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CONSTRUCTING REGULAR MAPS AND GRAPHS FROM PLANAR QUOTIENTS

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ABSTRACT. Let M be a map on an orientable surface. The generic regular map for M is, up to isomorphism, the unique regular map $M^{\#}$ such that $M^{\#}$ covers M and every regular map that covers M covers also $M^{\#}$. In this paper, we show that several interesting results concerning maps on surfaces and graphs can be established by constructing generic maps over appropriate quotients. Among them are simple proofs of theorems of Vince, MacBeath, and generalizations of results of Brown and Connelly, Archdeacon, and others. Using the same method we also show that for every integer $g \geq 3$ there exists an arc-transitive cubic graph whose girth equals g.

1. Introduction

The present paper deals with constructing highly symmetrical maps on closed surfaces by employing covering space techniques, a method which has already proved to be very fruitful. Various techniques have been used in explicit construction of coverings. These include voltage assignments (see Gross and Tucker [11; Chapters 2 and 4] for a detailed treatment of this concept), surgery, and others.

Here, we develop a different approach to the construction of symmetrical coverings. It is based on the well-known fact that every map can be covered by a regular map, one exhibiting the highest level of symmetry [14; Theorem 6.7]. Among regular maps that cover a given map, there is a universal smallest map which we call the *generic regular map*. Its characteristic property is that every regular map which covers the given map covers also its generic map.

We show in this paper that several important results in theory of maps on surfaces and in graph theory can be proved or reproved by constructing a generic

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regular map over an appropriate quotient map. For instance, Grünbaum's conjecture claiming that for every $p \geq 2$ and $q \geq 2$ there is a p-gonal orientable regular map with q-valent vertices (established by Vince [27] in 1983) can be proved by simply drawing certain trees in the plane. A similar proof with slightly more complicated base maps can be given for a result of MacBeath [16] about the existence of Hurwitz groups. We further employ generic maps to construct regular triangulations without short non-contractible cycles. Such maps are sometimes called "dense" and are of interest in topological graph theory. Corollaries of this result improve and generalize constructions of Brown and Connelly [5], and Archdeacon [1]. In addition, the same result implies that for any $g \geq 3$ there is an arc-transitive cubic graph whose girth is equal to g.

The generic map is usually constructed from an algebraic rather than combinatorial or geometric representation of the given map. It turns out, however, that a relatively small base map may have extremely large generic covering, and it is very difficult to control the size of the generic map by an appropriate choice of the base. Therefore, the value of our approach may be seen in proofs of existence results which use *dessins d'enfants* (children drawings, the term introduced by Grothendieck [12] for maps on surfaces to reflect their simplicity and concreteness) rather than in explicit constructions.

2. Definitions

Graphs considered in this paper are finite, non-trivial, connected, and may have both multiple edges and loops. For technical reasons, we allow our graphs to contain also semiedges, that is, edges that have one end-vertex and one free end.

Formally, a *graph* is a quadruple $G = (D, V; I, L)$, where $D = D(G)$ and $V = V(G)$ are non-empty finite sets, $I\colon D \to V$ is a surjective mapping, and $L = L_G$ is an involutory permutation on D. The elements of D and V are *arcs* and *vertices*, respectively, I is the incidence function assigning to every arc its *initial vertex*, and L is the *arc-reversing involution*; the orbits of the group $\langle L \rangle$ on D are *edges* of G. If an arc x is a fixed point of L, that is, $L(x) = x$, then the corresponding edge is a *semiedge*. If $IL(x) = I(x)$ but $L(x) \neq x$, then the edge is a *loop*. The remaining edges are *links*. Our graphs are thus essentially the same as those in Jones and Singerman [14]. Topologically they can be viewed as finite 1-dimensional cell complexes in which semiedges are identical with pendant edges except that the pendant point of a semiedge is not listed as vertex.

The usual graph-theoretical concepts such as cycles, connectedness, etc., easily translate to the present formalism. In particular, the *valency* of a vertex v is the number of arcs having v as their initial vertex.

A *map* is a connected topological graph cellularly embedded in some closed surface. In this paper, we only consider maps on orientable surfaces. We often replace a map by its combinatorial description. By a (combinatorial) *oriented map* we henceforth mean a pair (G, R), where G is a connected graph and R is a permutation of $D(G)$ called *rotation*, cyclically permuting arcs with the same initial vertex, that is, $IR(x) = I(x)$ for every $x \in D(G)$. Alternatively, one can describe a map M by specifying the arc-set $D = D(M)$, the rotation $R = R_M$, and the arc-reversing involution $L = L_M$. The vertices of the underlying graph $G = G_M$ of M then can be identified with the orbits of the group $\langle R\rangle$, and the incidence function I with the mapping which assigns to every arc of M the orbit of $\langle R\rangle$ it belongs to. The edges of G_M correspond to the orbits of $\langle L\rangle$, and the face-boundaries correspond to the orbits of $\langle RL\rangle$. The connectedness of G is guaranteed by the transitive action of the permutation group $\mathrm{Mon}(M) = \langle R, L\rangle$, the *monodromy group* of M, on the set $D(M)$. In this case, we write $M = (D;\, R, L)$.

As we deal with maps on orientable surfaces only, we usually omit the adjective "oriented".

Let $M_1 = (D_1;\, R_1, L_1)$ and $M_2 = (D_2;\, R_2, L_2)$ be two maps. A *map homomorphism* is a mapping $\varphi\colon D_1 \to D_2$ such that

$$\varphi R_1 = R_2 \varphi \quad \text{and} \quad \varphi L_1 = L_2 \varphi;$$

we write $\varphi\colon M_1 \to M_2$ to denote this homomorphism. Transitive actions of $\mathrm{Mon}(M_1)$ and $\mathrm{Mon}(M_2)$ ensure that every map homomorphism is surjective, and that it also induces a homomorphism of the underlying graphs. Topologically speaking, a map homomorphism is a graph preserving branched covering of the supporting oriented surfaces with branch points possibly at vertices, face centres, or free ends of semiedges. Therefore, we can say that a map $\tilde{M}$ *covers* M if there is a homomorphism $\tilde{M} \to M$.

With map homomorphisms we use also isomorphisms and automorphisms. In particular, the automorphism group $\mathrm{Aut}(M)$ of a map $M = (D;\, R, L)$ consists of all permutations in the full symmetry group $S(D)$ of D which commute with both R and L. It is well known that $|\mathrm{Aut}(M)| \le |D(M)|$ for every map M (see, for example, [14]). If $|\mathrm{Aut}(M)| = |D(M)|$, then the map M is called *regular*.

A map $M = (D;\, R, L)$ is called *reflexible* if it is isomorphic to its *mirror image* $(D;\, R^{-1}, L)$; an isomorphism $\alpha\colon (D;\, R, L) \to (D;\, R^{-1}, L)$ is called a *reflection* of M. Topologically speaking, map automorphisms preserve the chosen orientation of the underlying surface whereas reflections reverse it.

Our use of the term "regular map" coincides with that of Jones and Singerman [14], and Coxeter and Moser [7]. The same term is sometimes used in a slightly stronger meaning (for example in [27]), in which case "rotary map" or "orientably-regular map" represent the weaker concept.

3. Schreier representations and generic maps

Let H be a finite group generated by two elements r and l, where l is an involution (possibly trivial), and let S be a subgroup of H. Then we can construct a map $A(H/S;\, r,l) = (D';\, R',L')$ by taking the arc-set D' to be the set H/S of left cosets of S in H, and setting

$$R'(hS) = rhS\,,$$
$$L'(hS) = lhS\,,$$

for every arc $hS \in D' = H/S$. It is obvious that the monodromy group of the resulting map is a homomorphic image of H.

If $A(H/S;\, r,l)$ is isomorphic to a map M, then $A(H/S;\, r,l)$ is called a *Schreier representation* of M. On the other hand, given a map $M = (D;\, R,L)$, it is not difficult to find a Schreier representation for it. Indeed, let $H = \mathrm{Mon}(M) = \langle R,L\rangle$, let S be the stabilizer of a fixed arc $z \in D$ under the action of $\mathrm{Mon}(M)$ on D, and let $r = R$ and $l = L$. We show that there is a unique isomorphism $\lambda\colon M \to A(H/S;\, r,l)$ which takes the distinguished arc z to the coset S. This isomorphism is constructed as follows. Set $\lambda(z) = S$ and for any other arc x choose an arbitrary element $w_x \in \mathrm{Mon}(M)$ such that $w_x(z) = x$; then set $\lambda(x) = w_xS$. The labelling λ is well defined since for any two elements w and w' of $\mathrm{Mon}(M)$ with $w(z) = x = w'(z)$ we have $wS = w'S$. Moreover, $\lambda(Rx) = R\lambda(x)$ and $\lambda(Lx) = L\lambda(x)$ for any arc x of M. Hence, λ is the required isomorphism.

Consider a Schreier representation $A(H/S;\, r,l)$ of a map $M = (D;\, R,L)$, and assume that $T \le S \le H$. Then there is a natural projection $\pi\colon H/T \to H/S$, $hT \mapsto hS$, hT being an arbitrary coset of T in H. This mapping is in fact a map homomorphism $A(H/T;\, r,l) \to A(H/S;\, r,l)$. An important special case of this situation occurs when $T = 1$. In this case, $A(H/1;\, r,l)$ is a regular map. To see this, consider, for any element $h \in H$, the *right* translation τ_h of H by h, that is, the mapping given by the assignment $\tau_h\colon x \mapsto xh$ for any $x \in H$. Since τ_h commutes with both the rotation and the arc-reversing involution of $A(H/1;\, r,l)$, τ_h is an automorphism of $A(H/1;\, r,l)$. It follows that $\bigl|\mathrm{Aut}\bigl(A(H/1;\, r,l)\bigr)\bigr| \ge |H| = \bigl|D\bigl(A(H/1;\, r,l)\bigr)\bigr|$, so $A(H/1;\, r,l)$ is a regular map. It is easy to see that $\mathrm{Mon}\bigl(A(H/1; r,l)\bigr)$ coincides with H acting on itself by the *left* translation.

Let us return to the isomorphism $\lambda\colon M \to A\bigl(\mathrm{Mon}(M)/S; R,L\bigr)$, where $M = (D;\, R,L)$ is an arbitrary map, and S is the stabilizer of an arc $z \in D$

under the action of $\mathrm{Mon}(M)$ on D. If we identify M with $A(\mathrm{Mon}(M)/S;\ R, L)$ and set $M^{\#} = A(\mathrm{Mon}(M);\ R, L)$, we obtain a natural projection $\pi\colon M^{\#} \to M$, where $M^{\#} = A(\mathrm{Mon}(M);\ R, L)$ is a regular map with $\mathrm{Mon}(M^{\#}) = \mathrm{Mon}(M)$. In some sense, it is the smallest regular map for which there is a map homomorphism onto M. For if $\varphi\colon \tilde{M} \to M$ is a map homomorphism, then there is a homomorphism $\varphi'\colon \tilde{M} \to M^{\#}$ such that the diagram

$$\begin{tikzcd} \tilde{M} \arrow[rr, "\varphi'"] \arrow[dr, "\varphi"'] & & M^{\#} \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes. We call $M^{\#}$ the *generic regular map* for M.

It is the purpose of this paper to show that generic regular maps provide a very useful means both in map theory and graph theory. Further results about generic regular maps can be found in [21].

4. Maps of prescribed type

A map M is said to have *pattern* $(p_1, p_2, \dots, p_m; q_1, q_2, \dots, q_n)$ if the set of face-sizes of M is $\{p_1, p_2, \dots, p_m\}$, and the set of vertex-valencies of M is $\{q_1, q_2, \dots, q_n\}$. The *type* of M is the pair (p, q), where $p = \mathrm{lcm}\{p_1, p_2, \dots, p_m\}$ and $q = \mathrm{lcm}\{q_1, q_2, \dots, q_n\}$.

In 1976, Grünbaum [13] asked if for every pair of positive integers p and q with $1/p+1/q < 1/2$ (that is, in the hyperbolic case) there are infinitely many finite regular maps of type (p, q). He also remarked, however, that it was not even known whether for such p and q there was at least one map of that type. The question was answered in the affirmative by Vince [27] (1983) within a more general framework. His proof, based on a theorem of Mal'cev that every finitely generated matrix group is residually finite (see, for example, [15]), was highly non-elementary and non-constructive. Constructive proofs of Vince's theorem were subsequently given by Wilson and Gray and Wilson [10], [28], [29] along with some refinements.

One can observe that if M is a regular map of a hyperbolic type (p, q), then infinitely many regular maps of the same type can be constructed by covering space techniques, for example, by employing Surowski's voltage assignments which take values in 1-dimensional $\mathbb{Z}_n$-homology groups ([26]). Thus the crucial step is to construct at least one regular map of each given type (p, q). From this point of view, Vince's result can be derived from a theorem of Fox [8] (1952) saying that for any three given integers $a > 1$, $b > 1$, and $c > 1$ there can be found a permutation α of order a and a permutation β of order b such that c is the order of $\alpha\beta$. Indeed, if $b = 2$ and if α and β are the corresponding permutations, then the $A(\langle\alpha,\beta\rangle;\alpha\beta,\beta)$ is a regular map of type (a,c).

We establish the existence of regular maps of any hyperbolic type by explicitly constructing the permutations α and β in a very simple manner – by "drawing" certain trees on the plane and deriving the permutations from the resulting "dessin d'enfants". The striking feature of this proof is a systematic use of semiedges.

We need two simple lemmas.

LEMMA 1. *Let M be an oriented map with pattern $(p_1,p_2,\dots,p_m;q_1,q_2,\dots \dots,q_n)$. Then the generic regular map $M^\#$ for M has all faces p-gonal and all vertices q-valent, with $p = \operatorname{lcm}\{p_1,p_2,\dots,p_m\}$ and $q = \operatorname{lcm}\{q_1,q_2,\dots,q_n\}$. Consequently, the type of $M^\#$ is equal to the type of M.*

P r o o f. Let $M = (D;\, R,L)$. Clearly, all the vertices in $M^\# = (D';\, R',L')$ have the same valency q which, by the definition of $M^\#$, is equal to length of any cycle of R'. Since the cycles of R' have the form $(x,Rx,R^2x,\dots)$ for some $x \in D'$, q equals the order of R, that is, the least common multiple of the lengths of cycles in R. The length of any cycle of R coincides with the valency of the corresponding vertex, and so, $q = \operatorname{lcm}\{q_1,q_2,\dots,q_m\}$. The proof that $p = \operatorname{lcm}\{p_1,p_2,\dots,p_n\}$ is similar. □

The *n-semistar* Ss_n is the graph consisting of a single vertex and n incident semiedges.

LEMMA 2. *The only regular maps containing semiedges are the n-semistars Ss_n embedded into the sphere.*

THEOREM 3. (V i n c e) *For every pair of integers $p \geq 2$ and $q \geq 2$ there exists an orientable regular map of type (p,q) not containing semiedges.*

Remark. The statement of Theorem 3 can be slightly strengthened. It is actually sufficient to forbid the pairs $(p,1)$ and $(1,p)$ where $p > 2$.

P r o o f. Since the dual of a map of type (p,q) is a map of type (q,p) and is regular if and only if the original map is regular, we may assume that $q \geq p \geq 2$. We consider two cases.

Case 1: $p = q$. Let $M = (D;\, R,L)$ be a map obtained from the embedding of a pair of parallel edges in the sphere by adding $p-2$ semiedges to each vertex in such a way that both the inner and the outer face contain exactly $p-2$ of them on the boundary (see Figure 1). Clearly, the size of both faces and the valency of both vertices is p, so M has type (p,p). By Lemma 1, the generic regular map over M has the same type (p,p). As $M^\#$ has at least two vertices, Lemma 2 shows that it has no semiedges. This concludes Case 1.

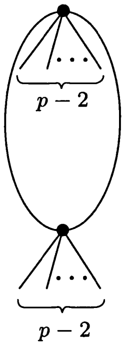

FIGURE 1.

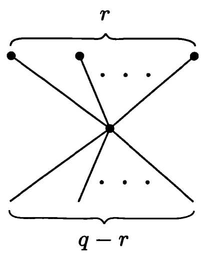

FIGURE 2.

Case 2: $p > q$. In this case, there exist integers $k \geq 1$ and $r \geq 0$, $r < q$, such that $p = kq + r$. Let P be a path on k vertices with end-vertices u and v.

If $k = 1$, then $u = v$, and $r > 0$ because $p \neq q$. Form a tree T by attaching r pendant links and $q - r$ semiedges to v (see Figure 2). Clearly, T has r vertices of valency 1 and a single vertex of valency q; the number of arcs in T is therefore $2r + (q - r) = p$. Any spherical embedding of T has a unique face of length p, and so, it gives rise to a map with pattern $(p; q, 1)$. By Lemma 1, the generic regular map has type (p, q), and since $p > q$, it has no semiedges.

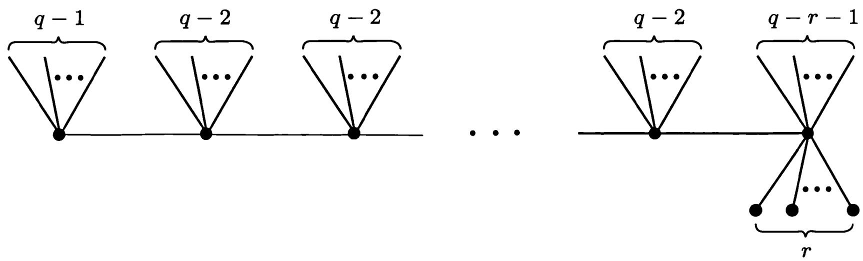

FIGURE 3.

If $k \geq 2$, then $u \neq v$. We form a tree by attaching $q - 1$ semiedges to u, $q - 2$ semiedges to each inner vertex of P, and r pendant links and $q - r - 1$ semiedges to v (see Figure 3). The resulting tree T has every vertex of valency q or 1, and the total number of arcs in T is the valency sum, that is,

$$q + (k-2)q + (1 + r + q - r - 1) + r = kq + r = p.$$

Since any spherical embedding of T has a unique face of length p, it produces a map with pattern $(p; q, 1)$. As before, the required regular map of type (p, q) is obtained applying Lemma 1. The regular map has no semiedges because $p > q$. □

A very similar proof has been independently found by Archdeacon, Gvozdjak and Širáň [2] (this issue p. 128). Their lifting argument is, nevertheless, slightly different: the base maps are endowed with a canonical voltage assignment which takes values in the monodromy group and subsequently lifted to the required regular maps.

In our approach to the construction of maps of an arbitrary type (p,q), it is possible to control the reflexibility of the resulting regular maps (cf. Gray and Wilson [10; p. 30–31]). This is enabled by taking an alternative approach to combinatorial description of maps. It is well known that a topological map M can be described by means of three involutions l, r and t acting on "flags" of M, mutually incident (vertex, edge, face) triples (see, for example, [2], [9]). The group $\langle l,r,t\rangle$ can be viewed as an "extended" monodromy group, and hence, the idea of construction of the generic map for M can be applied. The new map obtained in this way is the smallest reflexible regular map M^+ such that there exists a map homomorphism $M^+ \to M$. It is the *reflexible generic regular map* for M. In general, the maps M^+ and $M^\#$ are not isomorphic.

THEOREM 3′. *For every pair of integers $p \geq 2$ and $q \geq 2$ there exists a reflexible regular map of type (p,q) not containing semiedges.*

Proof. It is sufficient to interpret the topological maps M constructed in the proof of Theorem 3 by means of three involutions and apply the construction of M^+. □

5. Triangulations of given valency and arbitrarily large face-width

Let M be a map formed by embedding a graph G in a closed surface S other than the sphere. The *face-width* of M, $\mathrm{fw}(M)$, is the minimum of $|C \cap G|$ taken over all non-contractible (that is to say, homotopically non-null) simple closed curves C in S, and the *edge-width* of M, $\mathrm{ew}(M)$, is the length of a shortest non-contractible cycle in G. Face-width and edge-width have recently received wide attention due to their importance for the study of graph embeddings. This is well documented in Mohar's survey paper [17] published in this issue.

Obviously, $\mathrm{ew}(M) \geq \mathrm{fw}(M)$ for every map M. It is also easy to see that $\mathrm{ew}(M) = \mathrm{fw}(M)$ if M is a triangulation, and that the face-width of any map M equals the face-width of its dual M^* (see [17; Proposition 3.2]). In [1], maps where any of these invariants is large enough are called dense in order to reflect the fact that they measure how densely a graph is embedded on a surface.

Our next aim is to show that many remarkable dense maps can be constructed by our generic regular map construction. In particular, our method is employed to produce regular k-valent triangulations of arbitrarily large face-width. This result has a number of corollaries which reprove, improve or generalize several known results.

The following lemma can be proved by using elementary arguments from algebraic topology.

LEMMA 4. *Let $\psi\colon \tilde{S} \to S$ be a branched covering of surfaces. Let $D \subseteq S$ be an open disk which contains no branching point of ψ, and let $\tilde{D}$ be any connected component of $\psi^{-1}(D)$. Then the restriction $\psi|_{\tilde{D}} : \tilde{D} \to D$ is a homeomorphism.*

Now we are ready to prove Theorem 5.

THEOREM 5. *For every $k \geq 6$ and every $d \geq 1$ there exists a k-valent regular triangulation of some orientable surface whose face-width is $\geq d$.*

Proof. It is easy to construct 6-valent triangulations of the torus with arbitrarily large face-width, so we may assume that $k \geq 7$. Let T_k be the infinite regular k-valent triangulation of a hyperbolic plane, and let T be the triangulation of a disk D arising from T_k by taking the part of T_k induced by the vertices at distance not greater than d from a specified vertex v. We form a spherical map M_0 from two copies T' and T'' of T with different orientation by glueing their boundary circles identically. The cycle of M_0 that arises will be called the *equator*. The corresponding copies v' and v'' of the vertex v will be the *poles* of M_0. Obviously, M_0 is a spherical triangulation. It is k-valent apart from the vertices lying on the equator, where a sequence of $k-5$ consecutive vertices of valency 4 alternates with a single vertex of valency 6.

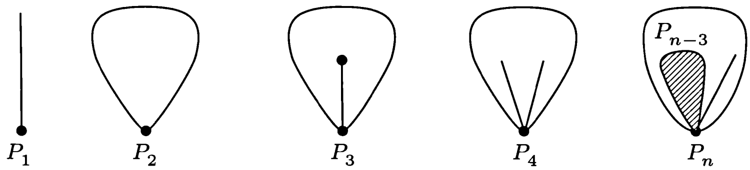

FIGURE 4.

Next we modify the triangulation M_0 by amending the valencies on the equator. The modification will result in a k-valent spherical map $M_{k,d}$ where all the faces are either 1-gons or 3-gons. In order to do this, consider the following sequence (P_n) of planar maps. The maps P_1, P_2, P_3 and P_4 are depicted in Figure 4. For $n \geq 5$, the map P_n is obtained from P_{n-3} by adding a semiedge and a loop as shown in Figure 4. Every P_n has a vertex of valency n, and at most one other vertex of valency 1. The vertex of valency n is incident with the loop bounding the outer face of P_n. We form $M_{k,d}$ as follows. If $k \geq 9$, we first replace every edge on the equator by two parallel edges. Inside every digon, we draw the map P_{k-8} or P_{k-6} depending on whether the "left" vertex in the digon has valency 8 or 6, respectively (see Figure 5). The cases $k = 7$ and $k = 8$ have to be solved separately; the solution is indicated in Figures 6 and 7.

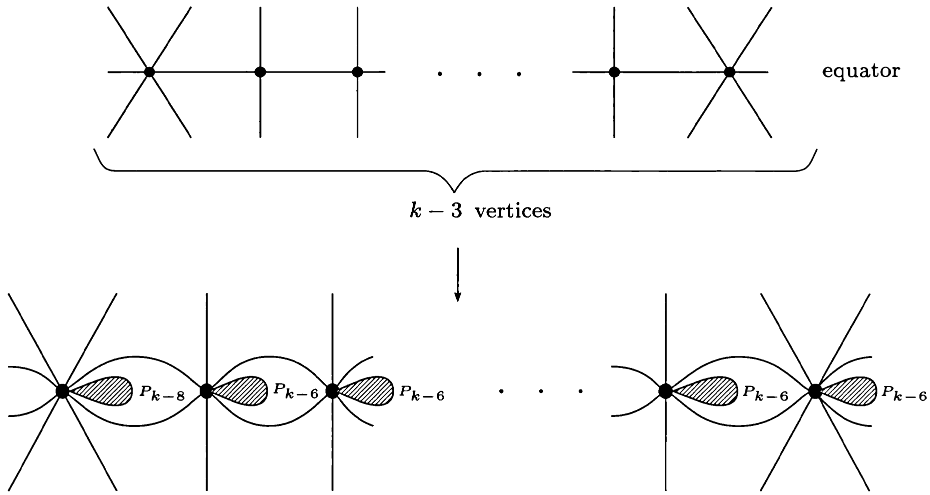

FIGURE 5. $k \geq 9$.

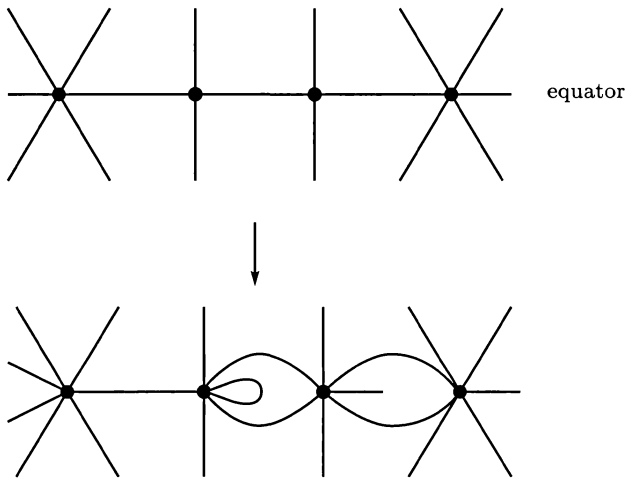

FIGURE 6. $k = 7$.

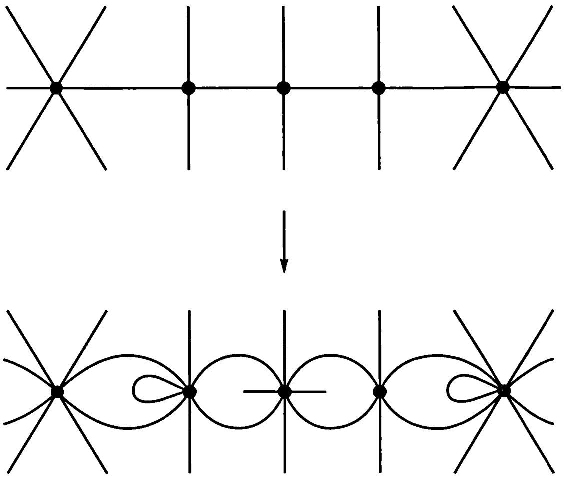

FIGURE 7. $k = 8$.

We claim that the generic triangulation $M^{\#}_{k,d}$ for $M_{k,d}$ has face-width at least d. Suppose this is not the case. Since $\mathrm{ew}(M^{\#}_{k,d}) = \mathrm{fw}(M^{\#}_{k,d})$, there is a non-contractible cycle C in $M^{\#}_{k,d}$ of length smaller than d. Recall that there is a natural branched covering projection $\pi\colon M^{\#}_{k,d} \to M_{k,d}$. As $M^{\#}_{k,d}$ is vertex-transitive (being a regular map), we can assume that C contains a preimage w of one of the poles, say v'. Consider the interior U of the disk D' supporting T' which contains v' (the "northern hemisphere" of $M_{k,d}$) and the component $\tilde{U}$ of $\pi^{-1}(U)$ containing w. From Lemma 4, it follows that the restriction of π on $\tilde{U}$ maps $\tilde{U}$ homeomorphically onto U. Since the length of C is less than d, C is wholly contained in $\tilde{U}$. Hence C is contractible, a contradiction. □

Theorem 5 shows, in particular, that for every fixed $k \geq 6$ there exist infinitely many k-valent regular triangulations (with increasing face-width). Especially interesting is the case $k = 7$ due to the relationship with Hurwitz groups. We call a map a *Hurwitz triangulation* if it is regular, 7-valent, and has all faces triangular. The automorphism group of a Hurwitz triangulation is a *Hurwitz group.* A Hurwitz group is finite and can be generated by two elements x and y which satisfy the relations

$$x^7 = 1\,, \quad y^2 = 1\,, \quad \text{and} \quad (xy)^3 = 1\,.$$

A short trip through an extensive literature about Hurwitz groups is offered in Conder's survey [6].

A classical result by Hurwitz (1893) says that for every map on an orientable surface of genus $g \geq 2$ the order of its orientation preserving automorphism group is bounded from above by $84(g-1)$. This extreme value is attained precisely when the map is a Hurwitz triangulation or the dual of a Hurwitz triangulation. The fact that there are infinitely many Hurwitz triangulations was first established by MacBeath in 1966 [16]. A map-theoretical proof was given by Surowski [26]. Our Theorem 5 yields another such proof.

THEOREM 6. (MacBeath) *There exist infinitely many Hurwitz triangulations and hence infinitely many Hurwitz groups.*

It is not difficult to give alternative constructions of Hurwitz triangulations. For instance, one can apply the generic regular map construction to an infinite sequence of irregular planar "dessins d'enfants" with pattern $(7,1;3,1)$, or pattern where one or both values 1 may be missing. Such a sequence can be constructed inductively from a certain base map of this kind which is then repeatedly expanded by adding a certain pattern. The first two members of such a sequence are shown in Figure 8.

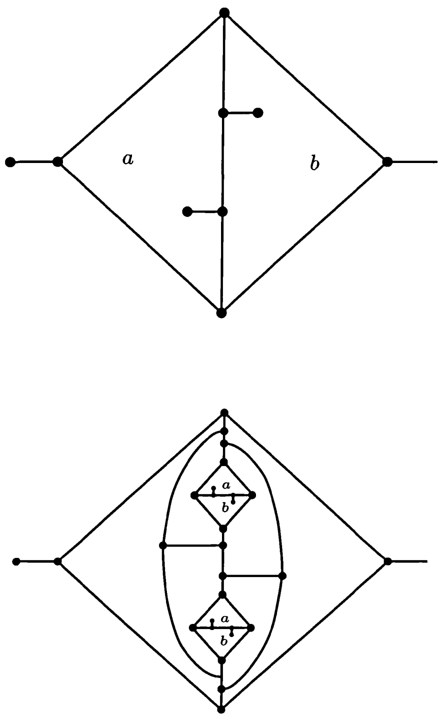

FIGURE 8.

An independent proof of MacBeath's theorem in a similar style can be found in Archdeacon, Gvozdjak and Širáň [2] (this issue p. 127). It employs toroidal base maps.

"Dessins d'enfants" can also be used to derive permutation representations of small Hurwitz groups. Note that if $M = (D;\, R, L)$ is a map for which $M^{\#}$ is a Hurwitz triangulation or the dual of a Hurwitz triangulation, then $\operatorname{Mon} M^{\#} = \langle R, L\rangle$ is a Hurwitz group. Let, for instance, M be the map in Figure 9 which has seven arcs and type $(7,3)$. If the arcs of M are labelled consistently with Figure 9, where a in the transposition (ab) denotes the arc in the indicated direction, we see that $R = (135)(467)$ and $L = (12)(34)$. These permutations generate a group of order 168 isomorphic to $PSL(2,7)$, the smallest of all Hurwitz groups. Permutation representations of the next three smallest Hurwitz groups of orders 504, 1092, and 1344 can be derived from maps in Figure 10 (a), (b) and (c), respectively.

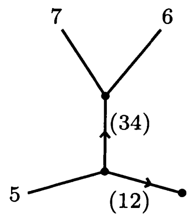

FIGURE 9.

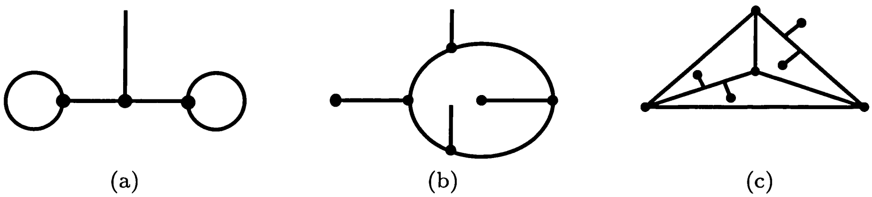

FIGURE 10.

We proceed to another application of Theorem 5. A graph G is said to be *vertex-locally* C_n (*edge-locally* C_n) if for every vertex v (for every edge e) of G the subgraph induced by the vertices at distance 1 from v (from e) is a cycle of length n. Parsons and Pisanski [23] proved that for $n \geq 6$ every vertex-locally C_n graph is the underlying graph of a uniquely determined n-valent triangulation of face-width at least 4, and vice versa. An edge analogue was established by Nedela [18]: for $n \geq 8$ every edge-locally C_n graph is the underlying graph of an m-valent triangulation of face-width at least 5, where

$n = 2m - 4$ and $m \geq 6$. With these facts in mind, Theorem 5 implies the following strengthening of an old result of Brown and Connelly [5].

THEOREM 7. *For every $n \geq 6$ there are infinitely many connected graphs that are locally C_n and edge-locally C_{2n-4}.*

Now we turn our attention to the duals of triangulations constructed in Theorem 5. A *polygonal graph* is a regular graph of girth g together with a set of g-cycles $\{Z_i\}$ such that every path of length two is in a unique Z_i. Most work has concentrated on cubic polygonal graphs. Graphs of girth not exceeding 9 were investigated by Perkel [24], [25] and Negami [22]. It was conjectured that no cubic polygonal graphs of girth $g \geq 10$ exist (see [1]). However, Archdeacon [1] constructed a cubic polygonal graph of girth g for every $g \equiv 0 \pmod 4$. We generalize this result by dropping the divisibility condition.

THEOREM 8. *For every integer $g \geq 6$ there exist infinitely many cubic arc-transitive polygonal graphs of girth g.*

Proof. Theorem 5 provides a k-valent regular triangulation $T_{k,d} = M^{\#}_{k,d}$ of face-width at least d for every $k \geq 6$ and $d \geq 1$. Consider the case when $d > k$, and let $G_{k,d}$ be the underlying graph of the dual map $T^*_{k,d}$. It follows that $G_{k,d}$ is cubic, arc-transitive (because $T^*_{k,d}$ is a regular map) and has girth at most k (since the face boundaries are k-cycles). Moreover, $G_{k,d}$ together with the set of face boundaries of $T^*_{k,d}$ is a polygonal graph. It remains to prove that the girth of $G_{k,d}$ equals k.

Let C be a shortest cycle in $G_{k,d}$, and let the length of C be g. We have $g \leq k$. Since

$$\mathrm{ew}(T^*_{k,d}) \geq \mathrm{fw}(T^*_{k,d}) = \mathrm{fw}(T_{k,d}) = d > k\,,$$

every cycle of length not exceeding k is contractible. Hence, C separates the underlying surface of $T^*_{k,d}$ into two components one of which is a disk D. We claim that the interior of D contains no vertices of $G_{k,d}$; in other words, C is the boundary of a face of $T^*_{k,d}$. Suppose not, and consider the subgraph K induced by the vertices in the interior of D. We distinguish two cases.

If K is acyclic, then it contains a vertex u of valency at most 1. It follows that there is a path P of length 2 joining two vertices on C and passing through u. However, $C \cup P$ obviously contains a shorter cycle – a contradiction.

If K contains a cycle, then the set of all edges with one end-vertex on C and the other end-vertex inside D constitutes a cycle-separating edge-cut X with $|X| < g$. On the other hand, we have proved in [19] (see also [20]) that the cyclic connectivity of a cubic vertex-transitive graph is equal to its girth. (Recall that an edge-cut X in a graph G is *cycle-separating* if at least two components of $G - X$ contain cycles. The *cyclic connectivity* of G is the smallest integer m such that G has a cycle-separating m-cut.) Thus we have a contradiction again.

Consequently, C bounds a face, and $g = k$. This completes the proof. $\square$

A lot of work has been done in the construction of cubic symmetrical graphs of arbitrarily large girth, see for example [3], [4]. From the known constructions, however, it has not been clear whether for any given integer $g \geq 6$ at least one arc-transitive cubic graph of girth g can be found. The previous theorem provides such a construction.

The result can still be improved: for every $g \geq 3$ we can actually construct a 2-arc-transitive cubic graph K_g of girth g. This can achieved by replacing the generic regular maps with the reflexible generic regular maps throughout. For every $k \geq 6$ and $d \geq 1$ we thus obtain a k-valent reflexible regular triangulation of face-width at least d (which is in fact Theorem 5′). Its dual is a reflexible regular map, and if the face-width is larger than k, then the girth of the underlying cubic graph equals k. As the map is reflexible, the graph is 2-arc-transitive [9; Theorem 4]. Finally, using the Platonic solids to cover the remaining small values of girth we obtain:

THEOREM 9. *For every integer $g \geq 3$ there exists a cubic 2-arc-transitive graph of girth g.*

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