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EXISTENCE RESULTS FOR SECOND ORDER VOLTERRA INTEGRODIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

LONGTU LI* — XIANGZHENG QIAN** — ZHICHENG WANG**

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ABSTRACT. In this paper, using the nonlinear alternative of Leray-Schauder and the Gronwall-Bellman-Bihari-type integral inequalities, we study the initial value problems of the second order Volterra integrodifferential equations with deviating arguments.

1. Introduction

In this paper, existence results are presented for the solutions of second order Volterra integrodifferential equation with deviating arguments

$$\big(p(t)u'(t)\big)' = f\Big(t, u\big(g_1(t)\big), \int_0^t k\big(t,s,u(g_2(s))\big)\,\mathrm{d}s, p(t)u'(t)\Big)\,, \qquad 0 \le t \le T\,, \tag{1.1}$$

with u satisfying the initial value condition

$$u(t) = \varphi(t)\,, \quad t \in [-r,0]\,, \qquad u'(0) = A\,, \tag{1.2}$$

where $p \in C[0,T]$, $p(t) > 0$, $f \in C\big([0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$, $k \in C\big([0,T] \times [0,T] \times \mathbb{R}, \mathbb{R}\big)$, $A \in \mathbb{R}$, $g_1, g_2 \in G$ (see Section 2), $\varphi \in C\big([-r,0], \mathbb{R}\big)$ and

$$r = -\min\Big\{ \min_{t\in[0,T]} g_1(t),\ \min_{t\in[0,T]} g_2(t) \Big\} > 0\,.$$

Here $u'(0)$ means the derivative on the right $u'(0^+)$.

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By using the nonlinear alternative of Leray-Schauder and suitable a priori estimates, we prove the global existence of a solution to (1.1)–(1.2) on the whole interval $[0,T]$.

The global existence of solutions for ordinary differential equations has been considered by many authors. By employing the topological transversaly theorem, D. O'Regan [4], and J. W. Lee and D. O'Regan [3], and Huaxing Xia and T. Spanily [6] have established global existence results for differential delay equations. We continue these consideration to a global existence problem (1.1)–(1.2) for the second order Volterra integrodifferential equation with deviating arguments. Since the application of the nonlinear alternative of Leray-Schauder involves a priori bounds of solutions, and the estimate of such bounds is more difficult for Volterra integrodifferential equation with deviating arguments, we use Gronwall-Bellman-Bihari-type inequalities ([1]) to establish a priori bounds.

Our paper is organized as follows. In Section 2, we present some preliminaries. The general global existence results are discussed in Section 3. Finally, in Section 4, we obtain some results on a priori bounds of solutions.

2. Preliminaries

Let C_r, $r > 0$, be the space of all continuous functions $u\colon [-r,0] \to \mathbb{R}$. For $\psi \in C_r$ we define the norm

$$\|\psi\|_{[-r,0]} = \sup_{\theta\in[-r,0]} |\psi(\theta)|\,.$$

For convenience, we introduce the following notations and definitions:

$$\begin{aligned}
\|u\|_0 &= \sup_{t\in[-r,0]} |u(t)|\,,\\
|u|_1 &= \max\{\|u\|_0,\ \|pu'\|_0\}\,,\\
C[-r,T] &= C([-r,T],\mathbb{R})\,,\\
K^1[-r,T] &= \{u \in C[-r,T]\cap C^1[0,T];\ u(0)=\varphi(0) \text{ and } |u|_1 < \infty\}\,.
\end{aligned}$$

DEFINITION 2.1. Denote by G the class of continuous functions $g\colon \mathbb{R} \to \mathbb{R}$ satisfying $g(t) \le t$.

DEFINITION 2.2. *H belongs to class G_1 if $H(u)$ is nonnegative, continuous, and nondecreasing for $u \ge 0$, $H\colon [0,\infty) \to [0,\infty)$, $H(u) > 0$ for $u > 0$, $H(0) = 0$, and $t^{-1}H(u) \le H(t^{-1}u)$ for $t \ge 1$ and $u \ge 0$.*

LEMMA 2.1. (**NONLINEAR ALTERNATIVE OF LERAY-SCHAUDER**) ([2]) *Assume U is a relatively open subset of a convex set K in a Banach space E. Let $G^*: \bar{U} \to K$ be a compact map, $p \in U$, and let $N_\lambda(u) = N(u, \lambda): \bar{U} \times [0,1] \to K$ be a family of compact maps (i.e., $N(\bar{U} \times [0,1])$ is contained in a compact subset of K, and $N: \bar{U} \times [0,1] \to K$ is continuous) with $N_1 = G^*$ and $N_0 = p$, the constant map to p. Then either*

(i) *G^* has a fixed point in $\bar{U}$;*

or

(ii) *there is a point $u \in \partial U$ and $\lambda \in [0,1]$ such that $u = N_\lambda u$.*

By the *solution of the IVP* (1.1)–(1.2) we mean a function $u \in K^1[-r,T]$ which satisfies the integrodifferential equation (1.1) and the initial value condition (1.2).

3. An existence principle

In this section, we present the general global existence result of a solution of the IVP (1.1)–(1.2).

THEOREM 3.1. *Let $f: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k: [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions, $p \in C[0,T]$, $p(t) > 0$, and assume $g_1, g_2 \in G$. Suppose that there is a constant M, independent of λ, such that $|u|_1 < M$ for any solution u to*

$$(p(t)u'(t))' = \lambda f\Big(t, u(g_1(t)), \int_0^t k(t,s,u(g_2(s)))\, \mathrm{d}s, p(t)u'(t)\Big),$$
$$0 \le t \le T, \quad 0 \le \lambda \le 1, \tag{1.1$_\lambda$}$$

$$u(t) = \varphi(t), \quad t \in [-r,0], \qquad u'(0) = A, \tag{1.2}$$

for each $\lambda \in (0,1)$. Then (1.1)–(1.2) has at least one solution $u \in K^1[-r,T]$ with

$$(pu')' \in C[0,T].$$

Proof. Solving $(1.1)_\lambda$–(1.2) is equivalent to finding a $u \in K^1[-r,T]$ such that satisfies

$$u(t) = \varphi(0) + \int_0^t \frac{1}{p(s)} \Big(Ap(0)$$
$$+ \lambda \int_0^s f\Big(r, u(g_1(r)), \int_0^r k(r,x,u(g_2(x)))\, \mathrm{d}x, p(r)u'(r)\Big)\, \mathrm{d}r\Big)\, \mathrm{d}s, \tag{3.1}$$

where

$$u(g_i(s)) = \begin{cases} u(g_i(s)), & g_i(s) \geq 0, \\ \varphi(g_i(s)), & g_i(s) \leq 0. \end{cases}$$

Define the operator $N_\lambda : C^1_B[0,1] \to C^1_B[0,1]$ by

$$(N_\lambda u)(t) = \varphi(0) + \int_0^t \frac{1}{p(s)} \Big(Ap(0) + \lambda \int_0^s f\Big(r, u(g_1(r)), \int_0^r k(r,x,u(g_2(x)))\,\mathrm{d}x, p(r)u'(r)\Big)\,\mathrm{d}r\Big)\,\mathrm{d}s.$$

Here $C^1_B[0,T] = \{u \in C[0,T] \cap C^1[0,T];\ u(0) = \varphi(0),\ u'(0) = A\}$. Of course, $(1.1)_\lambda$–(1.2) is equivalent to the fixed point problem $u = N_\lambda u$. Certainly, N_λ is continuous since f and k are completely continuous by the Arzela-Ascoli theorem. To see this, let $\Omega \subseteq C^1_B[0,T]$ be bounded, i.e., $|u|_1 \leq k^*$ for all $u \in \Omega$, where $k^* > 0$ is a constant. Let $K = \max\{\|\varphi\|_{[-r,0]}, k^*\}$. First $N_\lambda \Omega$ is uniformly bounded. This follows from the inequalities

$$|N_\lambda u(t)| \leq |\varphi(0)| + |A|p(0) \int_0^T \frac{\mathrm{d}s}{p(s)} + M_1 \int_0^T \frac{s\,\mathrm{d}s}{p(s)}$$

and

$$|p(t)(N_\lambda u)'(t)| \leq |A|p(0) + M_1 T,$$

where $M_1 = \sup |f(t,u_1,u_2,u_3)|$, where the supremum is computed over $[0,T] \times [-K,K] \times [K_1,K_1] \times [-K,K]$. Here $K_1 = \sup |k(t,s,w)|T$, where the supremum is computed over $[0,T] \times [0,T] \times [-K,K]$.

We next show the equicontinuity of $N_\lambda \Omega$ on $[0,T]$. For $u \in \Omega$ and $t_1, t_2 \in [0,T]$ we have

$$\begin{aligned} &|N_\lambda u(t_1) - N_\lambda u(t_2)| \\ &= \Big| \int_{t_2}^{t} \Big(\frac{1}{p(s)} Ap(0) + \lambda \int_0^s f\Big(r, u(g_1(r)), \int_0^r k(r,x,u(g_2(x)))\,\mathrm{d}x, p(r)u'(r)\Big)\,\mathrm{d}r\Big)\,\mathrm{d}s \Big|. \\ &\leq |A|\,p(0) \Big| \int_{t_2}^{t_1} \frac{\mathrm{d}s}{p(s)} \Big| + M_1 \Big| \int_{t_2}^{t_1} \frac{s\,\mathrm{d}s}{p(s)} \Big| \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&|p(t_1)(N_\lambda u)'(t_1) - p(t_2)(N_\lambda u)'(t_2)| \\
&= \left| \lambda \int_{t_2}^{t_1} f\Big(s, u(g_1(s)), \int_0^s k(s,r,u(g_2(r)))\, \mathrm{d}r, p(s)u'(s)\Big)\, \mathrm{d}s \right| \qquad (3.3)\\
&\le M_1 |t_1 - t_2| .
\end{aligned}$$

The equicontinuity of $N_\lambda \Omega$ on $[0,T]$ now follows from (3.2) and (3.3). Thus the Arzela-Ascoli theorem implies that N_λ is completely continuous. Let

$$M^* = \max\{M,\ \|\varphi\|_{[-r,0]}\},$$

and set

$$\begin{aligned}
U &= \{u \in C_B^1[0,T] :\ |u|_1 < M^* + 1\},\\
Q &= C_B^1[0,T],\\
E &= C[0,T] \cap C^1[0,T],\\
N_0 u(t) &= \varphi(0) + Ap(0) \int_0^t \frac{\mathrm{d}s}{p(s)} .
\end{aligned}$$

Note that $N(\bar{U} \times [0,1])$ is contained in a compact subset of Q. To see this, let $N(u_n, \lambda_n)$ be any sequence in $N(\bar{U} \times [0,1])$. Then, as above, $N(u_n, \lambda_n)$ is uniformly bounded and equicontinuous on $[0,T]$, so the Arzela-Ascoli theorem again yields the same results.

Apply Lemma 2.1 to deduce that N_1 has a fixed point, i.e., (1.1) – (1.2) has a solution $u \in K^1[-r,T]$. The fact $(p(t)u'(t))' \in C[0,T]$ follows from (3.1) with $\lambda = 1$.

The proof of Theorem 3.1 is complete. □

4. A priori bounds

In this section, we deal with a priori bounds for IVP to second order Volterra integrodifferential equations with deviating arguments. We shall use the Gronwall-Bellman-Bihari-type integral inequalities to derive a priori bounds to fulfil the conditions imposed in Theorem 3.1.

To simplify the notation, in the sequel, we will define

$$P(t) = \int_0^t \frac{1 + \int_0^s h_2(r) \exp\left(\int_r^s h_2(x)\, \mathrm{d}x \right) \mathrm{d}r}{p(s)}\, \mathrm{d}s\,,$$

$$W(t) = \|\varphi\|_{[-r,0]} + p(0)\, |A|\, P(t)\,.$$

THEOREM 4.1. *Let $f\colon [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k\colon [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions, $p \in C[0,T]$, $p(t) > 0$, and assume that $g_1, g_2 \in G$. Suppose that for every $t, s \in [0,T]$ and $u, v, w \in \mathbb{R}$*

$$\begin{gathered} |k(t,s,u)| \le h_1(s)|u| \qquad \text{and} \\ |f(t,u,v,w)| \le h(t)\big(|u| + |v| + h_2(t)|w|\big)\,, \end{gathered} \tag{H}$$

where h, h_1 and h_2 are continuous nonnegative real-valued functions on $[0,T]$.

Then the IVP (1.1) - (1.2) has at least one solution $u \in K^1[-r,T]$ with

$$(pu')' \in C[0,T]\,.$$

Proof. By Theorem 3.1, the IVP (1.1) - (1.2) has at least one solution in $K^1[-r,T]$ if there is a priori bound for solutions to $(1.1)_\lambda$ - (1.2).

Let $u(t)$ be a solution of $(1.1)_\lambda$ - (1.2). Then we have

$$p(t)u'(t) = p(0)A + \lambda \int_0^t f\Big(s, u(g_1(s)), \int_0^s k\big(s,x,u(g_2(x))\big)\, \mathrm{d}x, p(s)u'(s)\Big)\, \mathrm{d}s\,. \tag{4.1}$$

Applying (H), we obtain from (4.1)

$$\begin{aligned} |p(t)u'(t)| \le p(0)\,|A| &+ \int_0^t h(s)\Big(\big|u(g_1(s))\big| + \int_0^s h_1(x)\big|u(g_2(x))\big|\, \mathrm{d}x\Big)\, \mathrm{d}s \\ &+ \int_0^t h_2(s)|p(s)u'(s)|\, \mathrm{d}s\,. \end{aligned} \tag{4.2}$$

Applying Theorem 1 in [5] to the above inequality (4.2) we have

$$\begin{aligned} |p(t)u'(t)| \le &\left(1 + \int_0^t h_2(s) \exp\left(\int_s^t h_2(x)\, \mathrm{d}x \right) \mathrm{d}s \right) \cdot \\ &\cdot \left(p(0)|A| + \int_0^t h(s)\Big(\big|u(g_1(s))\big| + \int_0^s h_1(x)\big|u(g_2(x))\big|\, \mathrm{d}x\Big)\, \mathrm{d}s \right). \end{aligned} \tag{4.3}$$

Multiplying inequality (4.3) by $1/p(t)$ and integrating on $[0,t]$, we obtain

$$|u(t)| \le \|\varphi\|_{[-r,0]} + p(0)|A|P(t) + \int_0^t \big(P(t)-P(s)\big)h(s)\Big(\big|u(g_1(s))\big| + \int_0^s h_1(x)\big|u(g_2(x))\big|\,\mathrm{d}x\Big)\,\mathrm{d}s\,. \tag{4.4}$$

Using a similar method to that of the proof of Theorem 1 in [1], we get

$$\begin{aligned}|u(t)| &\le W(t)\exp\left(\int_0^t \frac{\big(P(t)-P(s)\big)h(s)W\big(g_1(s)\big)+h_1(s)W\big(g_2(s)\big)}{W\big(g_1(s)\big)}\,\mathrm{d}s\right)\\ &\le W(T)\exp\left(\int_0^T \frac{\big(P(t)-P(s)\big)h(s)W\big(g_1(s)\big)+h_1(s)W\big(g_2(s)\big)}{W\big(g_1(s)\big)}\,\mathrm{d}s\right)\\ &= N\,.\end{aligned} \tag{4.5}$$

From (4.3) and (4.5), we have

$$\begin{aligned}|p(t)u'(t)| &\le \left(1+\int_0^T h_2(s)\exp\Big(\int_s^T h_2(x)\,\mathrm{d}x\Big)\,\mathrm{d}s\right)\cdot\\ &\quad\cdot\left(p(0)|A| + N\int_0^T h(s)\Big(1+\int_0^s h_1(x)\,\mathrm{d}x\Big)\,\mathrm{d}s\right)\\ &= N_1\,.\end{aligned} \tag{4.6}$$

So we obtain

$$|u|_1 \le M = \max\{N, N_1\}\,.$$

Therefore, by Theorem 3.1, IVP (1.1)–(1.2) has at least one solution $u \in K^1[-r,T]$ and $\big(p(t)u'(t)\big)' \in C[0,T]$. □

THEOREM 4.2. *Let $f\colon [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $k\colon [0,T]\times[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous functions, $p\in C[0,T]$, $p(t)>0$, and assume that $g_1, g_2 \in G$. Suppose that*

(i) *f is as in assumption* (H) *of Theorem* 4.1 *and $q\in G_1$.*
(ii) *There exist a continuous nonnegative functions h^* on $[0,T]$ and n such that $n\colon [0,\infty)\to[0,\infty)$ is nondecreasing nonnegative submultiplicative for $u>0$ with $u(0)=0$ and*

$$\left|\int_0^t k\big(t,s,u(g_2(s))\big)\,\mathrm{d}s\right| \le h^*(t)n\big(\big|u(g_2(t))\big|\big)\,.$$

Then the IVP (1.1) – (1.2) *has at least one solution* $u \in K^1[-r,T]$ *with*

$$(pu')' \in C[0,T].$$

Proof. Let $u(t)$ be a solution of $(1.1)_\lambda$ – (1.2). Then we have

$$|p(t)u'(t)| = \left(1 + \int_0^t h_2(s) \exp\left(\int_s^t h_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right) \cdot \\ \cdot \left(p(0)|A| + \int_0^t h(s)q\big(|u(g_1(s))|\big)\,\mathrm{d}s + \int_0^t h_1(s)n\big(|u(g_2(s))|\big)\,\mathrm{d}s\right). \tag{4.7}$$

Multiplying inequality (4.7) by $1/p(t)$ and integrating on $[0,t]$, we obtain

$$\begin{aligned} |u(t)| &\le \|\varphi\|_{[-r,0]} + p(0)|A|P(t) \\ &\qquad + \int_0^t \big(P(t) - P(s)\big)h(s)q\big(|u(g_1(s))|\,\mathrm{d}s \\ &\qquad + \int_0^t \big(P(t) - P(s)\big)h_1(s)n\big(|u(g_2(s))|\big)\,\mathrm{d}s \\ &\le \max\big\{\|\varphi\|_{[-r,0]} + p(0)|A|P(T),\, 1\big\} \\ &\qquad + \int_0^t \big(P(T) - P(s)\big)h(s)q\big(|u(g_1(s))|\big)\,\mathrm{d}s \\ &\qquad + \int_0^t \big(P(T) - P(s)\big)h_1(s)n\big(|u(g_2(s))|\big)\,\mathrm{d}s. \end{aligned} \tag{4.8}$$

Applying Theorem 6 in [1], we get

$$|u(t)| \le Q^{-1}\left(Q(1) + \int_0^t \big(P(T) - P(s)\big)h(s)\,\mathrm{d}s\right) \cdot \\ \cdot N^{-1}\Bigg[N\big(\max\{\|\varphi\|_{[-r,0]} + p(0)|A|P(T),\, 1\}\big) \\ + \int_0^t \big(P(T) - P(s)\big)h_1(s)n\left(Q^{-1}\left(Q(1) + \int_0^t h(r)\big(P(T) - P(r)\big)\,\mathrm{d}r\right)\right)\mathrm{d}s\Bigg], \tag{4.9}$$

where

$$Q(v) = \int_{v_0}^{v} \frac{\mathrm{d}s}{q(s)}, \qquad v \geq v_0 \geq 0,$$

$$N(r) = \int_{r_0}^{r} \frac{\mathrm{d}s}{n(s)}, \qquad r \geq r_0 \geq 0,$$

and Q^{-1} and N^{-1} are the inverse of Q and N respectively. Clearly, Q^{-1} and N^{-1} are increasing functions. Using the hypotheses (i) and (ii), we see clearly that the right side of inequality (4.9) is bounded, which proves that there is a constant $M_1 > 0$ such that

$$|u(t)| \leq M_1 . \tag{4.10}$$

From (4.7) and (4.10), we know that there is a constant $M_2 > 0$ such that

$$|p(t)u'(t)| \leq M_2 .$$

So we obtain

$$|u|_1 \leq M = \max\{M_1, M_2\} .$$

Therefore, by Theorem 3.1, IVP (1.1)–(1.2) has at least one solution $u \in K^1[-r,T]$ and $\big(p(t)u'(t)\big)' \in C[0,T]$. □

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REFERENCES

[1] AKINYELE, O.: *On Gronwall-Bellman-Bihari-type integral inequalities in several variables with retardation*, J. Math. Anal. Appl. **104** (1984), 1–26.

[2] DUGUNDJI, J.—GRANAS, A.: *Fixed Point Theory, Vol. I.* Monogr. Mat., PNW, Warszawa, 1982.

[3] LEE, J. W.—O'REGAN, D.: *Existence results for differential delay equations I*, J. Differential Equations **102** (1993), 342–359.

[4] O'REGAN, D.: *Existence of solutions to some differential delay equations*, Nonlinear Anal. **20** (1993), 79–96.

[5] PACHPATTE, B. G.: *On some generalization of Bellman's lemma*, J. Math. Anal. Appl. **51** (1975), 141–150.

[6] XIA, HUAXING—SPANILY, T.: *Global existence problem for singular functional differential equations*, Nonlinear Anal. **20** (1993), 921–934.

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* *Department of Mathematics*
Wuyi University
Jiangmen, Guangdong, 529020
P. R. CHINA
E-mail: lilongtu@yahoo.com

** *Department of Applied Mathematics*
Hunan University
Changsha, Hunan 410082
P. R. ChINA
E-mail: jsyu@hunu.edu.cn