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Dedicated to the memory of Professor Milan Kolibiar

INVERSE SEMIRINGS WHOSE ADDITIVE ENDOMORPHISMS ARE MULTIPLICATIVE

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ABSTRACT. The purpose of this paper is to show that every inverse semiring whose additive endomorphisms are multiplicative is associative.

In [1], T. Kepka showed that every ring whose additive endomorphisms are ring endomorphisms is associative. The aim of this paper is to generalize this result for inverse semirings.

We shall fix the type $\tau = (t, \mathrm{ar})$ with $t = (+, \cdot, -)$, $\mathrm{ar}(+) = \mathrm{ar}(\cdot) = 2$ and $\mathrm{ar}(-) = 1$. An *inverse semiring* is a τ-algebra $\mathscr{S} = (S, \tau)$ satisfying the axioms:

(1) $(S, +, -)$ is a commutative inverse semigroup,
(2) multiplication "$\cdot$" distributes over addition "$+$" from either side,
(3) $0x + 0y = 0x \cdot 0y$, where we put $0z = z + (-z)$.

By $S(\mathscr{S})$, we denote the set of all elements of an inverse semiring. We put $E(\mathscr{S}) = \{x \in S(\mathscr{S}),\ x = x + x\}$ and $I(\mathscr{S}) = \{x \in S(\mathscr{S}),\ x = x^2\}$, where $x^2 = x \cdot x$. An inverse semiring $\mathscr{S}$ is said to be *associative* if $(S(\mathscr{S}), \cdot)$ is a semigroup.

According to (1), (2) and (3), it is easy to show (see [2]) the following:

(4) $-(x+y) = (-x) + (-y)$, $-(x \cdot y) = (-x) \cdot y = x \cdot (-y)$ and $-(-x) = x$.
(5) $0(x+y) = 0x + 0y = 0x \cdot 0y = 0(x \cdot y) = x \cdot 0y = 0x \cdot y$, $x + 0x = x = 0x + x$, $0x = 0(-x)$ and $0(0x) = 0x$.
(6) $E(\mathscr{S}) = \{x \in S(\mathscr{S}),\ x = 0x\} \subseteq I(\mathscr{S})$.

Associative inverse semirings were described in [3].

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DEFINITION 1. An inverse semiring $\mathscr{S}$ is said to be *distributive* if

(7) $x \cdot yz = xy \cdot xz$, $zy \cdot x = zx \cdot yx$ for all $x, y, z \in S(\mathscr{S})$.

Let us note that a product $u \cdot v$ we usually denoted by juxtaposition uv.

Let x be an element of a distributive inverse semiring $\mathscr{S}$. It follows from (7) that

(8) $x \cdot x^2 = (x^2)^2 = x^2 \cdot x$,

and so we have

(9) $x \cdot x^3 = x^2 \cdot x^3 = (x^3)^2 = x^3 \cdot x^2 = x^3 \cdot x = x^3$

if we put $x^3 = x \cdot x^2$.

Se [4].

LEMMA 1. *Let $\mathscr{S}$ be a distributive inverse semiring. If $x, y \in I(\mathscr{S})$, then:*

(i) $xy \in I(\mathscr{S})$,

(ii) $xy \cdot x = x \cdot yx$,

(iii) $x = -x$.

Proof. (i) and (ii) follow from (7).

(iii): By (8), (4) and (7), we have $x = (x^2)^2 = \left((-x)^2\right)^2 = (-x)(-x)^2 = -(xx^2) = -x$. □

For any elements x, y of an inverse semiring we put $x - y = x + (-y)$.

LEMMA 2. *Let $\mathscr{S}$ be a distributive inverse semiring. If $y = x - x^3$, where $x \in S(\mathscr{S})$, then $y^2 = x^2 - x^3$ and $y^3 = 0x$.*

Proof. According to (2), (1), (4), (9) and (5), we have $y^2 = (x - x^3)^2 = x^2 - x \cdot x^3 - x^3 \cdot x + (x^3)^2 = x^2 - x^3$ and $y^3 = (x^2 - x^3)(x - x^3) = x^3 - x^3 \cdot x - x^2 \cdot x^3 + (x^3)^2 = x^3 - x^3 = 0x^3 = 0x$. □

LEMMA 3. *Let $\mathscr{S}$ be a distributive inverse semiring. If $x, y, z \in S(\mathscr{S})$ and $x^3 \in E(\mathscr{S})$, then:*

$$x \cdot yz = 0x + 0y + 0z\,.$$

Proof. Using (7), (8) and (9) we obtain $x \cdot yz = xy \cdot xz = (x \cdot xz)(y \cdot xz) = (x^2 \cdot xz)(y \cdot xz) = (x^3 \cdot x^2z)(y \cdot xz)$. It follows from (6), (3), and (5) that $x^3 = 0x^3 = 0x$ and $x \cdot yz = (0x \cdot x^2z) = (y \cdot xz) = 0(xz) \cdot (y \cdot xz) = 0x + 0y + 0z$. □

LEMMA 4. *Let $\mathscr{S}$ be a distributive inverse semiring. If $x, y, z \in S(\mathscr{S})$, then:*

(10) $x \cdot yz = x^3 \cdot y^3z^3$.

Proof. First, we shall show that

(11) $x \cdot yz = x^3 \cdot yz$.

By (1), (2) and (5), we have $x \cdot yz = (x^3 + x - x^3) \cdot yz = x^3 \cdot yz + (x - x^3) \cdot yz$. It follows from Lemma 2, (5) and (6) that $(x - x^3)^3 \in E(\mathscr{S})$, and so, by Lemma 3 and (5), $(x - x^3) \cdot yz = 0x + 0y + 0z = 0(x^3 \cdot yz)$. Consequently, we have $x \cdot yz = x^3 \cdot yz + 0(x^3 \cdot yz) = x^3 \cdot yz$.

Now, we shall prove (10). According to (7), (8), (11) and its dual, we have $x \cdot yz = x^3 \cdot yz = x^3 y \cdot x^3 z = (x^3 y)^3 \cdot x^3 z = x^3 y^3 \cdot x^3 z = x^3 \cdot y^3 z = x^2 x \cdot (y^3 z)^3 = x^3 \cdot y^3 z^3$. □

THEOREM 1. *Let $\mathscr{S}$ be a distributive inverse semiring such that*

(12) $x \cdot yz = xz \cdot y = z \cdot xy$ *for all* $x, y, z \in S(\mathscr{S})$.

Then $\mathscr{S}$ is associative.

Proof. Let $\mathscr{S}$ be a distributive inverse semiring satisfying (12). First, we shall prove that

(13) $\mathcal{J} = \big(I(\mathscr{S}), \cdot\big)$ is a commutative semigroup.

It follows from Lemma 1 (i) that $\mathcal{J}$ is a groupoid. Let $x, y, z \in I(\mathscr{S})$. According to (12) and Lemma 1 (ii), we have $xy = xx \cdot y = x \cdot yx = xy \cdot x = y \cdot xx = yx$, and so the groupoid $\mathcal{J}$ is commutative. By (12), $x \cdot yz = z \cdot xy = xy \cdot z$. Thus the groupoid $\mathcal{J}$ is associative.

Now, we shall show that $\mathscr{S}$ is associative. Let $x, y, z \in S(\mathscr{S})$. Then, by Lemma 4, its dual, (9) and (13), we obtain $x \cdot yz = x^3 \cdot y^3 z^3 = x^3 y^3 \cdot z^3 = xy \cdot z$. □

DEFINITION 2. An *inverse AE-semiring* is an inverse semiring $\mathscr{S}$ such that every endomorphism of $\big(S(\mathscr{S}), +\big)$ is also an endomorphism of $\big(S(\mathscr{S}), \cdot\big)$.

THEOREM 2. *Every inverse AE-semiring is associative.*

Proof. Let $\mathscr{S}$ be an inverse AE-semiring. It is easy to show that $\mathscr{S}$ is distributive (see [1; Proposition 2.2 (i)]). According to Theorem 1, it remains to prove that $\mathscr{S}$ satisfies (12).

The mapping $x \mapsto xz + x$ is an endomorphism of $\big(S(\mathscr{S}), +\big)$, and so it is an endomorphism of $\big(S(\mathscr{S}), \cdot\big)$. Thus we have

$$(xz + x)(yz + y) = xy \cdot z + xy$$

for all $x, y, z \in S(\mathscr{S})$. Using (1) and (2) we get

$$xz \cdot yz + x \cdot yz + xz \cdot y + xy = xy \cdot z + xy .$$

By (7), we obtain

$$(xy \cdot z - xy \cdot z) + x \cdot yz + (xz \cdot y - xz \cdot y) + (xy - xy) = (xy \cdot z - xy \cdot z) - xz \cdot y + (xy - xy) .$$

According to (5), we have

$$0x + 0y + 0z + x \cdot yz = 0x + 0y + 0z - xz \cdot y .$$

Consequently,

$$0(x \cdot yz) + x \cdot yz = 0(-xz \cdot y) - xz \cdot y\,,$$

and so

$$x \cdot yz = -xz \cdot y\,.$$

It follows from [4; Theorem III.1.2 (ii)] (or directly from the dual of Lemma 4. (9) and Lemma 1 (i)) that $xz \cdot y \in I(\mathscr{S})$, and so, by Lemma 1 (iii), we get

$$x \cdot yz = -xz \cdot y = xz \cdot y\,.$$

Analogously, we can show that $z \cdot xy = xz \cdot y$ using the mapping $z \mapsto xz + z$ and the equality $(xz + z)(xy + y) = x \cdot zy + zy$. □

Note. It is easy to show that an inverse semiring $\mathscr{S}$ is a semilattice if an only if $E(\mathscr{S}) = S(\mathscr{S})$. Evidently, every semilattice is an inverse AE-semiring. In this note, we shall describe an inverse AE-semiring which is neither the semilattice nor the ring.

Let $\mathscr{S}$ be a τ-algebra, where $S(\mathscr{S}) = \{1, 0, h\}$ and

$+$	1	0	h
1	0	1	h
0	1	0	h
h	h	h	h

$-$	
1	1
0	0
h	h

$\cdot$	1	0	h
1	1	0	h
0	0	0	h
h	h	h	h

It is easy to verify that $\mathscr{S}$ is an inverse semiring. Let f be an additive endomorphism on $\mathscr{S}$. Then $f(0) \neq 1 \neq f(h)$. We have the following possibilities:

Case 1. $f(0) = h$. Then $f(h) = f(h) + f(0)$, $f(1) = f(1) + f(0)$, and so $f(h) = h = f(1)$.

Case 2. $f(0) = 0$ and $f(h) = h$. Then $f(0) = f(1) + f(1)$, and so $f(0) \neq h$.

Case 3. $f(0) = 0$ and $f(h) = 0$. Then $f(h) = f(h) + f(1)$, and so $f(1) = 0$.

From this we obtain that $\mathscr{S}$ has four additive endomorphisms:

	f_1	f_2	f_3	f_4
1	h	0	1	0
0	h	0	0	0
h	h	h	h	0

It is clear that every f_i $(i = 1, 2, 3, 4)$ is a multiplicative endomorphism on $\mathscr{S}$. Therefore $\mathscr{S}$ is an inverse AE-semiring.

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