D,\tilde{\alpha}))\le n+1$.

Suppose that $d_k(\sigma(D,\tilde{\alpha}))\le n$ and let ψ be an acyclic n-colouring of $\sigma(D,\tilde{\alpha})$. Let $I_{n-1} = \{1,2,\dots,n-1\}$. Property (ii) in the hypothesis of Lemma 2 implies that $J_v\cap\psi^{-1}(I_{n-1})$ is non empty, so choosing $u_v\in J_v\cap\psi^{-1}(I_{n-1})$ for $v\in V(D)$ we have that $\sigma(D,\tilde{\alpha})[\{u_v\}_{v\in V(D)}]\cong D$ and $n-1$ colourable. This yields a contradiction. It follows that $d_k(\sigma(D,\tilde{\alpha})) = n+1$. □

Lemma 3. *Let $d_k(D) = n > 1$, $g(D)\ge 4$ and $\tilde{\alpha} = ((\alpha_v, J_v))_{v\in V(D)}$ be a family such that α_v is a bipartite asymmetric and not acyclic digraph with bipartition $\{J_v, J_v' = V(\alpha_v) - J_v\}$ for $v\in V(D)$. Then $g(\sigma(D,\tilde{\alpha}))\ge 4$, $J' = \bigcup_{v\in V(D)} J_v'$ is independent, $d_k(\sigma(D,\tilde{\alpha})) = n$ and in every acyclic n-colouring of $\sigma(D,\tilde{\alpha})$, J' is not monochromatic.*

Proof. By Lemma 1, $g(\sigma(D,\tilde{\alpha}))\ge 4$. Clearly, J' is independent. Take $J = \bigcup_{v\in V(D)} J_v$. Let φ be any acyclic n-colouring φ of D and assign to each $w\in V(\sigma(D,\tilde{\alpha}))$ the colour $\varphi(v)$ when $w\in J_v$, and any colour different to $\varphi(v)$

when $w \in J_v'$. In this way we get an acyclic n-colouring of $\sigma(D,\widetilde{\alpha})$. It follows that $d_k(\sigma(D,\widetilde{\alpha})) = n$ since $\sigma(D,\widetilde{\alpha})$ contains an isomorphic copy of D.

Let ψ be any n-colouring of $\sigma(D,\widetilde{\alpha})$ and $i \in \{1,2,\dots,n\}$. Then there exists $v(i) \in V(D)$ such that $J_{v(i)}$ is monochromatic of colour i, for otherwise $\sigma(D,\widetilde{\alpha})[J-\psi^{-1}(i)]$ would be $(n-1)$-colourable which is impossible since it contains an isomorphic copy of D. Let $k \in \psi(J')$. We know that $J_{v(k)}$ is monochromatic of colour k hence $J'_{v(k)}$ is not monochromatic of colour k because $d_k(\alpha_k) \ge 2$. It follows that J' is not monochromatic. □

THEOREM 1. *Let D be a kernel-perfect digraph and $\widetilde{\alpha} = ((\alpha_v, J_v))_{v\in V(D)}$ a family such that α_v is a kernel-perfect digraph. Then $\sigma(D,\widetilde{\alpha})$ is a kernel-perfect digraph.*

Proof. Notice first that every connected induced subdigraph of $\sigma(D,\widetilde{\alpha})$ has the form $\sigma(D',\widetilde{\alpha}')$ for suitable D' and $\widetilde{\alpha}' = ((\alpha_v', J_v'))_{v\in V(D')}$ where D' and each α_v' are kernel-perfect digraphs (actually D' is an induced subdigraph of D and α_v' is an induced subdigraph of α_v for each $v \in V(D')$). Therefore we need to prove only that $\sigma(D,\alpha)$ has a kernel.

Choose a kernel N_v of α_v for each $v \in V(D)$ and take $Q = \{v \in V(D) \mid N_v \cap J_v \neq \emptyset\}$.

Let N be a kernel of $D[Q]$ and define

$$N^* = \bigcup_{v\in N\cup(V(D)-Q)} N_v \cup \bigcup_{v\in(Q-N)} N_v'$$

where N_v' is a kernel of $\alpha_v[V(\alpha_v) - J_v]$, $v \in (Q-N)$.

We will show that N^* is a kernel of $\sigma(D,\widetilde{\alpha})$.

First we will prove that N^* is independent.

Since N is an independent subset of $V(D)$ and each N_v is also independent, it follows from the definition of $\sigma(D,\widetilde{\alpha})$ that $\bigcup_{v\in N} N_v$ is independent.

The fact that for every $v \in (V(D)-Q)$ we have $N_v \cap J_v = \emptyset$ and the definition of $\sigma(D,\widetilde{\alpha})$ imply that $\bigcup_{v\in(V(D)-Q)} N_v$ is independent and for every $z \in Q$ there is neither $\Big(\bigcup_{v\in(V(D)-Q)} N_v\Big)N_z$-arc nor $N_z\Big(\bigcup_{v\in(V(D)-Q)} N_v\Big)$-arc. Hence

$$\bigcup_{v\in N} N_v \cup \bigcup_{v\in(V(D)-Q)} N_v$$

is independent.

On the other hand, N_v' is independent and the definition of $\sigma(D,\widetilde{\alpha})$ implies that for every $z \neq v$ there is neither $N_v'V(\alpha_z)$-arc nor $V(\alpha_z)N_v'$-arc. So

$\bigcup_{v\in(Q-N)} N'_v$ is independent and also

$$\bigcup_{v\in N\cup(V(D)-Q)} N_v \cup \bigcup_{v\in\left(Q-N\right)} N'_v$$

is independent.

It remains to show that N^* is absorbing.

Let $w \in V\sigma(D,\widetilde{\alpha}) - N^*$ and $v \in V(D)$ such that $w \in V(\alpha_v)$; we analyse several cases:

(a) $v \in \left(V(D) - Q\right)$.
Since $w \notin N^*$ it follows that $w \in \left(V(\alpha_v) - N_v\right)$. There exists a wN_v-arc as N_v is a kernel of α_v. Hence there exists wN^*-arc in $\sigma(D,\widetilde{\alpha})$.

(b) $v \in (Q - N)$.

(b.1) $w \in \left(V(\alpha_v) - J_v\right)$.
Since $w \notin N^*$ we have $w \notin N'_v$ and there exists a wN'_v-arc as N'_v is a kernel of $\alpha_v\left[V(\alpha_v) - J_v\right]$ and this is also a wN^*-arc.

(b.2) $w \in J_v$.
The definition of N and $v \in (Q-N)$ imply that there exists some $v' \in N$ such that $vv' \in F(D)$. Hence the definition of $\sigma(D,\widetilde{\alpha})$ implies that every $wJ_{v'}$-arc is in $\sigma(D,\widetilde{\alpha})$ and by the definition of Q we have $N_{v'} \cap J_{v'} \neq \emptyset$. We conclude that there exists a $wN_{v'}$-arc with $v' \in N$. So there exists a wN^*-arc.

(c) $v \in N$.
Since $w \notin N^*$ if follows that $w \in \left(V(\alpha_v) - N_v\right)$ and the definition of N_v implies that there exists a wN_v-arc which clearly is also a wN^*-arc.

□

THEOREM 2. *Let n be a natural number $n > 1$. There exists an asymmetric kernel-perfect digraph D_n such that $d_k(D_n) = n$ and $g(D_n) \geq 4$.*

P r o o f. We proceed by induction on n. Taking $D_2 = \vec{C}_4$, the directed cycle of length four, we see that Theorem 2 holds for $n = 2$.

Suppose that there has been constructed an asymmetric kernel-perfect digraph D_m for $m \geq 2$ such that $d_k(D_m) = m$ and $g(D_m) \geq 4$. Let α_v be the directed four cycle $(0_v, 1_v, 2_v, 3_v, 0_v)$, $J_v = \{1_v, 3_v\}$ and $J'_v = \{0_v, 2_v\}$ for each $v \in V(D_m)$. By Lemma 3, for $\widetilde{\alpha} = (\alpha_v, J_v)_{v\in V(D_m)}$, $d_k\left(\sigma(D_m,\widetilde{\alpha})\right) = m$, $g\left(\sigma(D_m,\widetilde{\alpha})\right) \geq 4$ and in every acyclic m-colouring of $\sigma(D_m,\widetilde{\alpha})$, $J' = \bigcup_{v\in V(D_m)} J'_v$ is not monochromatic. ($\widetilde{\alpha} = (\alpha_v, J_v)_{v\in V(D_m)}$).

Now we take $D \cong D_m$, $\widetilde{\alpha}' = \left((\alpha'_v, J'_v)\right)_{v\in V(D_m)}$ where $\alpha'_v \cong \sigma(D_m, \widetilde{\alpha})$, $J'_v = J'$, and $D_{m+1} \cong \sigma(D, \widetilde{\alpha}')$. It follows from Lemmas 1 and 2 that $d_k(D_{m+1}) = m+1$ and $g(D_{m+1}) \geq 4$. On the other hand both D_m and $\overrightarrow{C}_4$ are kernel-perfect digraphs, and hence by Theorem 1 the D_{m+1} is also kernel-perfect. □

THEOREM 3. *Every asymmetric kernel-perfect digraph without directed cycles of length three is an induced subdigraph of an asymmetric critical kernel-imperfect digraph without directed cycles of length three.*

Proof. Let D be any asymmetric kernel-perfect digraph without directed triangles and with $V(D) = \{1, 2, \dots, p\}$.

Denote by D^* the digraph defined as follows:

$$V(D^*) = \{(i,j) \mid i \in \{0,1,2,\dots,2p\},\ j \in \{0,1,\dots,4p\}\}.$$

In what follows we work with residue classes mod $2p+1$.

Let

$$\begin{aligned} F(D^*) = &\{((i,j),(i,j+1)) \mid i \in \{0,1,\dots,2p\},\ j \in \{0,1,\dots,4p-1\}\} \\ &\cup \{((i,4p),(i+r,2(r-1))) \mid i \in \{0,1,\dots,2p\}, \\ &\qquad\qquad r \in \{1,2,3,4,5,\dots,2p-1\}\} \\ &\cup \{((i,4p),(i+r,4p-2)) \mid i \in \{0,1,\dots,2p\}, \\ &\qquad\qquad r \in \{2,3,4,\dots,2p-1\}\} \\ &\cup \{((2i,4p),(2j,4p)) \mid (i,j) \in F(D)\}. \end{aligned}$$

Clearly, D is isomorphic to the subdigraph of D^* induced by

$$\{(2i,4p) \mid i \in \{1,2,\dots,p\}\}.$$

Define the digraph $C = \overrightarrow{C}_n(j_1, j_2, \dots, j_k)$ by $V(C) = \{0,1,\dots,n-1\}$ and $F(C) = \{uv \mid v-u \equiv j_s \pmod{n} \text{ for } s = 1,\dots,k\}$.

$D_0 = \overrightarrow{C}_{2p+1}\left(+1, \pm 2, \pm 3, \dots, \pm\left[\frac{2p+1}{2}\right]\right)$, $U = V(D_0)$, $A = (A_u)_{u\in U}$ where A_u is a directed path of length $2p$, γ_u^f a directed path of length two and $S_u = \emptyset$ for each $u \in U$, $f \in F(A_u)$ and U^+ a digraph isomorphic to D. It follows from [2; Theorem 2.4], [4; Lemma 2.1, Theorem 2.4] that D^* is an asymmetric critical kernel-imperfect digraph.

Finally we will prove that D^* has no directed triangles. We proceed by contradiction. Suppose that $\mathcal{C}$ is a directed triangle of D^*. It follows from the definition of D^* that $D^*\left[\{(i,4p) \mid i \in \{0,1,\dots,2p\}\}\right] \cong D \cup K^c_{p+1}$ (where K^c_{p+1} denotes the complement of the complete graph on $p+1$ vertices) and the hypothesis implies that $D \cup K^c_{p+1}$ has no directed triangles, hence there exists $(i,j) \in V(\mathcal{C})$ with $j \neq 4p$ such that $(i,j) \in \mathcal{C}$. Since $j \neq 4p$ the definition of

D^* implies that $(i, j+1) \in \mathcal{C}$. When $j+1 < 4p$ it follows from the definition of D^* that $(i, j+2) \in \mathcal{C}$ and in fact $\mathcal{C} = \big((i,j),(i,j+1),(i,j+2),(i,j)\big)$, so $\big((i,j+2),(i,j)\big) \in F(D^*)$ which is not possible. When $j+1 = 4p$, we have $j = 4p-1$, $\big((i,4p-1),(i,4p)\big) \in A(D^*)$ and since the only other vertex of D^* which is adjacent to $(i,4p-1)$ is $(i,4p-2)$ it follows that $\mathcal{C} = \big((i,4p-1),(i,4p),(i,4p-2),(i,4p-1)\big)$ which is in view of the definition of D^* impossible. □

COROLLARY 1. *Let n be a natural number, $n > 1$. There exists an asymmetric critical kernel-imperfect digraph D_n such that $d_k(D_n) = n$ and $\vec{g}(D_n) \geq 4$.*

Proof. It follows directly from Theorem 2 and 3. □

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