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ELIMINATION OF LOCAL BRIDGES

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ABSTRACT. Vertices of degree different from 2 in a graph K are called main vertices of K, and paths joining these vertices are branches of K. Let K be a subgraph of G. It is shown that if G is 3-connected (modulo K), then it is possible to replace branches of K by other branches joining the same pairs of main vertices of K such that G has no bridges with respect to the new subgraph whose vertices of attachment all lie on a single branch of K. We present a linear time algorithm that either performs such a task, or finds a Kuratowski subgraph K_5 or $K_{3,3}$ in a subgraph of G formed by a branch e and those bridges of K in G that are attached only to the branch e.

1. Introduction

Let K be a subgraph of a simple graph G. A K-*bridge* (or a *relative K-component*) is a subgraph of G which is either an edge $e \in E(G) \setminus E(K)$ (together with its endpoints) with both endpoints in K, or it is a connected component Q of $G - V(K)$ together with all edges (and their endpoints) between Q and K. Each edge of a relative K-component R having an endpoint in K is a *foot* of R. The vertices of $R \cap K$ are the *vertices of attachment* of R. A vertex of K whose degree in K is different from 2 is a *main vertex* of K. For convenience, if a connected component of K is a cycle, then we choose an arbitrary vertex of it and declare it to be a main vertex of K as well. A *branch* of K is any path (possibly a closed path) in K whose endpoints are main vertices, but no internal vertex on this path is a main vertex. If a relative K-component is attached to a single branch of K, it is said to be *local*. Otherwise, it is *global*.

A graph G is 3-*connected modulo* K if for every set of vertices $X \subset V(G)$ with at most 2 elements, every connected component of $G - X$ contains a

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main vertex of K. This is obviously equivalent to the following condition: If $G^+(K)$ is the graph obtained from G by adding three mutually adjacent new vertices whose additional neighbors are the main vertices of K, then $G^+(K)$ is 3-connected. On the other hand, if K is homeomorphic to a 3-connected graph, then G is 3-connected modulo K if and only if it is 3-connected.

In this paper, we study the problem of replacing a given subgraph K of G by a homeomorphic subgraph K' having the same set of main vertices such that no K'-bridge in G is local. Our main motivation comes from considering algorithmic aspects of embedding extension problems [7], [10]. Algorithms developed in [7], [10] rely on the theory of bridges: a subgraph K of G is embedded in the surface, and then this embedding is either extended to an embedding of G, or an obstruction for such extensions is found. One of the difficulties in achieving linear time complexity is the presence of local bridges.

Elimination of local bridges is a useful tool also in disjoint paths problems (cf. Ohtsuki [11], Robertson and Seymour [12]). Similar application is in graph drawing ([9]). We believe that our results can also be used in some other problems involving bridges (see, e.g., [13]).

In our algorithm, we need plane embeddings of graphs. These can be described combinatorially ([5]) by specifying a *rotation system*: for each vertex v of the graph G we have the cyclic permutation π_v of its neighbors, representing their circular order around v on the surface. In order to make a clear presentation of our algorithm, we have decided to use this description only implicitly. Whenever we say that we have an embedding, we mean such a combinatorial description.

There are very efficient (linear time) algorithms which for a given graph determine whether the graph is planar or not. The first such algorithm was obtained by Hopcroft and Tarjan [6] back in 1974. There are several other linear time planarity algorithms (Booth and Lueker [1], [8], Fraysseix and Rosenstiehl [4], Williamson [14], [15]). Extensions of original algorithms produce also an embedding (described by a rotation system) whenever the given graph is found to be planar ([2]), or find a small obstruction (a subgraph homeomorphic to K_5 or $K_{3,3}$) if the graph is non-planar ([14], [15]).

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model was introduced by Cook and Reckhow [3]. More precisely, our model is the *unit-cost* RAM where operations on integers, whose value is $\mathcal{O}(n)$, need only constant time (where n denotes the size of the given graph).

2. Elimination of local bridges

There are 2-connected graphs G, $K \subseteq G$, such that it is not possible to find a subgraph $K' \subseteq G$ homeomorphic to K without local bridges. Suppose, for example, that K contains a branch e with at least one local bridge and no global bridge attached to it. Then it is not possible to eliminate local bridges on e by replacing e by another branch. However, if G is 3-connected, such replacement is always possible. A strengthening of this fact is provided by the next result.

Proposition 2.1. *Let $K \subseteq G$ and let e be a branch of K joining main vertices x and y of K. Suppose that G is 3-connected modulo K. Then e can be replaced by a branch e' joining x and y which is internally disjoint from $K-e$ such that there are no local bridges of $K-e+e'$ attached to e'. Consequently, it is possible to replace K by a subgraph K' of G homeomorphic to K having the same set of main vertices and such that there are no local K'-bridges.*

Proof. Traversal of e from x toward y induces a linear order $\preceq$ on the vertices of e, where $p \preceq q$ if and only if p is encountered before q.

Let $B_1, \ldots, B_k$ be all K-bridges that are local on e. For every bridge B_i, let p_i and q_i be its endmost vertices of attachment, i.e., attachments closest to x and y, respectively. Let H be the graph consisting of the branch e together with $B_1, \ldots, B_k$. The proof is by induction on the number of edges $|E(H)|$.

If there are no local bridges on e, the result is obvious. Otherwise, we claim that there is a local bridge B_i such that the open segment of e from p_i to q_i, denoted by (p_i, q_i), contains an attachment of a global bridge. Suppose that this is not the case. Let $u_1 = x \preceq u_2 \preceq \cdots \preceq u_k = y$ be the attachments of global bridges on e. Take a local bridge B. By the assumption, all its attachments to e lie in a closed segment $[u_i, u_{i+1}]$, where $1 \leq i < k$. But then the connected component of $G - u_i - u_{i+1}$ that contains both B and the segment (u_i, u_{i+1}) contains no main vertex of K. This is a contradiction with 3-connectivity modulo K.

Let B_i be a local bridge such that there is an attachment of a global bridge on a segment (p_i, q_i). Replace the segment of e from p_i to q_i by a path in B_i connecting p_i and q_i that is internally disjoint from e and denote the new branch by e''. After this replacement, at least edges from $[p_i, q_i]$ become part of a global bridge. Therefore the graph consisting of e'' together with all $(K-e+e'')$-bridges local on e'' has fewer edges than H, and the induction hypothesis applies. As e' we take the branch e'' from the last replacement. □

3. A linear time algorithm

Unfortunately, the proof of Proposition 2.1 yields a quadratic time algorithm for the elimination of local bridges. It is possible to improve it into an $\mathcal{O}(n \log n)$ algorithm by some additional more sophisticated methods. However, in various applications (see, e.g., [7], [10]), a linear time procedure is desired. A solution that is suitable for the applications in surface embedding algorithms is presented in this section. If L is a subgraph of G homeomorphic to K_5 or to $K_{3,3}$, we say that L is a *Kuratowski subgraph* of G. If H is a graph and $x, y \in V(H)$, denote by $H + xy$ the graph obtained from H by adding a new edge between x and y.

LEMMA 3.1. *Let $K \subseteq G$, and let e be a branch of K joining main vertices x and y of K. Suppose that G is 3-connected modulo K. There is a linear time algorithm that performs one of the following:*

(1) *Replaces e by a branch e' joining x and y which is internally disjoint from $K - e$ such that there are no local bridges of $K - e + e'$ attached to e'.*
(2) *Finds a Kuratowski subgraph L of $G + xy$ such that $L \cap K \subseteq e$.*

Proof. Let N be the graph obtained from the branch e by adding all local bridges attached to it. If the graph $N + xy$ is planar, consider one of its plane embeddings. Let W be the facial walk of one of the faces containing xy. Since G is 3-connected modulo K, it follows easily that $N + xy$ is 2-connected, and hence W is a (simple) cycle. Now we replace e by $e'' := W - xy$. The set of local bridges is modified accordingly. Some of the previous local bridges might merge together into a new local bridge, others might become global with respect to the changed subgraph K of G (and are therefore removed from consideration). But since the graph G is 3-connected modulo K, no new local bridges arise. Let N' be the modified graph of local bridges. By the above, $N' + xy \subseteq N + xy$. Using the induced plane embedding of $N' + xy$, we repeat the above procedure by selecting the "other" facial walk W' of the face containing xy on its boundary. Let $e' := W' - xy$ be the new branch replacing e''. One can show that e' has no local bridges attached to it.

Otherwise, let L be a Kuratowski subgraph from the planarity test for $N + xy$. Note that L can be obtained in linear time by the algorithm of Williamson [14], [15]. It is clear that L fits (2). □

Now we are ready to present our main result.

THEOREM 3.2. *Let $K \subseteq G$, and let e be a branch of K joining main vertices x and y of K. Suppose that G is 3-connected modulo K. There is a linear time algorithm that either replaces e by a branch e' joining x and y such that e'*

is internally disjoint from $K-e$*, and there are no local bridges of* $K-e+e'$ *attached to* e'*, or finds a Kuratowski subgraph* L *of* G *such that* $L\cap K\subseteq e$.

Proof. Let N be the graph obtained from the branch e by adding all local bridges attached to it. If N is not planar, its Kuratowski subgraph L, obtained by the algorithm of [14] and [15] in linear time, has the property stated in the theorem.

Suppose now that N is planar. Traversing e from x towards y we get the first vertex x_1 with a local bridge attached to it. (If there is no such vertex, we can stop.) Among all local bridges at x_1, we select a subset containing those bridges whose "rightmost" attachment on e is as close to y as possible. Denote this other extreme attachment by y_1. If among the selected bridges there is an edge x_1y_1, then let B_1 be this edge. Otherwise, let B_1 be any of the selected bridges.

Suppose now that we have constructed a sequence $B_1,\dots,B_k$ of local bridges at e with the following property. If x_j and y_j are the "leftmost" (i.e., closest to x) and the "rightmost" (i.e., closest to y) attachments of B_j $(1\le j\le i)$, then $x_1\prec x_2\prec y_1\preceq x_3\prec y_2\preceq\cdots\preceq x_i\prec y_{i-1}\prec y_i$, where the relation $\prec$ (and $\preceq$) stands for "being closer to x on e". Moreover, every bridge of K attached strictly between x_1 and y_{i-1} has all its attachments on the closed segment $[x_1,y_i]$ of e. (Case $i=1$ with B_1, x_1 and y_1 defined as above is assumed to fulfil these conditions.) If some global bridge is attached strictly between x_i and y_i, then we terminate the construction of the sequence $B_1,B_2,\dots,B_i$. The obtained sequence will be used later. Let us remark that we reach this point sooner or later for G is 3-connected modulo K. Suppose now that no global bridge is attached between x_i and y_i. By the 3-connectivity modulo K of the graph G and the properties of $B_1,\dots,B_i$, there is a local bridge attached strictly between x_i and y_i which has an attachment closer to y than y_i. Among all such bridges, let B_{i+1} be the bridge attached between x_i and y_i obtained as follows. We first determine the "rightmost" vertex y_{i+1} that is an attachment of such a bridge, among the candidates attached at y_{i+1}, we select those which have an attachment x_{i+1} as close as possible to x, and in the obtained subset, we choose as B_{i+1} the edge $x_{i+1}y_{i+1}$ if possible, and otherwise, we choose as B_{i+1} any of these candidates. By the properties of the sequence $B_1,\dots,B_i$, x_{i+1} cannot precede y_{i-1} on e. Now it is easy to see that the bridges $B_1,\dots,B_{i+1}$ fulfil the "inductive" requirements for the sequence $B_1,\dots,B_{i+1}$.

Upon terminating, the time spent in the above procedure is proportional to the number of edges of G in the segment of e from x_1 to the last vertex, say y_k, plus the number of edges in the local bridges attached to this segment. After changing the segment from x_1 to the last vertex y_k, we will not use the new segment in the above procedure any more. Therefore the overall time spent by this part of the algorithm is linear.

Suppose that we obtained the sequence $B_1, \dots, B_k$ by the above procedure. Our goal is to replace the segment from x_1 to y_k by a path in $B_1 \cup \dots \cup B_k \cup e$ such that the new segment will have no local bridges attached to it. This will be done in two steps. In the first step, we define a path f from x_1 to y_k and replace the corresponding segment of e by f. In the second step, we remove the remaining local bridges by applying the algorithm of Lemma 3.1.

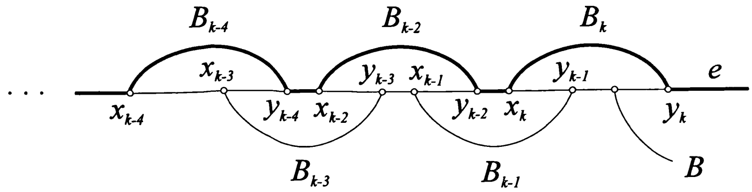

FIGURE 1.

For $i = 1, \dots, k$, let f_i be a path in B_i from x_i to y_i which is internally disjoint from e. Let f be the path composed of $f_k, f_{k-2}, f_{k-4}, \dots$ together with segments on e between y_{k-2} and x_k, y_{k-4} and x_{k-2}, etc. (together with the segment of e from x_1 to x_2 if k is even). Recall that there is a global bridge B attached between y_{k-1} and y_k (possibly at y_{k-1}); see Figure 1, where the path f is shown in bold. By the property of our sequence $B_1, \dots, B_k$, it follows that after the above replacement of the segment of e from x_1 to y_k by f, the bridges $B_{k-1}, B_{k-3}, B_{k-5}, \dots$ are all merged with B into a single global bridge.

Consider the local bridges with respect to the new graph that are attached to f. Since f and all considered local bridges are contained in N, we can take the induced plane embedding of the graph H consisting of f together with its local bridges. We claim that there exists a plane embedding of H such that every local bridge at f is attached to f from one side only (with respect to our embedding). The local bridges are of two types. They are either local at e as well (in which case they are attached to some of the segments of $e \cap f$), or they emerge as subgraphs of bridges $B_k, B_{k-2}, B_{k-4}, \dots$. By our choice of B_i, when constructing the sequence $B_1, \dots, B_k$, a local bridge attached at the segment from y_{i-2} to x_i ($i \equiv k \pmod 2$) has all its attachments on this segment, and it is easy to see that under the plane embedding of N, all such local bridges are attached from the same side of e (otherwise, the path f_{i-1} in B_{i-1} and the local bridge would intersect). If k is even, the local bridges attached to the segment of e from x_1 to x_2 may be attached to both sides of e. However, this can happen only at the vertex x_1. But bridges attached to x_1 can be easily re-embedded in such a way that they attach to e from one side only. The new local bridges that are contained in B_i (where $i \equiv k \pmod 2$) may attach to f

from both sides. But if this is the case, then either $i = k$, or $i = 1$, and the other sides can be attained only in x_1 or y_k. (To see this, consider a simple closed curve consisting of a path in Q joining the feet q_1, q_2 of Q attached to different sides of f together with the corresponding segment on f. In the plane, this curve intersects f_{i-1}, f_{i+1}, or the segment of e between x_i and y_i.) Again, these bridges can be re-embedded in such a way that each of them is attached to f from one side only. The obtained plane embeddings of local bridges at f enable us to use Lemma 3.1 (since the addition of the edge x_1y_k will not destroy the plane embedding) to replace f by a path f' without local bridges. Note that the actual re-embeddings will be done automatically by the planarity testing of the corresponding graph in the algorithm of Lemma 3.1. Again, the overall time spent for this purpose is linear.

If there are additional local bridges attached to e at the segment from y_k to y, we repeat the whole procedure. □

COROLLARY 3.3. *Let $K \subset G$, and suppose that G is 3-connected modulo K. There is a linear time algorithm that either replaces every branch of K by another branch joining the same pair of main vertices and such that G has no local bridges with respect to the new subgraph, or finds a Kuratowski subgraph L of G such that $L \cap K$ is contained in a single branch of K.*

P r o o f. Apply Theorem 3.2 to every branch of K separately. After changing one of the branches, the local bridges attached to other branches do not change. Therefore, the total time spent by the procedure is linear in the size of the input. □

REFERENCES

[1] BOOTH, K. S.—LUEKER, G. S.: *Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-trees*, J. Comput. System Sci. **13** (1976), 335–379.

[2] CHIBA, N.—NISHIZEKI, T.—ABE, S.—OZAWA, T.: *A linear algorithm for embedding planar graphs using PQ-trees*, J. Comput. System Sci. **30** (1985), 54–76.

[3] COOK, S. A.—RECKHOW, R. A.: *Time bounded random access machines*, J. Comput. System Sci. **7** (1976), 354–375.

[4] de FRAYSSEIX, H.—ROSENSTIEHL, P.: *A depth-first search characterization of planarity*, Ann. Discrete Math. **13** (1982), 75–80.

[5] GROSS, J. L.—TUCKER, T. W.: *Topological Graph Theory*, Wiley-Interscience, New York, 1987.

[6] HOPCROFT, J. E.—TARJAN, R. E.: *Efficient planarity testing*, J. Assoc. Comput. Mach. **21** (1974), 549–568.

[7] JUVAN, M.—MARINČEK, J.—MOHAR, B.: *A linear time algorithm for embedding graphs in the torus* (Submitted).

[8] LEMPEL, A.—EVEN, S.—CEDERBAUM, I.: *An algorithm for planarity testing of graphs.* In: Theory of Graphs: International Symposium, Rome, July 1966 (P. Rosenstiehl, ed.), Gordon and Breach, New York, 1967, pp. 215–232.

[9] MOHAR, B.: *Convex representations of maps on the torus and other flat surfaces*, Discrete Comput. Geom. **11** (1994), 83–95.

[10] MOHAR, B.: *A linear time algorithm for embedding graphs in an arbitrary surface*, SIAM J. Discrete Math. (To appear).

[11] OHTSUKI, T.: *The two disjoint paths problem and wire routing design.* In: Graph Theory and Algorithms (N. Saito, T. Nishizeki, eds.), Lecture Notes in Comput. Sci. 108, Springer, New York, 1980, pp. 207–216.

[12] ROBERTSON, N.—SEYMOUR, P. D.: *An outline of a disjoint paths algorithm.* In: Paths, Flows, and VLSI-Layout, (B. Korte, L. Lovász, H. J. Prömel, A. Schrijver, eds.), Springer-Verlag, Berlin, 267–292.

[13] VOSS, H.-J.: *Cycles and Bridges in Graphs*, Kluwer, Dordrecht, 1991.

[14] WILLIAMSON, S. G.: *Embedding graphs in the plane – algorithmic aspects*, Ann. Discrete Math. **6** (1980), 349–384.

[15] WILLIAMSON, S. G.: *Depth-first search and Kuratowski subgraphs*, J. Assoc. Comput. Mach. **31** (1984), 681–693.

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