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REMARKS ON THE (R)-DENSITY OF SETS OF NUMBERS, II

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ABSTRACT. This paper is an extension of the previous paper [BUKOR, J.—ŠALÁT, T.—TÓTH, J. T.: *Remarks on* (R)*-density of sets of numbers*, Tatra Mt. Math. Publ **11** (1997), 159–165] and contains decomposition of the set $\mathbb{N}$ having non dense ratio sets of sum of any two components.

Introduction

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and the set of all positive real numbers respectively. For $A \subseteq \mathbb{R}^+$ we put $R(A) = \left\{\frac{a}{b} : a, b \in A\right\}$. The set $R(A)$ is said to be the ratio set of A. The set A is said to be (R)-dense provided that $R(A)$ is dense in $\mathbb{R}^+$.

These concepts were introduced in [3], [7] and [8] for subsets of $\mathbb{N}$. We extend their meaning to sets $A \subseteq \mathbb{R}^+$.

This paper consists of four parts. In the first part, a certain class of decompositions of $\mathbb{N}$ related to the concept of dense ratio sets will be investigated. In the second part, we shall give some results on ratio sets constructed by prime numbers. In the third part, we shall introduce an iteration of $R(A)$. The paper concludes with open problems.

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We mention with grief the death of Prof. P. Erdős with whom we discussed the paper some weeks before his death.

1. A class of decompositions of $\mathbb{N}$

In this part of the paper, we shall study a class of decompositions of the set $\mathbb{N}$ depending on density of ratio sets of subsume of components.

Firstly we generalized, for $A \subseteq \mathbb{R}^+$, the result [2; Theorem 2] that if $A \subseteq \mathbb{N}$, then at least one of the sets A, $\mathbb{N} \setminus A$ is an (R)-dense set.

THEOREM 1.1. *Suppose that $X = \{x_1 < x_2 < \dots < x_n < \dots\} \subseteq \mathbb{R}^+$ is an unbounded set with*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1\,. \tag{1}$$

If the set $A \subseteq X$ is not (R)-dense, then $X \setminus A$ is an (R)-dense set.

Proof. We proceed indirectly. Suppose that the sets $A \subseteq X$, $B = X \setminus A$ are not (R)-dense. Then there exist the numbers $1 < a < b$ and $\varepsilon > 0$ such that

$$(a-\varepsilon, a+\varepsilon) \cap R(A) = \emptyset = (b-\varepsilon, b+\varepsilon) \cap R(B)\,. \tag{2}$$

By (1), there is an $n_0 \in \mathbb{N}$ such that for each k with $x_k \geq \frac{x_{n_0}}{ab}$ we have

$$\frac{x_{k+1}}{x_k} < 1 + \frac{\varepsilon}{b}\,. \tag{3}$$

Let $n \geq n_0$, $x_n \in A$, $x_{n+1} \in B$. Put $s = \frac{x_n}{ab}$. Let i be the greatest j with $x_j \leq s$. Choosing $l = i+1$ we see from (3) that

$$1 < \frac{x_l}{s} < 1 + \frac{\varepsilon}{b}\,. \tag{4}$$

The following possibilities can occur:

a) $x_l \in A$.
b) $x_l \in B$.

a) It follows from Propositions 1 and 2 in [3] that the set $A = \{a_1 < a_2 < \dots\} \subseteq \mathbb{N}$ is (R)-dense if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Further, in [2], it is proved that the set $Y = \{1 \leq y_1 < y_2 < \dots\} \subseteq \mathbb{R}^+$ with $\lim\limits_{n\to\infty} y_n = \infty$ is (R)-dense if and only if the set $\{[y_1] \leq [y_2] \leq \dots\} \subseteq \mathbb{N}$ is (R)-dense. From these results, it can be easily deduced that the set X is (R)-dense. Therefore there is a $t \in \mathbb{N}$ such that

$$1 < \frac{x_t}{as} < 1 + \frac{\varepsilon}{b}\,. \tag{5.1}$$

Using (5.1), (4) we get

$$\left|\frac{x_t}{x_l} - a\right| = a \left|\frac{\frac{x_t}{as} - \frac{x_l}{s}}{\left(\frac{x_l}{s}\right)}\right| < a\frac{\varepsilon}{b} < \varepsilon\,.$$

From this because of (2) and Condition a), we see that $x_t \in B$. So we have $x_t \in B$ and also $x_{n+1} \in B$.

We shall estimate the number $\left|\frac{x_{n+1}}{x_t} - b\right|$. Using (3), (5.1) and the equality $b = \frac{x_n}{as}$ we obtain

$$\left|\frac{x_{n+1}}{x_t} - b\right| = b\left|\frac{x_{n+1}}{x_n}\frac{sa}{x_t} - 1\right| < b\max\left\{\left(1+\frac{\varepsilon}{b}\right) - 1, 1 - \frac{1}{1+\frac{\varepsilon}{b}}\right\} = b\frac{\varepsilon}{b} = \varepsilon .$$

Thus $\frac{x_{n+1}}{x_t} \in R(B) \cap (b-\varepsilon, b+\varepsilon)$, which contradicts (2).

b) Choose a $t \in \mathbb{N}$ such that

$$1 < \frac{x_t}{bs} < 1 + \frac{\varepsilon}{b} . \tag{5.2}$$

Then, by (4) and (5.2), we get

$$\left|\frac{x_t}{x_l} - b\right| = b\,\frac{\left|\frac{x_t}{sb} - \frac{x_l}{s}\right|}{\left(\frac{x_l}{s}\right)} < b\frac{\varepsilon}{b} = \varepsilon .$$

From this and (2), we see that $x_t \in A$. Then, by a simple calculation, we get from (5.2) (using $a = \frac{x_n}{bs}$):

$$\left|\frac{x_n}{x_t} - a\right| = \left|\frac{x_n}{bs}\frac{bs}{x_t} - a\right| = a\left|\frac{bs}{x_t} - 1\right| = a\left(1 - \frac{bs}{x_t}\right) < a\left(1 - \frac{b+\varepsilon}{b}\right) = \varepsilon\frac{a}{b+\varepsilon} < \varepsilon ,$$

which contradicts (2). The theorem follows. □

Remark 1.1. The condition (1) in the previous theorem cannot be omitted. This is shown in the following example.

EXAMPLE 1.1. Suppose that the sequence $(s_n)_{n=1}^{\infty}$ of rational numbers greater than 1 contains every rational number greater than 1 in infinitely many places. Define $X = \{x_1, x_2, \dots\}$ as follows:

$$x_1 = 2 ,$$
$$x_n = \begin{cases} \left[x_{n-1}s_{\frac{n}{2}}\right] & \text{for } n \text{ even,} \\ 2^{x_{n-1}} & \text{for } n \text{ odd,} \end{cases} \qquad \text{for} \quad n \geq 2 .$$

Obviously the sequence $\left(\frac{x_{n+1}}{x_n}\right)_{n=1}^{\infty}$ does not satisfy (1). We now show that the set X is an (R)-dense set. It suffices to prove that each rational number greater than 1 is an accumulation point of the set $R(X)$.

Let $s > 1$, $s \in \mathbb{Q}$. Then there exists a sequence $n_1 < n_2 < \cdots$ of positive integers such that $s_{n_k} = s$ $(k = 1, 2, \dots)$ (see the definition of $(s_j)_{j=1}^{\infty}$), and

$$\frac{x_{2n_k}}{x_{2n_k-1}} = \frac{[x_{2n_k-1} \cdot s_{n_k}]}{x_{2n_k-1}} \to s \qquad (k \to \infty).$$

Thus s is an accumulation point of $R(X)$. Put $A = \{x_1 < x_3 < \cdots < x_{2n-1} < \dots\}$, $B = \{x_2 < x_4 < \cdots < x_{2n} < \dots\}$. Then $X = A \cup B$ is a decomposition of X, and neither of the sets A, B is (R)-dense, moreover the only accumulation points of $R(A)$, $R(B)$ are 0, $+\infty$.

In the following, we shall return to $A \subseteq \mathbb{N}$ without loss of generality. Theorem 1.1 suggest the following definition.

DEFINITION 1.1. Let $k \in \mathbb{N}$. Denote by $f(k)$ the least positive integer such that the set $\mathbb{N}$ can be decomposed into $f(k)$ (pair-wise disjoint) sets $A_1, \dots, A_{f(k)}$ in such a way that the union of k arbitrarily chosen sets from $A_1, \dots, A_{f(k)}$ is not an (R)-dense set.

Example 1 in [2] and Theorem 1.1 show that $f(1) = 3$. From Theorem 1.1, we conclude that $f(2) \geq 5$. The following example shows that $f(2) \leq 7$.

EXAMPLE 1.2. Put

$$A_k^i = \{[2^k \cdot 2^{\frac{i-1}{7}}]+1,\ [2^k \cdot 2^{\frac{i-1}{7}}]+2, \dots, [2^k \cdot 2^{\frac{i}{7}}]\} \quad (i = 1, 2, \dots, 7,\ k = 1, 2, \dots).$$

Further, we set $A_1 = \{1, 2, \dots, 8\} \cup \bigcup\limits_{k=3}^{\infty} A_k^1$ and $A_i = \bigcup\limits_{k=3}^{\infty} A_k^i$ for $i = 2, 3, \dots, 7$.

Obviously, $\mathbb{N} = \bigcup\limits_{i=1}^{7} A_i$ is a decomposition of $\mathbb{N}$.

It can be checked that

$$R(A_i \cup A_{i+1}) \cap (2^{\frac{2}{7}}, 2^{\frac{5}{7}}) = \emptyset \qquad \text{for} \quad i = 1, 2, \dots, 6,$$

$$\left.\begin{aligned} R(A_1 \cup A_7) \cap (2^{\frac{2}{7}}, 2^{\frac{5}{7}}) &= \emptyset \\ R(A_i \cup A_{i+2}) \cap (2^{\frac{3}{7}}, 2^{\frac{4}{7}}) &= \emptyset \end{aligned}\right\} \qquad \text{for} \quad i = 1, 2, \dots, 5.$$

Further we have

$$R(A_i \cup A_{i+3}) \cap (2^{\frac{1}{7}}, 2^{\frac{2}{7}}) = \emptyset \qquad \text{for} \quad i = 1, 2, 3, 4,$$

$$R(A_i \cup A_{i+4}) \cap (2^{\frac{1}{7}}, 2^{\frac{2}{7}}) = \emptyset \qquad \text{for} \quad i = 1, 2, 3.$$

The last equalities can be verified by similar methods. We shall illustrate this method by proving the equality

$$R(A_1 \cup A_4) \cap (2^{\frac{1}{7}}, 2^{\frac{2}{7}}) = \emptyset.$$

Let $a, a' \in A_1 \cup A_4$ and $a > a'$. The following two cases can occur.

1) $a, a' \in A_k^1 \cup A_k^4$ for a suitable k,
2) $a \in A_l^1 \cup A_l^4$, $a' \in A_k^1 \cup A_k^4$, where $l > k$.

1) If $a, a' \in A_k^1$ or $a, a' \in A_k^4$, then $\frac{a}{a'} \le 2^{\frac{1}{7}}$. Further, if $a \in A_k^4$, $a' \in A_k^1$, then

$$\frac{a}{a'} \ge \frac{2^k \cdot 2^{\frac{3}{7}}}{2^k \cdot 2^{\frac{1}{7}}} = 2^{\frac{2}{7}}.$$

2) We have

$$\frac{a}{a'} \ge \frac{2^{k+1}}{2^k \cdot 2^{\frac{4}{7}}} = 2^{\frac{3}{7}}.$$

Finally we can check that

$$R(A_i \cup A_{i+5}) \cap (2^{\frac{3}{7}}, 2^{\frac{4}{7}}) = \emptyset \qquad \text{for} \quad i = 1, 2.$$

We prove this for $i = 1$. Let $a > a'$, $a, a' \in A_1 \cup A_6$. Then the following two cases can occur.

1) If $a, a' \in A_k^1$ or $a, a' \in A_k^6$, then $\frac{a}{a'} \le 2^{\frac{1}{7}}$.
   If $a \in A_k^6$, $a' \in A_k^1$, then $2^{\frac{4}{7}} \le \frac{a}{a'} \le 2^{\frac{6}{7}}$.
2) If $a \in A_{k+1}^1$, $a' \in A_k^6$, then $2^{\frac{1}{7}} \le \frac{a}{a'} \le 2^{\frac{3}{7}}$.
   If $a \in A_{k+1}^1$, $a' \in A_k^1$, then

$$\frac{a}{a'} \ge \frac{2^{k+1}}{2^k . 2^{\frac{1}{7}}} = 2^{\frac{6}{7}}.$$

   If $a \in A_{k+1}^6$, $a' \in A_k^6$, then

$$\frac{a}{a'} \ge \frac{2^{k+1} \cdot 2^{\frac{5}{7}}}{2^k \cdot 2^{\frac{6}{7}}} = 2^{\frac{6}{7}}.$$

2. Ratio sets involving prime numbers

In this part of the paper, $P = \{p_1 < p_2 < \dots < p_n < \dots\}$ stands for the set of all primes. Put

$$A = A(\alpha_1, \alpha_2, \dots) = \{p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_n^{\alpha_n}, \dots\},$$

where $(\alpha_n)_1^\infty$ is a sequence of positive real numbers.

In [9], the following problem is formulated.

Characterize sequences $(\alpha_n)_1^\infty$ of positive reals for which the set $A(\alpha_1, \alpha_2, \dots)$ is (R)-dense.

Note that if $\alpha_n = 1$ $(n = 1, 2, \dots)$, then $A(1, 1, \dots)$ is (R)-dense according to a well-known result of A. Schinzel (cf. [6; p. 155]).

A partial answer to this problem is given in the following theorem.

THEOREM 2.1. *Suppose that the sequence $(\alpha_n)_1^\infty$ of positive reals satisfies the following conditions.*

(i) $\alpha_n = O(n^{3/8})$,

(ii) $\alpha_{n+1} - \alpha_n = O(n^{-\varepsilon})$ *for some* $\varepsilon > 0$.

Then the set $A = A(\alpha_1, \alpha_2, \dots)$ *is* (R)*-dense.*

Proof. By Theorem 1.1, it suffices to prove that

$$\lim_{n\to\infty} \frac{p_{n+1}^{\alpha_{n+1}}}{p_n^{\alpha_n}} = 1 .$$

By a simple arrangement, we get

$$\frac{p_{n+1}^{\alpha_{n+1}}}{p_n^{\alpha_n}} = p_{n+1}^{\alpha_{n+1}-\alpha_n} \left(\frac{p_{n+1}}{p_n}\right)^{\alpha_n} . \tag{6}$$

According to (ii), there exist a $c > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$-\frac{c}{n^\varepsilon} \leq \alpha_{n+1} - \alpha_n \leq \frac{c}{n^\varepsilon}$$

for some $\varepsilon > 0$. From this, we get

$$\mathrm{e}^{-c\frac{\log p_{n+1}}{n^\varepsilon}} \leq p_{n+1}^{\alpha_{n+1}-\alpha_n} \leq \mathrm{e}^{c\frac{\log p_{n+1}}{n^\varepsilon}} .$$

Since $p_n \sim n \log n$ $(n \to \infty)$, we see that

$$\lim_{n\to\infty} \frac{\log p_{n+1}}{n^\varepsilon} = 0 ,$$

and so

$$\lim_{n\to\infty} p_{n+1}^{\alpha_{n+1}-\alpha_n} = 1 . \tag{7}$$

It still suffices to show that (see (6)):

$$\lim_{n\to\infty} \left(\frac{p_{n+1}}{p_n}\right)^{\alpha_n} = 1 . \tag{7'}$$

According to a lemma of Ingham, for all sufficiently large x there is a prime between x and $x + x^{5/8}$ (cf. [4]). By this lemma, we have

$$1 \leq \left(\frac{p_{n+1}}{p_n}\right)^{\alpha_n} \leq \left(\frac{p_n + p_n^{5/8}}{p_n}\right)^{\alpha_n} \leq (1 + p_n^{-3/8})^{\alpha_n} . \tag{8}$$

According to (i), there exist $n_1 \in \mathbb{N}$ and $c_1 > 0$ such that

$$\alpha_n \leq c_1 n^{3/8} \qquad \text{for} \quad n > n_1 .$$

But then

$$(1+p_n^{-3/8})^{\alpha_n} \le \left((1+p_n^{-3/8})^{p_n^{3/8}}\right)^{c_1(\frac{n}{p_n})^{3/8}}. \tag{9}$$

Since $\lim\limits_{n\to\infty} \dfrac{n}{p_n} = 0$, from (8) and (9), we get the inequality (7'). The theorem now follows from (6), (7) and (7'). □

Remark 2.1. Condition (ii) in the previous theorem cannot be replaced by the following weaker condition:

$$\alpha_{n+1} = \alpha_n + 1/\log n \qquad (n > 1).$$

In this case, we in fact have

$$p_{n+1}^{\alpha_{n+1}-\alpha_n} = p_{n+1}^{\frac{1}{\log n}} = \mathrm{e}^{\frac{\log p_{n+1}}{\log n}} \to \mathrm{e} \qquad \text{if} \quad n\to\infty\,,$$

thus

$$\lim_{n\to\infty} \frac{p_{n+1}^{\alpha_{n+1}}}{p_n^{\alpha_n}} = \mathrm{e}\,,$$

and so, by [9; Proposition 3], the set $A(\alpha_1,\alpha_2,\dots)$ cannot be (R)-dense.

Put $\alpha_n = n^\beta$, $n=1,2,\dots$, where $\beta \ge 0$. If $\beta = 0$, then $A = A(\alpha_1,\alpha_2,\dots) = P$, and so A is an (R)-dense set. For $\beta = 1$ the set $A(\alpha_1,\alpha_2,\dots) = A(1,2,3,\dots)$ is not (R)-dense (cf. [9; Proposition 3]). Hence the following question appears to be of interest:

Does the equality

$$\sup\{\beta\ge 0:\ A(1^\beta,2^\beta,\dots,n^\beta,\dots) \text{ is (R)-dense}\} = 1$$

hold?

Remark 2.2. Put $\alpha_n = n^{3/8}$ $(n=1,2,\dots)$. Then, by the previous theorem,

$$S = \sup\{\beta\ge 0:\ A(1^\beta,2^\beta,\dots,n^\beta,\dots) \text{ is (R)-dense}\} \ge \frac{3}{8}\,.$$

Using a result of R. C. Baker and G. Harman [1] instead of the afore mentioned lemma of Ingham, we see that Condition (i) in Theorem 2.1 can be replaced by $\alpha_n = O(n^{0{,}465})$, which implies $S \ge 0{,}465$.

In [4; p. 364], it is observed that using Lindelöf's conjecture it can be shown that for all sufficiently large x there is a prime between x and $x + x^{1/2+\varepsilon}$ (for every $\varepsilon > 0$). Hence, if we assume the Lindelöf conjecture, we can show that S can be arbitrarily closely to $1/2$.

On the other hand, assuming Schinzel's conjecture (cf. [5]) according to which in each interval $\bigl(x, x+(\log x)^2\bigr)$ there is a prime (for $x > x_0$), we obtain $S = 1$.

3. Iteration of $R(A)$

Let $A \subseteq \mathbb{R}^+$. Define recurrently

$$R^0(A) = A\,, \qquad R^1(A) = R(A)\,, \qquad R^{n+1}(A) = R\big(R^n(A)\big) \quad \text{for} \quad n \geq 0\,.$$

It is obvious that from the density of $R^n(A)$ the density of $R^{n+1}(A)$ follows.

In connection with this, the question arises of whether for each $n \geq 0$ there exists a set $A = \{a_1 < a_2 < \dots\} \subseteq \mathbb{N}$ such that $R^n(A)$ is not dense in $\mathbb{R}^+$, but $R^{n+1}(A)$ is already dense in $\mathbb{R}^+$.

Denote by $\mathcal{A}_n$ $(n \geq 0)$ the class of all sets $A \subset \mathbb{N}$ having the previous property (i.e., $R^n(A)$ is not dense in $\mathbb{R}^+$, but $R^{n+1}(A)$ is dense in $\mathbb{R}^+$). Note that $\mathcal{A}_0 \neq \emptyset$ (e.g., the set $\mathbb{N}$ belongs to $\mathcal{A}_0$), and this property has every class $\mathcal{A}_n$.

THEOREM 3.1. *For each $n \geq 0$ we have $\mathcal{A}_n \neq \emptyset$.*

Proof. It suffices to prove that for each $n \in \mathbb{N}$ there is a set $A \subseteq \mathbb{N}$ such that $R^n(A)$ is not a dense set in $\mathbb{R}^+$, but $R^{n+1}(A)$ is a dense set in $\mathbb{R}^+$. Define by fixed $n \in \mathbb{N}$ the set

$$A = \{a_1 < a_2 < \dots\} \subset \mathbb{N}$$

in this way

$$a_1 = 1 \quad \text{and} \quad a_{k+1} = a_k^{2^{n-1}}(k+1)\,, \quad k = 1, 2, \dots\,. \tag{10}$$

We show that

$$R^n(A) \cap (1,2) = \emptyset\,. \tag{11}$$

Let $r \in R^n(A)$, $r > 1$. Then

$$r = \frac{c_1 c_2 \dots c_{2^{n-1}}}{b_1 b_2 \dots b_{2^{n-1}}}\,, \qquad \text{where} \quad c_i, b_i \in A \ \ (i = 1, 2, \dots, 2^{n-1})\,,$$

and

$$c_1 \leq c_2 \leq \dots \leq c_{2^{n-1}}\,, \qquad b_1 \leq b_2 \leq \dots \leq b_{2^{n-1}}\,. \tag{12}$$

Denote by l the greatest positive integer with $c_l \neq b_l$. l exists since $r > 1$. We have the following possibilities:

(1) $c_l > b_l$,
(2) $c_l < b_l$.

(1) In this case, $c_l = a_{k+1}$ $(k \in \mathbb{N})$, and it is easy to verify that $a_k \geq b_l$. Replacing each c_j $(j < l)$ by 1 and each b_j $(j \leq l)$ by a_k, then, from (10), (12), we get the estimate

$$r \geqq \frac{a_{k+1}}{a_k^{2^{n-1}}} = k + 1 \geqq 2\,.$$

(2) In this case, $b_l = a_{j+1}$, $j \in \mathbb{N}$, and $a_j \geqq c_l$. From (10), (12), we have

$$r \leqq \frac{a_j^{2^{n-1}}}{a_{j+1}} = \frac{1}{j+1} \leqq \frac{1}{2}.$$

Thus (11) holds.

We show that $R^{n+1}(A)$ is dense in $\mathbb{R}^+$. For this it suffices to prove that

$$R^n(A) \supset \{2, 3, \dots, n, \dots\}.$$

For each $k \in \mathbb{N}$ we have $\{1, a_k, a_{k+1}\} \subset A$. Then $\left\{a_k, \frac{1}{a_k}, \frac{a_{k+1}}{a_k}\right\} \subset R(A)$, further $\left\{a_k^2, \frac{1}{a_k^2}, \frac{a_{k+1}}{a_k^2}\right\} \subset R^2(A)$.

So we get

$$\left\{a_k^{2^{n-2}}, \frac{1}{a_k^{2^{n-2}}}, \frac{a_{k+1}}{a_k^{2^{n-2}}}\right\} \subset R^{n-1}(A).$$

Thus

$$k+1 = \frac{a_{k+1}}{a_k^{2^{n-1}}} \in R^n(A) \qquad \text{for each} \quad k \in \mathbb{N}.$$

□

4. Open problems

To conclude the paper, let us describe some open problems associated with this topic.

1. Determine the exact value of $f(2)$ (for the definition of the function $f(n)$ see above). We know that $5 \leq f(2) \leq 7$. Almost nothing is known about the behaviour of $f(n)$ for large n.

2. Denote by $R(A)^d$ the set of all accumulation points of the ratio set $R(A)$. Determine the greatest α such that in every partition of $\mathbb{N}$ into the sets A_1, A_2, A_3, one of the sets $R(A_i)^d$ $(i = 1, 2, 3)$ covers an interval of length at least α.

3. Let $\mathbb{N} = A_1 \cup A_2 \cup A_3$ be a partition of $\mathbb{N}$, and assume that $R(A_1)^d$ covers an interval of the length δ, $0 < \delta < 1$. What can be said about the sets $R(A_2)^d$, $R(A_3)^d$?

4. Is it true that in every partition of $\mathbb{N}$ into the sets A_1, A_2, A_3, there is an interval of arbitrary length contained in one of the sets $R(A_i)^d$ $(i = 1, 2, 3)$?

5. Is it true that for every partition of $\mathbb{N}$ into the sets A_1, A_2, A_3 there exists an $i \in \{1, 2, 3\}$ such that the set $\{x - y : \ x, y \in R(A_i)^d\}$ is equal to the set of all real numbers?

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