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Dedicated to the memory
of Professor Milan Kolibiar

CLONE COMPATIBLE IDENTITIES AND CLONE EXTENSIONS OF ALGEBRAS

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ABSTRACT. In this paper, we consider algebras of a given type τ with the set F of fundamental operation symbols and without nullary operations. An identity $\varphi \approx \psi$ of type τ we call clone compatible if φ and ψ is the same variable or neither φ nor ψ is a variable, and we have the same fundamental operation symbols in φ and ψ. For a variety V we denote by V^c the variety defined by all clone compatible identities from $\mathrm{Id}(V)$. In this paper, we assume $|F| \geq 2$. First we study properties of clone compatible identities, then we define a construction called a clone extension of an algebra. Using this construction we represent algebras from V^c by means of algebras from V if V satisfies some assumptions on terms. Further we define equational bases of V^c, and we apply these results to the varieties of lattices, Boolean algebras and some others.

0. Preliminaries

We shall consider algebras of a given type $\tau\colon F \to \mathbb{N}$, where F is a set of fundamental operation symbols, and $\mathbb{N}$ is the set of positive integers, i.e., *we do not admit nullary operations*. For a term φ of type τ we denote by $\mathrm{Var}(\varphi)$ the set of all variables occurring in φ, and by $F(\varphi)$ – the set of all fundamental operation symbols occurring in φ. If φ is a term of type τ, then writing $\varphi(x_{i_1}, \dots, x_{i_m})$ instead of φ we shall mean that $\mathrm{Var}(\varphi) = \{x_{i_1}, \dots, x_{i_m}\}$.

In [4] the notion of regular identity was introduced. Namely, an identity $\varphi = \psi$ is regular if $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$. In [7]–[9], we considered so-called biregular and uniform identities of type τ defined as follows: $\varphi \approx \psi$ is *biregular* if it is regular and $F(\varphi) = F(\psi)$; $\varphi \approx \psi$ is *uniform* if $F(\varphi) = F(\psi)$ and if

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$F(\varphi) \neq F$, then $\varphi \approx \psi$ is regular. In [9], we gave some representation theorems for algebras from varieties defined by uniform and biregular identities. In [7], we studied varieties defined by uniform and biregular identities from bisemilattices. An identity $\varphi \approx \psi$ of type τ is called *normal* (see [1], cf. [3], or *non-trivializing* in [6]) if it is of the form $x \approx x$ or $F(\varphi) \neq \emptyset \neq F(\psi)$. In [8], we mentioned about some other kind of identities, namely: $\varphi \approx \psi$ we called *operationally regular* if $F(\varphi) = F(\psi)$. In this paper, such identities we prefer to call *clone compatible*, however, we exclude $x \approx y$.

For a variety V of type τ we denote by V^c the variety defined by all clone compatible identities from $\mathrm{Id}(V)$. If V is the variety of lattices, then an identity $\varphi \approx \psi$ is clone compatible if and only if it is uniform, so studying them reduces to results from [7]. However, in Section 5, we use representation theorem also for bisemilattices. If $|F| = 1$, then a clone compatible identity is normal, and this case reduces to results of [3] or [2].

Therefore, in this paper, *we assume* $|F| \geq 2$. Among others, we want to consider the case of $\mathbf{B}^c$, where $\mathbf{B}$ is the variety of Boolean algebras with fundamental operation symbols $+$, $\cdot$, $'$. This we do in Sections 1 and 5, however, we prove more general theorems. In Section 1, we study clone compatible identities in varieties. In Section 2, we define a construction called a clone extension of an algebra. In Section 3, we give representation theorems for algebras from varieties V^c under some assumptions. In Section 4, we find equational bases of V^c. In Section 5, we apply theorems from Section 3 for some varieties.

1. Clone compatible identities

First we observe that the set of clone compatible identities of a variety V need not be an equational theory. In fact, the identity $x + x \cdot y \approx x + x \cdot z$ is clone compatible in the variety $\mathbf{B}$ of Boolean algebras, but $x + x \cdot y \approx x + x \cdot z'$ is not, although it is a consequence of the previous one. So let us try to find out what an equational theory generates the set of clone compatible identities from $\mathrm{Id}(\mathbf{B})$. It means that we want to find the form of identities from $\mathrm{Id}(\mathbf{B}^c)$.

If $(\varphi \approx \psi) \in \mathrm{Id}(V)$, we shall write $V \models \varphi \approx \psi$.

LEMMA 1.1. *Let V be a variety of type τ, $|F| \geq 2$, and for every $f, g \in F$, $f \neq g$, there exists a term $p_{f,g}(x,y)$ of type τ such that $F\big(p_{f,g}(x,y)\big) = \{f, g\}$ and*

$$V \models p_{f,g}(x,y) \approx x . \tag{1.1}$$

Then we have: if $V \models \varphi \approx \psi$, where $|F(\varphi)| > 1$ and $|F(\psi)| > 1$, then $V^c \models \varphi \approx \psi$.

Proof. If $|F| = 2$, then the statement holds automatically. Let $|F| > 2$, $f, g \in F(\varphi)$, $f \neq g$, and let $h, t \in F(\psi)$, $h \neq t$. Let us fix $p_{f,g}(x,y)$ and $p_{h,t}(x,y)$. Then

$$V^c \models p_{f,g}(x,y) \approx p_{f,g}(x,z) \quad \text{and} \quad V^c \models p_{h,t}(x,y) \approx p_{h,t}(x,z). \tag{1.2}$$

By (1.1),

$$V^c \models \varphi \approx p_{f,g}(\varphi,\varphi) \tag{1.3}$$

since this identity is clone compatible and belongs to $\mathrm{Id}(V)$. Similarly,

$$V^c \models \psi \approx p_{h,t}(\psi,\psi). \tag{1.4}$$

Further, by (2),

$$V^c \models p_{f,g}(\varphi,\varphi) \approx p_{f,g}(\varphi,\psi) \quad \text{and} \quad V^c \models p_{h,t}(\psi,\psi) \approx p_{h,t}(\psi,\varphi). \tag{1.5}$$

Since $V \models \varphi \approx \psi$, so by (1),

$$V^c \models p_{f,g}(\varphi,\psi) \approx p_{h,t}(\psi,\varphi) \tag{1.6}$$

as it is clone compatible. Now, by (1.3), (1.5), (1.4), (1.5), (1.6), we get the statement. □

Let V be a variety of type τ, consider the following condition:

(1.i) Every identity $\varphi \approx \psi$ from $\mathrm{Id}(V)$ is regular whenever $F(\varphi) = F(\psi) = \{f\}$, $f \in F$.

THEOREM 1.2. *If a variety V of type τ satisfies assumptions of Lemma 1.1 and condition (1.i), then the equational theory $\mathrm{Id}(V^c)$ consists exactly of the union of three disjoint sets E_1, E_2, E_3 defined as follows:*

E_1 consists of all identities from $\mathrm{Id}(V)$ satisfying (1.i);
E_2 is the set of all identities $\varphi \approx \psi$,
where $|F(\varphi)| > 1$, $|F(\psi)| > 1$ and $V \models \varphi \approx \psi$;
E_3 is the set of all identities $x_i \approx x_i$, where x_i is a variable.

Proof. We denote by $\mathrm{cc}(V)$ the set of all clone compatible identities from $\mathrm{Id}(V)$. Since $\mathrm{Id}(V^c)$ is the smallest equational theory generated by $\mathrm{cc}(V)$, so to prove the theorem, it is enough to show that the set $E = E_1 \cup E_2 \cup E_3$ is an equational theory containing $\mathrm{cc}(V)$ and $E \subseteq \mathrm{Id}(V^c)$. Obviously, $\mathrm{cc}(V) \subseteq E$. One can easily check that E is an equational theory, i.e., it is closed under five Birkhoff's derivation rules. Obviously, $E_1 \cup E_3 \subseteq \mathrm{Id}(V^c)$. By Lemma 1.1, $E_2 \subseteq \mathrm{Id}(V^c)$, what completes the proof. □

COROLLARY 1.3. *The equational theory* $\mathrm{Id}(\mathbf{B}^c)$ *consists exactly of the union of three disjoint sets* E_1, E_2, E_3 *as in Theorem* 1.2, *where* $V = \mathbf{B}$ *is a variety of Boolean algebras and* $F = \{+, \cdot, '\}$.

Proof. Put

$$p_{+,\cdot}(x,y) = x + x \cdot y\,, \quad p_{+,'}(x,y) = x + (y + y')'\,, \quad p_{\cdot,'}(x,y) = x \cdot (y \cdot y')'. \tag{1.7}$$

Further, $\mathbf{B}$ satisfies (1.i). So assumptions of Theorem 1.2 are satisfied. □

Remark 1.4. The second assumption of Theorem 1.2 is essential.

In fact, let V be the variety of groups with fundamental operations $\cdot$, $^{-1}$ satisfying $x^n \approx y^n$. Then the identity $x^n \approx (y \cdot y^{-1})^n$ belongs to $\mathrm{Id}(V^c)$ and does not belong to E.

For further considerations, it is useful to consider for every variety V the variety $\overline{V^c}$ defined by all identities $\varphi \approx \psi$ satisfied in V for which $F(\varphi) = F(\psi) = \{f\}$ for $f \in F$, or for which both $|F(\varphi)|$ and $|F(\psi)|$ is greater than 1. In fact, many important varieties of groups, rings, lattices and Boolean algebras satisfy Lemma 1.1 (see Section 5), and we have:

LEMMA 1.5. *If a variety* V *satisfies assumptions of Lemma* 1.1, *then* $V^c = \overline{V^c}$.

In fact, we observe that we have always $\overline{V^c} \subseteq V^c$.

For fixed $f \in F$ we put $\{f\}' = F \setminus \{f\}$. An identity $\varphi \approx \psi$ of type τ will be called f*-normal* if it is one of the following forms:

$$\varphi \approx \psi\,, \qquad \text{where} \quad F(\varphi) \cup F(\psi) \subseteq \{f\}\,; \tag{1.8}$$

$$\varphi \approx \psi\,, \qquad \text{where} \quad F(\varphi) \cap \{f\}' \neq \emptyset \neq F(\psi) \cap \{f\}'\,. \tag{1.9}$$

For a variety V of type τ we denote by V_f the variety of type τ defined by all f-normal identities from $\mathrm{Id}(V)$. Further, we put $N_f(V) = \mathrm{Id}(V_f)$.

PROPOSITION 1.6. *If every identity of the form* (1.8) *from* $\mathrm{Id}(V)$ *is regular, then the set* $N_f(V)$ *is an equational theory.*

The proof is left to the reader since it is similar to that of Theorem 1.2.

If $q(x)$ is a unary term of type τ with $F(q) = F_0$ for some $F_0 \subseteq F$, then in the sequel, we shall write $q_{F_0}(x)$ instead of $q(x)$, and we shall write $q_f(x)$ if $F_0 = \{f\}$ for some $f \in F$.

Let $V^{(f)}$ denote the variety defined by the set $\mathrm{Id}(\overline{V^c}) \cup \{q_f(x) \approx x\}$.

Lemma 1.7. *If V is a variety of type τ, and there exists a unary term $q_f(x)$ such that*

$$V \models q_f(x) \approx x\,, \tag{1.10}$$

then $V^{(f)} = V_f$.

Proof. Since $\operatorname{Id}(\overline{V^c}) \subseteq \operatorname{Id}(V^{(f)})$, and V_f satisfies (1.10), so $V_f \subseteq V^{(f)}$. If $\varphi \approx \psi$ is of the form (1.8), and $V_f \models \varphi \approx \psi$, then

$$V^{(f)} \models \varphi \approx q_f(\varphi) \approx q_f(\psi) \approx \psi\,. \tag{1.11}$$

If $\varphi \approx \psi$ is of the form (1.9), and $V_f \models \varphi \approx \psi$, then we have again (1.11). Thus $\operatorname{Id}(V_f) \subseteq \operatorname{Id}(V^{(f)})$, and consequently, $V^{(f)} \subseteq V_f$. □

Theorem 1.8. *Let V satisfy the following condition:*

There exists a term $q_{\{f,g\}}(x)$ such that $f, g \in F$, $f \neq g$ and $V \models q_{\{f,g\}}(x) \approx x$.

Then the variety V'' of the type τ defined by $\operatorname{Id}(\overline{V^c}) \cup \{q_{\{f,g\}}(x) \approx x\}$ is equal to V.

Proof. Let $V \models \varphi \approx \psi$. Then $V'' \models \varphi \approx q_{\{f,g\}}(\varphi) \approx q_{\{f,g\}}(\psi) \approx \psi$. □

Corollary 1.9. *Let V be a variety of algebras for which there exist unary terms $q_f(x)$ and $q_g(x)$ with $V \models \big(q_f(x) \approx x \approx q_g(x)\big)$. Then V is a variety defined by $\operatorname{Id}(\overline{V^c}) \cup \{q_f(x) \approx x,\ q_g(x) \approx x\}$.*

Lemma 1.10. *Let a variety V of type τ satisfy the following condition:*

(1.iv) *There exists a term $q_f(x,y)$ of type τ such that the identity*

$$q_f(x,y) \approx x \tag{1.12}$$

is satisfied in V.

Then $V_f = V$.

Proof. Since we have $\operatorname{Id}(V_f) \subseteq \operatorname{Id}(V)$, so we have to prove the converse inclusion.

Let $V \models \varphi \approx \psi$. If $\varphi \approx \psi$ is of the form (1.8) or (1.9), then it belongs to $\operatorname{Id}(V_f)$ by the definition of V_f. Suppose that $F(\varphi) \subseteq \{f\}$, and there is $g \in \{f\}'$, where $g \in F(\psi)$. Obviously, identity (1.12) is satisfied in V_f. So we have:

$$V_f \models q_f(\varphi,\psi) \approx \varphi\,, \quad V_f \models q_f(\psi,\varphi) \approx \psi\,, \quad V_f \models q_f(\varphi,\psi) \approx q_f(\psi,\varphi)$$

since the last identity is of the form (1.9), and it is satisfied in V. Thus $V_f \models \varphi \approx \psi$. □

Corollary 1.11. *If V is the variety of groups with fundamental operation symbols $\cdot$ and $^{-1}$ satisfying $x^n \approx y^n$, then $V_\cdot$ and V coincide.*

In fact, take $q_\cdot(x,y) = x \cdot y^n$.

2. An f-clone extension and a clone extension of an algebra

Let $|F| \geq 2$ and let $\mathcal{B} = (B; F^{\mathcal{B}})$ be an algebra of type τ, and let r be a retraction of $\mathcal{B}$, i.e., an idempotent endomorphism. We shall say that $\mathcal{B}$ is an *f-clone extension of an algebra $\mathcal{A} = (A; F^{\mathcal{A}})$ with respect to r* if the following conditions are satisfied:

(2.i) $A = r(B)$;

(2.iiα) If $g \in \{f\}'$, $a_{i_1}, \ldots, a_{i_{\tau(g)}} \in B$, then
$$g^{\mathcal{B}}(a_{i_1}, \ldots, a_{i_{\tau(g)}}) = g^{\mathcal{A}}(r(a_{i_1}), \ldots, r(a_{i_{\tau(g)}}));$$

(2.iiβ) If $a_{i_1}, \ldots, a_{i_{\tau(f)}} \in B$ and $\{a_{i_1}, \ldots, a_{i_{\tau(f)}}\} \cap r(B) \neq \emptyset$, then
$$f^{\mathcal{B}}(a_{i_1}, \ldots, a_{i_{\tau(f)}}) = f^{\mathcal{A}}(r(a_{i_1}), \ldots, r(a_{i_{\tau(f)}})).$$

We shall say that $\mathcal{B}$ is an *f-clone extension of an algebra $\mathcal{A}$* if it is an f-clone extension of an algebra $\mathcal{A}$ with respect to some r.

(2.iii) For every $q \in F$ we have $q^{\mathcal{B}}|_A = q^{\mathcal{A}}$, consequently, $\mathcal{A}$ is a subalgebra of $\mathcal{B}$.

In fact, if $q \in F$, $a_{i_1}, \ldots, a_{i_{\tau(q)}} \in A$, then, by (2.ii$\alpha$) or (2.ii$\beta$), we have $q^{\mathcal{B}}(a_{i_1}, \ldots, a_{i_{\tau(q)}}) = q^{\mathcal{A}}(r(a_{i_1}), \ldots, r(a_{i_{\tau(q)}})) = q^{\mathcal{A}}(a_{i_1}, \ldots, a_{i_{\tau(q)}})$.

Let us observe that every algebra $\mathcal{C} = (C; F^{\mathcal{C}})$ is an f-clone extension of itself if we accept r to be the identity.

LEMMA 2.1. *If $\mathcal{B} = (B; F^{\mathcal{B}})$ is an f-clone extension of an algebra $\mathcal{A}$, and $\varphi(x_{i_1}, \ldots, x_{i_m})$ is a term of type τ with $F(\varphi) \cap \{f\}' \neq \emptyset$, then for every $a_{i_1}, \ldots, a_{i_m} \in B$ we have $\varphi^{\mathcal{B}}(a_{i_1}, \ldots, a_{i_m}) = \varphi^{\mathcal{A}}(r(a_{i_1}), \ldots, r(a_{i_m}))$.*

P r o o f . Using the definition of an f-clone extension we can verify the statement of the lemma by induction on complexity of a term φ. $\square$

COROLLARY 2.2. *If $\mathcal{B}$ is an f-clone extension of an algebra $\mathcal{A}$, and $\varphi(x_{i_1}, \ldots, x_{i_m}) \approx \psi(x_{j_1}, \ldots, x_{j_n})$ is an identity of type τ with $F(\varphi) \cap \{f\}' \neq \emptyset \neq F(\psi) \cap \{f\}'$, then $\varphi \approx \psi$ is satisfied in $\mathcal{B}$ if and only if it is satisfied in $\mathcal{A}$.*

This follows at once from Lemma 2.1.

Let $\tau \colon F \to \mathbb{N}$ be a type of algebras with $|F| \geq 2$. Let S be a nonempty set, and $\{r_f\}_{f \in F}$ be an indexed family of mappings with $r_f \colon S \to S$ satisfying the following conditions:

(2.iv) $r_f \circ r_f = r_f$ for every $f \in F$;
$r_f \circ r_g = r_g \circ r_f$ for every $f, g \in F$;
$r_f \circ r_g = r_s \circ r_t$ for every $f, g, s, t \in F$; $f \neq g$, $s \neq t$.

Such family will be called *a concentrating family of mappings on* S.

EXAMPLE 2.3. Let $S = \{a_1, a_2, a_3, a_4\}$, $F = \{f, g\}$. We put $r_f(a_1) = a_2$, $r_f(a_2) = a_2$, $r_f(a_3) = a_4$, $r_f(a_4) = a_4$, $r_g(a_1) = a_3$, $r_g(a_2) = a_4$, $r_g(a_3) = a_3$, $r_g(a_4) = a_4$.

We put $h = r_f \circ r_g$ for some $f, g \in F$, $f \neq g$. By (2.iv), h does not depend on the choice of f and g. We have

(2.v) The mapping h is idempotent, and if $a \in h(S)$, then $r_f(a) = h(a) = a$ for every $f \in F$.

In fact, $h \circ h = (r_f \circ r_g) \circ (r_f \circ r_g) = (r_f \circ r_f) \circ (r_g \circ r_g) = r_f \circ r_g = h$ for some $f, g \in F$, $f \neq g$. If $a \in h(S)$, then $a = h(b)$ for some $b \in S$. So $a = h(b) = h(h(b)) = h(a)$. Further, $r_f(a) = r_f(h(a)) = r_f(r_f(r_g(a))) = r_f(r_g(a)) = h(a) = a$.

Put $A_f = r_f(S)$ for every $f \in F$ and put $A = h(S)$.

(2.vi) $\bigcap_{f \in F} A_f = A \neq \emptyset$, and if $f, g \in F$, $f \neq g$, then $A_f = A_g$ if and only if $A_f = A = A_g$.

In fact, if $a \in \bigcap_{f \in F} A_f$, then we have $r_f(a) = a$ for every $f \in F$. So, for $f \neq g$, $h(a) = r_f(r_g(a)) = r_f(a) = a$. Consequently, $a \in A$. If $a \in A$, then, for $f \in F$, $r_f(a) = a$ by (2.v). Thus $a \in \bigcap_{f \in F} A_f$. If $A_f = A_g$ for some $f \neq g$, $a \in A_f$, then we have $h(a) = r_f(r_g(a)) = r_f(a) = a$, so $a \in A$. Now, by the first statement $A = A_f$.

Put $F_0 = \{f \in F : A_f = A\}$. By (2.iv) and (2.vi), we have:

(2.vii) For every $f, g \in F \setminus F_0$, $f \neq g$, we have $(A_f \setminus A) \cap (A_g \setminus A) = \emptyset$. So, if $a \notin A$, then there exists at most one $f \in F$ such that $a \in A_f$.

(2.viii) If for some $f \in F$, $a \in S$ we have $r_f(a) \in A$, then $r_f(a) = h(a)$.

In fact, by (2.v), $r_f(a) = r_g(r_f(a)) = r_f(r_g(a)) = h(a)$, where $f \neq g$.

(2.ix) If $a \in A_f$ and $g \in \{f\}'$, then $r_g(a) = h(a)$.

In fact, $r_g(a) = r_g(r_f(a)) = r_f(r_g(a)) = h(a)$.

If $\mathcal{A}_1 = (A_1; F^{\mathcal{A}_1})$ and $\mathcal{A}_2 = (A_2; F^{\mathcal{A}_2})$ are two algebras, then we shall write $\mathcal{A}_1 = \mathcal{A}_2$ if $A_1 = A_2$ and $f^{\mathcal{A}_1} = f^{\mathcal{A}_2}$ for every $f \in F$.

Let $\mathbf{S} = \big(S, \{r_f\}_{f \in F}, \{\mathcal{A}_f\}_{f \in F}, \mathcal{A}\big)$ be a system satisfying the following conditions:

(2.x) S is a nonempty set;

(2.xi) $\{r_f\}_{f \in F}$ is a concentrating family of mappings, i.e., satisfying (2.iv);

(2.xii) $\mathcal{A}$ and $\mathcal{A}_f$ are algebras of type τ for every $f \in F$, where $\mathcal{A} = (A; F^{\mathcal{A}})$, $A = h(S)$, where $h = r_f \circ r_g$ for some $f \neq g$, $\mathcal{A}_f = (A_f; F^{\mathcal{A}_f})$, $A_f = r_f(S)$;

(2.xiii) For every $f \in F$, $h|A_f$ is a retraction of $\mathcal{A}_f$ such that $\mathcal{A}_f$ is an f-clone extension of $\mathcal{A}$ by h.

The system $\mathbf{S}$ we shall call a *concentrating system.* We define a new algebra $\mathcal{A}(\mathbf{S}) = \big(S; F^{\mathcal{A}(\mathbf{S})}\big)$ of type τ, where the fundamental operations in $\mathcal{A}(\mathbf{S})$ are defined by condition:

(2.xiv) If $f \in F$ and $a_{i_1}, \dots, a_{i_{\tau(f)}} \in S$, then

$$\begin{aligned} f^{\mathcal{A}(\mathbf{S})}\big(a_{i_1}, \dots, a_{i_{\tau(f)}}\big) &= f^{\mathcal{A}(\mathbf{S})}\big(r_f(a_{i_1}), \dots, r_f(a_{i_{\tau(f)}})\big) \\ &= f^{\mathcal{A}_f}\big(r_f(a_{i_1}), \dots, r_f(a_{i_{\tau(f)}})\big) . \end{aligned}$$

The algebra $\mathcal{A}(\mathbf{S})$ will be called a *clone extension of the algebra* $\mathcal{A}$ *by the concentrating system* $\mathbf{S}$, or briefly, a *clone extension of the algebra* $\mathcal{A}$.

By (2.iv) and (2.v), h and $h|A_f$ are uniquely defined. Further,

$$h(A_f) = r_g\big(r_f\big(r_f(S)\big)\big) = r_g\big(r_f(S)\big) = h(S) = A .$$

So, (2.i) is satisfied for $r = h$ and every $f \in F$, and therefore $f^{\mathcal{A}(\mathbf{S})}$ is well defined.

(2.xv) For every $f \in F$ we have $f^{\mathcal{A}(\mathbf{S})}|A_f = f^{\mathcal{A}_f}$, and r_f is a retraction of $\mathcal{A}(\mathbf{S})$. So $\mathcal{A}_f$ is a subalgebra of $\mathcal{A}(\mathbf{S})$.

In fact, let $f \in F$ and $a_{i_1}, \dots, a_{i_{\tau(f)}} \in A_f$. By (2.xiv), we have

$$f^{\mathcal{A}(\mathbf{S})}\big(a_{i_1}, \dots, a_{i_{\tau(f)}}\big) = f^{\mathcal{A}_f}\big(r_f(a_{i_1}), \dots, r_f\big(a_{i_{\tau(f)}}\big)\big) = f^{\mathcal{A}_f}\big(a_{i_1}, \dots, a_{i_{\tau(f)}}\big) .$$

Let $b_{i_1}, \dots, b_{i_{\tau(f)}} \in S$, $f, g \in F$. If $f = g$, then, by (2.xiv), we have

$$\begin{aligned} r_f\big(f^{\mathcal{A}(\mathbf{S})}(b_{i_1}, \dots, b_{i_{\tau(f)}})\big) &= r_f\big(f^{\mathcal{A}_f}\big(r_f(b_{i_1}), \dots, r_f\big(b_{i_{\tau(f)}}\big)\big)\big) \\ &= f^{\mathcal{A}_f}\big(r_f\big(r_f(b_{i_1})\big), \dots, r_f\big(r_f\big(b_{i_{\tau(f)}}\big)\big)\big) \\ &= f^{\mathcal{A}(\mathbf{S})}\big(r_f(b_{i_1}), \dots, r_f\big(b_{i_{\tau(f)}}\big)\big) . \end{aligned}$$

If $g \neq f$, then we have by (2.xiv), (2.xiii), (2.iv) that

$$\begin{aligned} r_g\big(f^{\mathcal{A}(\mathbf{S})}(b_{i_1},\ldots,b_{i_{\tau(f)}})\big) &= r_g\big(f^{\mathcal{A}_f}\big(r_f(b_{i_1}),\ldots,r_f(b_{i_{\tau(f)}})\big)\big) \\ &= r_g\big(f^{\mathcal{A}_f}\big(r_f\big(r_f(b_{i_1})\big),\ldots,r_f\big(r_f(b_{i_{\tau(f)}})\big)\big)\big) \\ &= r_g\big(r_f\big(f^{\mathcal{A}_f}\big(r_f(b_{i_1}),\ldots,r_f(b_{i_{\tau(f)}})\big)\big)\big) \\ &= h\big(f^{\mathcal{A}_f}\big(r_f(b_{i_1}),\ldots,r_f(b_{i_{\tau(f)}})\big)\big) \\ &= f^{\mathcal{A}_f}\big(h\big(r_f(b_{i_1})\big),\ldots,h\big(r_f(b_{i_{\tau(f)}})\big)\big) \\ &= f^{\mathcal{A}_f}\big(r_f\big(r_g(b_{i_1})\big),\ldots,r_f\big(r_g(b_{i_{\tau(f)}})\big)\big) \\ &= f^{\mathcal{A}(\mathbf{S})}\big(r_g(b_{i_1}),\ldots,r_g(b_{i_{\tau(f)}})\big)\,. \end{aligned}$$

(2.xvi) For every $f \in F$, $f^{\mathcal{A}(\mathbf{S})}|_A = f^{\mathcal{A}}$, and h is a retraction of $\mathcal{A}(\mathbf{S})$. So $\mathcal{A}$ is a subalgebra of $\mathcal{A}(\mathbf{S})$.

Indeed, h is an endomorphism since $h = r_f \circ r_g$ for some $f \neq g$, and it is idempotent by (2.v). By (2.xiv), (2.iiα), (2.iiβ) and (2.v), we have that if $q \in F$ and $a_{i_1},\ldots,a_{i_{\tau(q)}} \in A$, then

$$\begin{aligned} q^{\mathcal{A}(\mathbf{S})}\big(a_{i_1},\ldots,a_{i_{\tau(q)}}\big) &= q^{\mathcal{A}_q}\big(r_q(a_{i_1}),\ldots,r_q(a_{i_{\tau(q)}})\big) \\ &= q^{\mathcal{A}_q}\big(a_{i_1},\ldots,a_{i_{\tau(q)}}\big) = q^{\mathcal{A}_q}\big(h(a_{i_1}),\ldots,h(a_{i_{\tau(q)}})\big) \\ &= q^{\mathcal{A}}\big(h\big(h(a_{i_1})\big),\ldots,h\big(h(a_{i_{\tau(q)}})\big)\big) = q^{\mathcal{A}}\big(a_{i_1},\ldots,a_{i_{\tau(q)}}\big)\,. \end{aligned}$$

Every algebra $\mathcal{C} = (C; F^{\mathcal{C}})$ is a clone extension of itself since it is enough to put $S = C$ and to accept r_f to be the identity map in C for every $f \in F$.

(2.xvii) If $\mathcal{B} = (B; F^{\mathcal{B}})$ is an f-clone extension of an algebra $\mathcal{A} = (A; F^{\mathcal{A}})$, then it is a clone extension of the algebra $\mathcal{A}$.

In fact, it is enough to put $S = B$, $\mathcal{A}_f = \mathcal{B}$, $\mathcal{A}_g = \mathcal{A}$ for every $g \in \{f\}'$, r_f to be an identity, and $r_g = r$ for every $g \in \{f\}'$.

(2.xviii) Let for some $f, g \in F$, $f \neq g$, we have $r_f = r_g = \mathrm{id}$, where id is the identity map. Then $\mathcal{A}(\mathbf{S}) = \mathcal{A}$, i.e., $\mathcal{A}(\mathbf{S})$ is the trivial clone extension.

In fact, then for every $a \in S$ we have $h(a) = r_f\big(r_g(a)\big) = r_f(a) = a$. So $\mathcal{A}(\mathbf{S}) = \mathcal{A}$.

(2.xix) If $\{a_{i_1},\ldots,a_{i_{\tau(f)}}\} \cap \bigcup_{g\in\{f\}'} r_g(S) \neq \emptyset$, then

$$\begin{aligned} f^{\mathcal{A}(\mathbf{S})}\big(a_{i_1},\ldots,a_{i_{\tau(f)}}\big) &= f^{\mathcal{A}(\mathbf{S})}\big(h(a_{i_1}),\ldots,h(a_{i_{\tau(f)}})\big) \\ &= f^{\mathcal{A}}\big(h(a_{i_1}),\ldots,h(a_{i_{\tau(f)}})\big)\,. \end{aligned}$$

Indeed, let $a_{i_k} \in \bigcup_{g\in\{f\}'} r_g(S) = \bigcup_{g\in\{f\}'} A_g$ for some $k \in \{1,\dots,\tau(f)\}$. So there is $q \in \{f\}'$ such that $a_{i_k} \in A_q$. Then, by (2.xiii), (2.xiv), (2.iiα), (2.iiβ), we have

$$\begin{aligned}
&f^{\mathcal{A}(\mathbf{S})}\big(a_{i_1},\dots,a_{i_{\tau(f)}}\big)\\
&=f^{\mathcal{A}(\mathbf{S})}\big(a_{i_1},\dots,a_{i_{k-1}},r_q(a_{i_k}),a_{i_{k+1}},\dots,a_{i_{\tau(f)}}\big)\\
&=f^{\mathcal{A}_f}\big(r_f(a_{i_1}),\dots,r_f(a_{i_{k-1}}),r_f\big(r_q(a_{i_k})\big),r_f(a_{i_{k+1}}),\dots,r_f(a_{i_{\tau(f)}})\big)\\
&=f^{\mathcal{A}_f}\big(r_f(a_{i_1}),\dots,r_f(a_{i_{k-1}}),h(a_{i_k}),r_f(a_{i_{k+1}}),\dots,r_f(a_{i_{\tau(f)}})\big)\\
&=f^{\mathcal{A}}\big(h(a_{i_1}),\dots,h(a_{i_{k-1}}),h\big(h(a_{i_k})\big),h(a_{i_{k+1}}),\dots,h(a_{i_{\tau(f)}})\big)\\
&=f^{\mathcal{A}}\big(h(a_{i_1}),\dots,h(a_{i_{k-1}}),h(a_{i_k}),h(a_{i_{k+1}}),\dots,h(a_{i_{\tau(f)}})\big)\,.
\end{aligned}$$

Obviously, a clone extension of an algebra $\mathcal{A}$ depends on the structure of every $\mathcal{A}_f$. However, in the further considerations, we require something more from algebras $\mathcal{A}_f$ to obtain representation theorems.

LEMMA 2.4. *If $\varphi(x_{i_1},\dots,x_{i_m})$ is a term of type τ such that $F(\varphi)=\{f\}$ and $a_{i_1},\dots,a_{i_m}\in S$, then*

$$\varphi^{\mathcal{A}(\mathbf{S})}(a_{i_1},\dots,a_{i_m}) = \varphi^{\mathcal{A}_f}\big(r_f(a_{i_1}),\dots,r_f(a_{i_m})\big)\,.$$

This follows from (2.xiv) by easy induction on the complexity of φ.

LEMMA 2.5. *If $\varphi(x_{i_1},\dots,x_{i_m})$ is a term of type τ such that $|F(\varphi)|>1$ and $a_{i_1},\dots,a_{i_m}\in S$, then*

$$\varphi^{\mathcal{A}(\mathbf{S})}(a_{i_1},\dots,a_{i_m}) = \varphi^{\mathcal{A}}\big(h(a_{i_1}),\dots,h(a_{i_m})\big)\,.$$

P r o o f. If $\varphi = f\big(x_{i_1},\dots,x_{i_{k-1}},g(y_{j_1},\dots,y_{j_{\tau(g)}}),x_{i_{k+1}},\dots,x_{i_{\tau(f)}}\big)$ for some $f,g\in F$, $f\neq g$, then, since $g^{\mathcal{A}(\mathbf{S})}\big(b_{j_1},\dots,b_{j_{\tau(g)}}\big) = g^{\mathcal{A}_g}\big(r_g(b_{j_1}),\dots,r_g(b_{j_{\tau(g)}})\big)$ and $g^{\mathcal{A}_g}\big(r_g(b_{j_1}),\dots,r_g(b_{j_{\tau(g)}})\big)\in r_g(S)$, so, by (2.xiv), (2.ii$\alpha$), (2.ii$\beta$) and (2.xix), we have

$$\begin{aligned}
&f^{\mathcal{A}(\mathbf{S})}\big(a_{i_1},\dots,a_{i_{k-1}},g^{\mathcal{A}(\mathbf{S})}(b_{j_1},\dots,b_{j_{\tau(g)}}),a_{i_{k+1}},\dots,a_{i_{\tau(f)}}\big)\\
&=f^{\mathcal{A}}\big(h(a_{i_1}),\dots,h(a_{i_{k-1}}),h\big(g^{\mathcal{A}(\mathbf{S})}(b_{j_1},\dots,b_{j_{\tau(g)}}),h(a_{i_{k+1}}),\dots,h(a_{i_{\tau(f)}})\big)\big)\\
&=f^{\mathcal{A}}\big(h(a_{i_1}),\dots,h(a_{i_{k-1}}),g^{\mathcal{A}(\mathbf{S})}\big(h(b_{j_1}),\dots,h(b_{j_{\tau(g)}})\big),h(a_{i_{k+1}}),\dots,h(a_{i_{\tau(f)}})\big)\\
&=f^{\mathcal{A}}\big(h(a_{i_1}),\dots,h(a_{i_{k-1}}),g^{\mathcal{A}}\big(h(b_{j_1}),\dots,h(b_{j_{\tau(g)}})\big),h(a_{i_{k+1}}),\dots,h(a_{i_{\tau(f)}})\big)\,.
\end{aligned}$$

In general, we proceed by induction on the complexity of φ. If $\varphi = f(\varphi_{i_1},\dots$ $\dots,\varphi_{i_{\tau(f)}})$, then there exists $g\in F$ such that $g\neq f$, and for some k, $k\in\{1,\dots,\tau(f)\}$, we have $g\in F(\varphi_{i_k})$. If $F(\varphi_{i_k})=\{g\}$, then using Lemma 2.4 we infer as above. If $|F(\varphi_{i_k})|>1$, then we use the inductional assumption. □

LEMMA 2.6. *If* $\varphi(x_{i_1},\dots,x_{i_m}) \approx \psi(x_{j_1},\dots,x_{j_s})$ *is an identity of type* τ *with* $F(\varphi(x_{i_1},\dots,x_{i_m})) = F(\psi(x_{j_1},\dots,x_{j_s})) = \{f\}$ *for some* $f \in F$, *then it is satisfied in* $\mathcal{A}(\mathbf{S})$ *if and only if it is satisfied in* $\mathcal{A}_f$.

This follows from Lemma 2.4.

LEMMA 2.7. *If* $\varphi(x_{i_1},\dots,x_{i_m}) \approx \psi(x_{j_1},\dots,x_{j_s})$ *is an identity of type* τ *with* $|F(\varphi(x_{i_1},\dots,x_{i_m}))| > 1$ *and* $|F(\psi(x_{j_1},\dots,x_{j_s}))| > 1$, *then it is satisfied in* $\mathcal{A}(\mathbf{S})$ *if and only if it is satisfied in* $\mathcal{A}$.

This follows from Lemma 2.5.

The above definition of a clone extension of an algebra $\mathcal{A}$ is a kind of construction. Now we want to give a little simpler equivalent definition of this notion which is rather a kind of description.

Let $\tau\colon F \to \mathbb{N}$ be a type of algebras with $0 \notin \tau(F)$ and $|F| \geq 2$.

DEFINITION 2.8. An algebra $\mathcal{B} = (B; F^{\mathcal{B}})$ of type $\tau\colon F \to \mathbb{N}$ is a *clone extension of an algebra* $\mathcal{A}$ *of type* τ *by means of a family* $\{\mathcal{A}_f\}_{f\in F}$ *of algebras of type* τ if the following conditions are satisfied:

(2.xx) There exists a concentrating family $\{r_f\}_{f\in F}$ of retractions of $\mathcal{B}$.

(2.xxi) $\mathcal{A}_f$, $f \in F$ and $\mathcal{A}$ are subalgebras of $\mathcal{B}$, where $\mathcal{A}_f = \left(r_f(B);\ F^{\mathcal{B}}|_{r_f(B)}\right)$ and $\mathcal{A} = \left(r_s(r_t(B));\ F^{\mathcal{B}}|_{r_s(r_t(B))}\right)$ for fixed $s,t \in F$, $s \neq t$.

(2.xxii) If $f \in F$, $a_{i_1},\dots,a_{i_{\tau(f)}} \in r_f(B)$ and $\{a_{i_1},\dots,a_{i_{\tau(f)}}\} \cap r_s(r_t(B)) \neq \emptyset$, then

$$f^{\mathcal{A}_f}(a_{i_1},\dots,a_{i_{\tau(f)}}) = f^{\mathcal{A}}\big(r_s(r_t(a_{i_1})),\dots,r_s(r_t(a_{i_{\tau(f)}}))\big).$$

If $b_{i_1},\dots,b_{i_{\tau(f)}} \in B$, then

$$f^{\mathcal{B}}(b_{i_1},\dots,b_{i_{\tau(f)}}) = f^{\mathcal{A}_f}(r_f(b_{i_1}),\dots,r_f(b_{i_{\tau(f)}})).$$

By (2.xv), (2.xvi) and (2.iiβ), the conditions of the previous definition imply (2.xx) – (2.xxii) and checking the converse is easy to verify.

3. Representation theorems

For a variety V of type τ we denote by $\Sigma_f(V)$ the set of all identities of the form (1.8) belonging to $\mathrm{Id}(V)$.

THEOREM 3.1. *Let V be a variety of type τ. If an algebra $\mathcal{B}$ is an f-clone extension of an algebra $\mathcal{A}$ from V and $\mathcal{B}$ satisfies $\Sigma_f(V)$, then $\mathcal{B}$ belongs to V_f.*

Proof. If $\varphi \approx \psi$ is of the form (1.9) and $V \models \varphi \approx \psi$, then $\mathcal{A}$ satisfies $\varphi \approx \psi$ and, by Corollary 2.2, $\mathcal{B}$ satisfies $\varphi \approx \psi$. If $\varphi \approx \psi$ is of the form (1.8), then $\mathcal{B}$ satisfies $\varphi \approx \psi$ by the assumption. Thus $\mathcal{B} \in V_f$. □

THEOREM 3.2. *Let V be a variety of type τ. If $\overline{\mathcal{A}}$ is a clone extension of an algebra $\mathcal{A}$ from V where for every $f \in F$ the algebra $\mathcal{A}_f$ satisfies $\Sigma_f(V)$, then $\overline{\mathcal{A}}$ belongs to $\overline{V^c}$ and consequently, to V^c.*

Proof. If $\varphi \approx \psi$ is satisfied in V, where $|F(\varphi)|, |F(\psi)| > 1$, then it is satisfied in $\mathcal{A}$, and by Lemma 2.7, it is satisfied in $\overline{\mathcal{A}}$. If $\varphi \approx \psi$ is satisfied in V, where $F(\varphi) = F(\psi) = \{f\}$ for some $f \in F$, then it belongs to $\Sigma_f(V)$, so by assumption, it is satisfied in $\mathcal{A}_f$. By Lemma 2.6, $\varphi \approx \psi$ is satisfied in $\overline{\mathcal{A}}$. □

For a variety V of type τ let us consider the following condition.

(3.i) For every $f \in F$, there exists a term $q_f(x)$ such that $V \models q_f(x) \approx x$.

THEOREM 3.3. *Let V be a variety of type τ satisfying condition* (3.i). *If $\mathcal{A}^*$ belongs to $\overline{V^c}$, then $\mathcal{A}^*$ is a clone extension of an algebra $\mathcal{A}$ from V, where for every $f \in F$ the algebra $\mathcal{A}_f$ satisfies $\Sigma_f(V)$.*

Proof. Let $\mathcal{A}^* = (A^*; F^{\mathcal{A}^*})$ belong to $\overline{V^c}$. Put $r_f(a) = q_f{}^{\mathcal{A}^*}(a)$ for every $a \in A^*$. So, by (3.i), conditions (2.iv) and (2.xi) hold. In fact, by (3.i), we have

$$\overline{V^c} \models q_f\big(q_f(x)\big) \approx q_f(x) \qquad \text{for every } f \in F$$
$$\overline{V^c} \models q_f\big(q_g(x)\big) \approx q_g\big(q_f(x)\big) \qquad \text{for } f, g \in F;$$
$$\overline{V^c} \models q_f\big(q_g(x)\big) \approx q_s\big(q_t(x)\big) \qquad \text{for } f, g, s, t \in F, \ f \neq g, \ s \neq t.$$

Put $q_h = q_f\big(q_g(x)\big)$ for some $f \neq g$ and $h(a) = q_h{}^{\mathcal{A}^*}(a)$ for every $a \in A^*$. So h is idempotent by (2.v). Put $\mathcal{A} = \big(A; F^{\mathcal{A}^*}|_A\big)$, where $A = h(A^*)$. The algebra $\mathcal{A}$ is well defined. In fact, for every $f \in F$

$$\overline{V^c} \models q_h\big(f(x_1, \ldots, x_{\tau(f)})\big) \approx f\big(q_h(x_1), \ldots, q_h(x_{\tau(f)})\big).$$

So, if $a_{i_1}, \ldots, a_{i_{\tau(f)}} \in A$, then $h\big(f^{\mathcal{A}^*}(a_{i_1}, \ldots, a_{i_{\tau(f)}})\big) = f^{\mathcal{A}^*}\big(h(a_{i_1}), \ldots, h(a_{i_{\tau(f)}})\big)$ $= f^{\mathcal{A}^*}\big(a_{i_1}, \ldots, a_{i_{\tau(f)}}\big)$. Consequently, $f^{\mathcal{A}^*}\big(a_{i_1}, \ldots, a_{i_{\tau(f)}}\big) \in h(A^*) = A$.

We prove that $\mathcal{A}$ belongs to V. If $V \models \varphi(x_{i_1},\dots,x_{i_m}) \approx \psi(x_{j_1},\dots,x_{j_n})$, then $\overline{V^c} \models \varphi(q_h(x_{i_1}),\dots,q_h(x_{i_m})) \approx \psi(q_h(x_{j_1}),\dots,q_h(x_{j_n}))$. So, if $a_{i_1},\dots,a_{i_m}$, $b_{j_1},\dots,b_{j_n} \in h(A^*)$, then, since h is the identity map on $h(A^*)$, we have

$$\begin{aligned}\varphi^{\mathcal{A}}(a_{i_1},\dots,a_{i_m}) &= \varphi^{\mathcal{A}}(h(a_{i_1}),\dots,h(a_{i_m}))\\ &= \psi^{\mathcal{A}}(h(b_{j_1}),\dots,h(bj_n)) = \psi^{\mathcal{A}}(b_{j_1},\dots,b_{j_n}).\end{aligned}$$

Thus $\mathcal{A}$ belongs to V.

Let $\mathcal{A}_f = (A_f; F^{\mathcal{A}^*}|A_f)$, where $A_f = r_f(A^*)$. The algebra $\mathcal{A}_f$ is well defined for every $f \in F$. In fact,

$$\overline{V^c} \models q_f(f(x_1,\dots,x_{\tau(f)})) \approx f(q_f(x_1),\dots,q_f(x_{\tau(f)})),$$

and further we infer as for $\mathcal{A}$ and q_h. We prove that $\mathcal{A}_f$ belongs to $V_{f'}$ for every $f \in F$. Since r_f is the identity on $r_f(A^*)$, so $\mathcal{A}_f$ satisfies $q_f(x) \approx x$. Since $\mathcal{A}_f$ belongs to $\overline{V^c}$, thus, by Lemma 1.7, $\mathcal{A}_f$ belongs to V_f and satisfies $\Sigma_f(V)$.

It remains to prove (2.xiii) and (2.xiv). (2.xiv) is satisfied in $\mathcal{A}^*$, since $\mathcal{A}^*$ belongs to V^c and

$$V^c \models f(x_1,\dots,x_{\tau(f)}) \approx f(q_f(x_1),\dots,q_f(x_{\tau(f)})).$$

We prove (2.xiii). We have $h(A_f) = r_g(r_f(A_f)) = r_g(r_f(r_f(A^*))) = r_g(r_f(A^*)) = h(A^*) = A$. So (2.i) holds. Let $g \in f'$, $a_{i_1},\dots,a_{i_{\tau(g)}} \in A_f$. Then $a_{i_k} = r_f(a_{i_k})$ for $k = 1,\dots,\tau(g)$. By (2.xiv), we have

$$\begin{aligned}g^{\mathcal{A}_f}(a_{i_1},\dots,a_{i_{\tau(g)}}) &= g^{\mathcal{A}_f}(r_f(a_{i_1}),\dots,r_f(a_{i_{\tau(g)}}))\\ &= g^{\mathcal{A}_f}(r_g(r_f(a_{i_1})),\dots,r_g(r_f(a_{i_{\tau(g)}})))\\ &= g^{\mathcal{A}^*}(h(a_{i_1}),\dots,h(a_{i_{\tau(g)}})) = g^{\mathcal{A}}(h(a_{i_1}),\dots,h(a_{i_{\tau(g)}})).\end{aligned}$$

So we proved (2.iiα). If $b_{i_1},\dots,b_{i_{\tau(f)}} \in A_f$ and $\{b_{i_1},\dots,b_{i_{\tau(f)}}\} \cap A \neq \emptyset$, then for some $1 \le k \le \tau(f)$ we have $b_{i_k} \in A$. Then, since

$$\begin{aligned}\overline{V^c} \models\ & f(x_1,\dots,x_{k-1},q_h(x_k),x_{k+1},\dots,x_{\tau(f)})\\ &\approx f(q_h(x_1),\dots,q_h(x_{k-1}),q_h(x_k),q_h(x_{k+1}),\dots,q_h(x_{\tau(f)})),\end{aligned}$$

so

$$\begin{aligned}f^{\mathcal{A}_f}(b_{i_1},\dots,b_{i_{\tau(f)}}) &= f^{\mathcal{A}_f}(b_{i_1},\dots,b_{i_{k-1}},h(b_{i_k}),b_{i_{k+1}},\dots,b_{i_{\tau(f)}})\\ &= f^{\mathcal{A}_f}(h(b_{i_1}),\dots,h(b_{i_{k-1}}),h(b_{i_k}),h(b_{i_{k+1}}),\dots,h(b_{i_{\tau(f)}}))\\ &= f^{\mathcal{A}^*}(h(b_{i_1}),\dots,h(b_{i_{k-1}}),h(b_{i_k}),h(b_{i_{k+1}}),\dots,h(b_{i_{\tau(f)}}))\\ &= f^{\mathcal{A}}(h(b_{i_1}),\dots,h(b_{i_{k-1}}),h(b_{i_k}),h(b_{i_{k+1}}),\dots,h(b_{i_{\tau(f)}})).\end{aligned}$$

Thus, we proved (2.iiβ). □

THEOREM 3.3′. *Let V be a variety of type τ satisfying assumptions of Lemma* 1.1 *and condition* (3.i). *If $\mathcal{A}^*$ belongs to V^c, then $\mathcal{A}^*$ is a clone extension of an algebra $\mathcal{A}$ from V, where for every $f \in F$ the algebra $\mathcal{A}_f$ satisfies $\Sigma_f(V)$.*

This follows from Theorem 3.3 and Lemma 1.5.

THEOREM 3.4. *Let V be a variety of type τ satisfying condition* (3.i). *If an algebra $\overline{\mathcal{A}} = (\overline{A};\, F^{\overline{\mathcal{A}}})$ belongs to V_f, then $\overline{\mathcal{A}}$ is an f-clone extension of an algebra $\mathcal{A}$ belonging to V, and $\overline{\mathcal{A}}$ satisfies $\Sigma_f(V)$.*

Proof. Let $\overline{\mathcal{A}} = (\overline{A};\, F^{\overline{\mathcal{A}}})$ belongs to V_f. Since $V_f \subseteq \overline{V^c}$, so $\overline{\mathcal{A}}$ belongs to $\overline{V^c}$. As, in the proof of Theorem 3.3, we put $r_g(a) = q_g{}^{\overline{\mathcal{A}}}(a)$ for every $a \in \overline{A}$. Then, by the argumentation from the proof of Theorem 3.3, we conclude that $\overline{\mathcal{A}}$ is a clone extension of an algebra $\mathcal{A}$ from V, where $\mathcal{A}_g \in V_g$ for every $g \in F$. The identity $q_f(x) \approx x$ is of the form (1.8) and belongs to $\mathrm{Id}(V)$, so it belongs to $\mathrm{Id}(V_f)$, and consequently , it is satisfied in $\overline{\mathcal{A}}$. Hence $\overline{\mathcal{A}} = \mathcal{A}_f$. Thus $\overline{\mathcal{A}}$ satisfies $\Sigma_f(V)$. If s, t belong to $\{f\}'$, then the identity $q_s(x) \approx q_t(x)$ is of the form (1.9) and, by (3.i), belongs to $\mathrm{Id}(V)$. Thus, it belongs to $\mathrm{Id}(V_f)$, and consequently, it is satisfied in $\overline{\mathcal{A}}$. Thus $r_s(a) = r_t(a)$ for every $a \in \overline{A}$. Finally, $h = r_s \circ r_t = r_s \circ r_s = r_t \circ r_t = r_s = r_t$, so $\mathcal{A}_s = \mathcal{A}$ for every $s \in \{f\}'$. Now the proof that $\overline{\mathcal{A}} = \mathcal{A}_f$ is an f-clone extension of $\mathcal{A}$ is analogous to that of the end of Theorem 3.3. □

THEOREM 3.5. *Let V satisfy condition* (3.i). *Then an algebra $\overline{\mathcal{A}}$ belongs to V_f if and only if $\overline{\mathcal{A}}$ is an f-clone extension of an algebra $\mathcal{A}$ belonging to V, and $\overline{\mathcal{A}}$ satisfies $\Sigma_f(V)$.*

This follows from Theorems 3.1 and 3.4.

4. Equational bases

Let V be a variety of type τ satisfying condition (3.i). Let B be an equational base of V, and B_f be an equational base of $\Sigma_f(V)$ for every $f \in F$. Let B^c be a set of identities of type τ defined as follows:

(b₁) For every $f \in F$ the identity $q_f(q_f(x)) \approx q_f(x)$ belongs to B^c.
(b₂) For every $f, g \in F$ the identity $q_f(g(x_1, \ldots, x_{\tau(g)})) \approx g(q_f(x_1), \ldots$
$\ldots, q_f(x_{\tau(g)}))$ belongs to B^c.
(b₃) For every $f, g \in F$ the identity $q_f(q_g(x)) \approx q_g(q_f(x))$ belongs to B^c.

(b_4) For every $f, g, s, t \in F$, where $f \neq g$ and $s \neq t$, the identity $q_f\big(q_g(x)\big) \approx q_s\big(q_t(x)\big)$ belongs to B^c.

(b_5) If an identity $\varphi(x_{i_1}, \dots, x_{i_m}) \approx \psi(x_{j_1}, \dots, x_{j_n})$ belongs to B, then the identity $\varphi\big(q_f(q_g(x_{i_1})), \dots, q_f(q_g(x_{i_m}))\big) \approx \psi\big(q_f(q_g(x_{j_1})), \dots, q_f(q_g(x_{j_n}))\big)$ belongs to B^c for some $f, g \in F$, $f \neq g$.

(b_6) For every $f \in F$ the identity $f\big(x_1, \dots, x_{\tau(f)}\big) \approx f\big(q_f(x_1), \dots, q_f(x_{\tau(f)})\big)$ belongs to B^c.

(b_7) If an identity $\varphi(x_{i_1}, \dots, x_{i_m}) \approx \psi(x_{j_1}, \dots, x_{j_n})$ belongs to B_f, then the identity $\varphi\big(q_f(x_{i_1}), \dots, q_f(x_{i_m})\big) \approx \psi\big(q_f(x_{j_1}), \dots, q_f(x_{j_n})\big)$ belongs to B^c.

(b_8) For every $f, g \in F$, $f \neq g$ and $k = 1, \dots, \tau(f)$, the identity $f\big(x_1, \dots \dots, x_{k-1}, q_f\big(q_g(x_k)\big), x_{k+1}, \dots, x_{\tau(f)}\big) \approx f\big(q_f\big(q_g(x_1)\big), \dots, q_f\big(q_g(x_{k-1})\big), q_f\big(q_g(x_k)\big), q_f\big(q_g(x_{k+1})\big), \dots, q_f\big(q_g(x_{\tau(f)})\big)\big)$ belongs to B^c.

THEOREM 4.1. *If V is a variety satisfying condition* (3.i), *then B^c is an equational base of $\overline{V^c}$.*

Proof. Let us denote by $C(V)$ the class of all clone extensions of algebras from V, where for every $f \in F$ the algebra $\mathcal{A}_f$ satisfies $\Sigma_f(V)$. Put $V_* = \mathrm{Mod}(B^c)$. Since every identity from B^c belongs to $\mathrm{Id}\big(\overline{V^c}\big)$, so $B^c \subseteq \mathrm{Id}\big(\overline{V^c}\big)$, hence $\overline{V^c} \subseteq V_*$. To complete the proof, it is enough to show that if $\mathcal{A}^* = \big(A^*;\, F^{\mathcal{A}^*}\big)$ belongs to V_*, then it belongs to $C(V)$, see Theorem 3.2. Let $\mathcal{A}^* \in V_*$. As in the proof of Theorem 3.3, we put $r_f(a) = {q_f}^{\mathcal{A}^*}(a)$ for every $a \in A^*$. By (b_1)–(b_4), condition (2.iv) is satisfied. We put $h(a) = q_f^{\mathcal{A}^*}\big(q_f^{\mathcal{A}^*}(a)\big)$ for every $a \in A^*$ and some $f, g \in F$ with $f \neq g$. Then h is a retraction of $\mathcal{A}^*$ by (b_1) and (b_2). We put $\mathcal{A} = \Big(h(A^*);\, F^{\mathcal{A}^*}\big|h(A^*)\Big)$. We prove that $\mathcal{A} \in V$. If an identity $\varphi(x_{i_1}, \dots, x_{i_m}) \approx \psi(x_{j_1}, \dots, x_{j_n})$ belongs to B and $a_{i_1}, \dots, a_{i_m}, a_{j_1}, \dots, a_{j_n}$ belong to $h(A^*)$, then since h is an identity on $h(A^*)$, we have by (b_5) the equalities

$$\begin{aligned}\varphi^{\mathcal{A}}(a_{i_1}, \dots, a_{i_m}) &= \varphi^{\mathcal{A}^*}\big(h(a_{i_1}), \dots, h(a_{i_m})\big)\\ &= \psi^{\mathcal{A}^*}\big(h(a_{j_1}), \dots, h(a_{j_n})\big) = \psi^{\mathcal{A}}(a_{j_1}, \dots, a_{j_n}).\end{aligned}$$

Thus $\mathcal{A}$ satisfies B and, consequently, belongs to V. Similarly, using (b_2) and (b_7) we show that $\mathcal{A}_f = \Big(r_f(A^*);\, F^{\mathcal{A}^*}\big|r_f(A^*)\Big)$ satisfies B_f, so it satisfies $\Sigma_f(V)$. By (b_6), condition (2.xiv) is satisfied. Similarly as in the proof of Theorem 3.3, we show that since (b_6) and (b_8) belong to B^c, so $\mathcal{A}_f$ is the f-clone extension of $\mathcal{A}$ with respect to $h\big|A_f$. □

THEOREM 4.1′. *If V is a variety satisfying assumptions of Theorem* 3.3′, *then B^c is a equational base of V^c.*

This follows from Lemma 1.5 and Theorem 4.1.

COROLLARY 4.2. *If a variety V satisfies assumptions of Theorem 4.1, F is finite, V is finitely based, and $\Sigma_f(V)$ has a finite base for every $f \in F$, then $\overline{V^c}$ is finitely based.*

COROLLARY 4.2′. *If a variety V satisfies assumptions of Theorem 4.1′, F is finite, V is finitely based, and $\Sigma_f(V)$ has a finite base for every $f \in F$, then V^c is finitely based.*

Let V be a variety of type τ satisfying assumption of Theorem 3.5. Let B be an equational base of V, and B_f be an equational base of $\Sigma_f(V)$. We define the set B^f of identities as follows:

(c_1) $B_f \subseteq B^f$;
(c_2) $q_g(x) \approx q_s(x)$ belongs to B^f for every $g, s \in \{f\}'$;
(c_3) $q_g(q_g(x)) \approx q_g(x)$ belongs to B^f for every $g \in \{f\}'$;
(c_4) $q_g(s(x_1, \ldots, x_{\tau(s)})) \approx s(q_g(x_1), \ldots, q_g(x_{\tau(s)}))$ belongs to B^f for every $g \in \{f\}'$, $s \in F$;
(c_5) if $\varphi(x_{i_1}, \ldots, x_{i_m}) \approx \psi(x_{j_1}, \ldots, x_{j_n})$ belongs to B, then the identity $\varphi(q_g(x_{i_1}), \ldots, q_g(x_{i_m})) \approx \psi(q_g(x_{j_1}), \ldots, q_g(x_{j_n}))$ belongs to B^f for some $g \in \{f\}'$;
(c_6) for every $g \in \{f\}'$ the identity $g(x_1, \ldots, x_{\tau(g)}) \approx g(q_g(x_1), \ldots, q_g(x_{\tau(g)}))$ belongs to B^f;
(c_7) for $g \in \{f\}'$ the identities $f(x_1, \ldots, x_{k-1}, q_g(x_k), x_{k+1}, \ldots, x_{\tau(f)}) \approx f(q_g(x_1), \ldots, q_g(x_{k-1}), q_g(x_k), q_g(x_{k+1}), \ldots, q_g(x_{\tau(f)}))$ belong to B^f.

THEOREM 4.3. *If a variety V satisfies assumption of Theorem 3.5, B is an equational base of V, and B_f is an equational base of $\Sigma_f(V)$, then B^f is an equational base of V_f.*

P r o o f. Denote by V^f the variety defined by B^f. Since every identity from B^f belongs to $\mathrm{Id}(V_f)$, so $V_f \subseteq V^f$. Denote by $C_f(V)$ the class of all f-clone extensions of algebras from V satisfying $\Sigma_f(V)$. Now, by Theorem 3.1, it is enough to prove that if an algebra $\overline{\mathcal{A}} = (\overline{A};\ F^{\overline{\mathcal{A}}})$ belongs to V^f, then it belongs to $C_f(V)$. But $\overline{\mathcal{A}}$ satisfies $\Sigma_f(V)$ by (c_1). We put, for $a \in \overline{A}$, $r(a) = q_g^{\overline{\mathcal{A}}}(a)$ for fixed $g \in \{f\}'$. By (c_2), r is well defined, and it is idempotent by (c_3). We put $\mathcal{A} = \left(r(\overline{A});\ F^{\overline{\mathcal{A}}}|_{r(\overline{A})}\right)$, and $r(\overline{A}) = A$. By (c_4), r is a retraction, so $\mathcal{A}$ is well defined (the proof of Theorem 3.3).

If $\varphi(x_{i_1}, \ldots, x_{i_m}) \approx \psi(x_{j_1}, \ldots, x_{j_n})$ belongs to B, then by (c_5), $\varphi(q_g(x_{i_1}), \ldots$ $\ldots, q_g(x_{i_m})) \approx \psi(q_g(x_{j_1}), \ldots, q_g(x_{j_n}))$ holds in $\overline{\mathcal{A}}$. So, by (c_2) and (c_3), for $a_{i_1}, \ldots, a_{i_m}, b_{j_1}, \ldots, b_{j_n} \in A$ we have $\varphi^{\mathcal{A}}(a_{i_1}, \ldots, a_{i_m}) = \varphi^{\overline{\mathcal{A}}}(r(a_{i_1}), \ldots, r(a_{i_m}))$

$= \psi^{\overline{\mathcal{A}}}(r(b_{j_1}), \dots, r(b_{j_n})) = \psi^{\mathcal{A}}(b_{j_1}, \dots, b_{j_n})$. Thus $\mathcal{A} \in V$ and (2.i) holds. If $g \in \{f\}'$, then by (c_6) and (c_2), it is easy to show that (2.iiα) holds. Similarly, by (c_7), we prove that (2.iiβ) holds for $\overline{\mathcal{A}}$. Consequently, $\overline{\mathcal{A}}$ belongs to $C_f(V)$, what completes the proof. □

Corollary 4.4. *If a variety V satisfies assumptions of Theorem 3.5, F is finite, V is finitely based, and $\Sigma_f(V)$ is finitely based, then V_f is finitely based.*

5. Examples

Example 5.1. Let V be a variety of bisemilattices, i.e., the variety of algebras of type $\tau\colon \{+, \cdot\} \to \mathbb{N}$, where $\tau(+) = \tau(\cdot) = 2$ and both $+$ and $\cdot$ is idempotent, symmetrical and associative. Put $q_+(x) = x + x$ and $q_.(x) = x \cdot x$. By Theorem 3.3′, an algebra $\mathcal{A}$ belongs to V^c if and only if $\mathcal{A}$ is a clone extension of a bisemilattice $\mathcal{B}$, where the following holds. The algebra $\mathcal{B}_+$ is the $+$-clone extension of $\mathcal{B}$ with respect to $h|A_+$, and $+$ is a join semilattice operation in $\mathcal{B}_+$. The algebra $\mathcal{B}_.$ is the $\cdot$-clone extension of $\mathcal{B}$ with respect to $h|A_.$, and $\cdot$ is a meet semilattice operation in $\mathcal{B}_.$. By Corollaries 4.2′ and 4.4, V_+, $V_.$ and V^c is finitely based if V is finitely based.

In particular, if V is degenerated variety of bisemilattices (it satisfies $x \approx y$), then $\mathcal{B}$ is 1-element and $\mathcal{B}_+ = \mathcal{B}_. = \mathcal{B}$. Further, for every $a, b \in \mathcal{A}$ we have $a + b = a \cdot b = c$, where c is the only element of $\mathcal{B}$.

Example 5.2. Let V be a variety of Boolean algebras. Then we obtain the similar conclusions as for lattices, where $q_+(x) = x + x$, $q_.(x) = x \cdot x$ and $q_{\prime}(x) = (x')'$. $\mathcal{B}$ is a Boolean algebra, $\mathcal{A}$ is a clone extension of $\mathcal{B}$, where $\mathcal{B}_+$, $\mathcal{B}_.$ are described as in Example 5.1, and in $\mathcal{B}_{\prime}$ the operation $'$ is an involution, $\Sigma_+(V)$, $\Sigma_.(V)$, $\Sigma_{\prime}(V)$ are finitely based, and V^c is finitely based since V is finitely based.

Example 5.3. Let V be a variety of groups with operations $\cdot$ and $^{-1}$ satisfying $x^n \approx y^n$. Put $q_.(x) = x^{n+1}$ and $q_{-1}(x) = (x^{-1})^{-1}$. Then we obtain analogous results as in Examples 5.1 and 5.2. In particular, V^c is finitely based. Moreover, if $\mathcal{B}$ belongs to V^c, then it is a clone extension of a group $\mathcal{A}$ by means of a family $\{\mathcal{A}_., \mathcal{A}_{-1}\}$, where $\mathcal{A}_. = \mathcal{A}$. In fact, by Corollary 1.10, we have $V_. = V$, so $V_. \models (x^{-1})^{-1} \approx x^{n+1} \approx x$, and consequently, $r_.$ and r_{-1} are identities in V. Now, by (2.xvii) and (2.vi), $\mathcal{A}_. = \mathcal{A}$.

Example 5.4. Let $\tau\colon \{\oplus, \odot\} \to \mathbb{N}$ be a type of algebras with $\tau(\oplus) = \tau(\odot) = 2$. Let V be the variety of algebras of type τ generated by $Z_p = (\{0, 1, \dots, p-1\}; \oplus, \odot)$, where $\oplus$ is the addition modulo p, $\odot$ is the multiplication modulo p, and p is prime. Then V is a nontrivial variety of rings satisfying $(p+1)x \approx x \approx x^p$. Consequently, assumptions of Theorems 3.3′ and 4.1′ are satisfied and V^c is finitely based.

EXAMPLE 5.5. Let V be a variety of some type τ satisfying (3.i), where $|F| = 2$. Let V_r be the variety of type τ defined by all regular identities from $\mathrm{Id}(V)$. Then V_r also satisfies (3.i), and we can apply Theorems 3.3′ and 4.1′. This observation we can apply to Examples 5.1, 5.3 and 5.4.

Obviously not every variety satisfies assumptions of Lemma 1.1. However, if it satisfies condition (3.i), then we can apply Theorem 3.3, Theorem 4.1 and Corollary 4.2 as in the following example for $n > 2$.

EXAMPLE 5.6. Let $\tau\colon \{\circ_1, \dots, \circ_n\} \to \mathbb{N}$ be a type of algebras with $\tau(\circ_k) = 2$ for $k = 1, \dots, n$ and $2 \le n < \omega$. Let L_n be the variety of n-lattices (see [5]), i.e., the variety of type τ defined by the following identities: $x \circ_k x \approx x$, $x \circ_k y \approx y \circ_k x$, $(x \circ_k y) \circ_k z \approx x \circ_k (y \circ_k z)$ for $k = 1, \dots, n$, and $x \circ_{i_1} (x \circ_{i_2} (\dots (x \circ_{i_n} y) \dots))$ $\approx x$ for every permutation $i_1, \dots, i_n$ of indices $1, \dots, n$. By Theorem 3.3 and Corollary 4.2, the variety $\overline{L}_n$ is finitely based, and every algebra from $\overline{L}_n$ can be represented as a clone extension of an algebra from L_n.

Some other results concerning clone compatible identities will be published in the future.

REFERENCES

[1] GRACZYŃSKA, E.: *On regular identities*, Algebra Universalis **17** (1983), 369–375.

[2] HAŁKOWSKA, K.: *On some operator defined on equational classes of algebras*, Arch. Math. (Brno) **12** (1976), 209–212.

[3] MEĽNIK, J. J.: *Nilpotent shifts of varieties*, Mat. Zametki **14(5)** (1973) [English translation: Math. Notes **14** (1973), 962–966]. (Russian)

[4] PŁONKA, J.: *On a method of construction of abstract algebras*, Fund. Math. **61** (1966), 183–189.

[5] PŁONKA, J.: *On distributive n-lattices and n-quasilattices*, Fund. Math. **62** (1968), 293–300.

[6] PŁONKA, J.: *On the subdirect product of some equational classes of algebras*, Math. Nachr. **63** (1974), 303–305.

[7] PŁONKA, J.: *Biregular and uniform identities of bisemilattices*, Demonstratio Math. **20** (1987), 95–107.

[8] PŁONKA, J.: *On varieties of algebras defined by identities of some special forms*, Houston J. Math. **14** (1988), 253–263.

[9] PŁONKA, J.: *Biregular and uniform identities of algebras*, Czechoslovak Math. J. **40(115)** (1990), 367–387.

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