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Dedicated to the memory of Professor Milan Kolibiar

MODULAR MEDIAN ALGEBRAS GENERATED BY SOME PARTIAL MODULAR MEDIAN ALGEBRAS

HILDA DRAŠKOVIČOVÁ

(*Communicated by Tibor Katriňák*)

ABSTRACT. Let $\mathcal{M}$ denote the variety of algebras with one ternary operation (abc) satisfying the identities $(abb) = b$ and $((abc)dc) = (ac(dcb))$. The subvariety $\mathcal{T}$ of the variety $\mathcal{M}$ is given by the identity $((abc)de) = ((ade)(bde)(cde))$. It is known that the lattice of subvarieties of the variety $\mathcal{T}$ forms a strictly increasing sequence (a chain) of varieties $\mathcal{T}_n$, $n = 1, 2, \dots, \omega$, and $\mathcal{T} = \mathcal{T}_\omega$. For each $\mathcal{T}_n$, $1 < n < \omega$, it is given a finite base of identities. The free algebra $F_{\mathcal{M}}(3)$ on three generators over the variety $\mathcal{M}$ belongs to the variety $\mathcal{T}$. Since we do not know anything about the free algebra $F_{\mathcal{M}}(4)$ on four generators over $\mathcal{M}$, we give results about the algebras in $\mathcal{M}$ or in $\mathcal{T}$, respectively, which are generated by some partial algebras.

Introduction

Denote by $\mathcal{M}$ the variety of algebras A with a single ternary operation (xyz) (notation $A = (A; ())$) satisfying the identities

(1) $(abb) = b$,

(2) $((abc)dc) = (ac(dcb))$.

The algebras from $\mathcal{M}$ are called *modular median algebras* (shortly *m.m. algebras*) as in the papers [6] and [8]. Denote by $\mathcal{D}$ the subvariety of $\mathcal{M}$ given by the identity

(D) $(abc) = (bac)$.

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The variety $\mathcal{M}$ was studied by M. Kolibiar and T. Marcisová in [15]. They have shown that the varieties $\mathcal{M}$ and $\mathcal{D}$ are related to the varieties of modular and distributive lattices, respectively: In a modular lattice L, the ternary operation

$$\text{(o)} \quad (xyz) = \big(x \wedge (y \vee z)\big) \vee (y \wedge z) = \big(x \vee (y \wedge z)\big) \wedge (y \vee z)$$

satisfies the identities (1) and (2). Moreover, if L is distributive, then also (D) is satisfied. Also a partial converse is true (see [15]): Consider an algebra $A \in \mathcal{M}$ which contains two specific elements 0, 1 and satisfies the identity $(0x1) = x$. Then the algebra $(A; \wedge, \vee)$, where $x \wedge y = (x0y)$, $x \vee y = (x1y)$, is a modular lattice in which 0 and 1 are the least and the greatest element, respectively, and the identity (o) holds. This lattice is distributive if $A \in \mathcal{D}$.

The study of ternary algebras related to distributive lattices was initiated by G. Birkhoff and S. A. Kiss [5] and followed by M. Sholander (in [19], [20], [21]) and many other authors (e.g., [1], [15], [13]; a survey can be found in [3]).

The study of ternary algebras related to modular lattices was initiated by J. Hashimoto [10] and followed by other authors (e.g., [15], [11], [12], [13], [6], [8]). More general ternary algebras were investigated by J. R. Isbell [13] and J. Hedlíková [12].

Denote by $\mathcal{T}$ and $\mathcal{U}$ the subvariety of the variety $\mathcal{M}$ satisfying the identity

$$\text{(T)} \quad \big((abc)de\big) = \big((ade)(bde)(cde)\big)$$

and

$$\text{(U)} \quad \big((abc)ad\big) = \big(ab(cad)\big),$$

respectively.

E. Fried and A. F. Pixley [9] introduced the notion of a dual discriminator variety. It was shown in [6] that $\mathcal{T}$ is a dual discriminator variety, $\mathcal{T}$ has equationally definable principal congruences, $\mathcal{T}$ has congruence extension property, and any algebra from $\mathcal{T}$ can be embedded in a modular lattice. Independently, the variety $\mathcal{T}$ appeared as a special subvariety of media introduced by J. R. Isbell [13] (he called them isotropic media). The identity (U) appeared in an algebraic description of block graphs (alias Husimi trees) performed by L. Nebeský [18]. Both identities (T) and (U) are used (see [4; Theorem 3]) in a characterization (solely by algebraic identities) of quasi-median algebras, i.e., algebras associated with quasi-median graphs introduced by H. M. Mulder in [17].

It was shown in [8; Theorem 1] that the varieties $\mathcal{T}$ and $\mathcal{U}$ coincide. It holds $\mathcal{D} \subset \mathcal{T}$, $\mathcal{D} \neq \mathcal{T}$ (see, e.g., [8]). Denote by $\mathcal{L}(\mathcal{M})$ the lattice of all subvarieties of the variety of $\mathcal{M}$. It was shown in [8; Theorem 2, Theorem 3] that each of the identities (D) and (T) splits the lattice $\mathcal{L}(\mathcal{M})$ into two parts. The free algebra $F_{\mathcal{M}}(3)$ on three generators over the variety $\mathcal{M}$ has six elements and can be embedded in the free modular lattice on three generators (cf. [13; Corollary to 2.2]). Moreover, $F_{\mathcal{M}}(3)$ belongs to the variety $\mathcal{T}$ (cf. [13; below 5.14]). We do not know anything about the free algebra $F_{\mathcal{M}}(4)$ on four generators from the variety $\mathcal{M}$. We know from [13; 5.14] that the variety $\mathcal{T}$ is locally finite.

In the present paper, some results are given about an algebra $A \in \mathcal{T}$ generated by a partial algebra of order four (Theorem 2 and Theorem 3 below) and $A \in \mathcal{M}$ generated by a partial algebra of order five (Theorem 1), respectively. It is given a finite base of identities for each subvariety of the variety $\mathcal{T}$ (Theorem 4 below).

Preliminary results

LEMMA A. ([15; Lemma]) *The following identities and implications hold in each $A \in \mathcal{M}$.*

(3) $(aba) = a$,
(4) $(abc) = (acb)$,
(5) $(aab) = a$,
(6) $\big((abc)bc\big) = (abc)$,
(7) $\big((abc)ac\big) = \big(ac(abc)\big) = (abc)$,
(8) $\big(ab(cab)\big) = (abc)$,
(9) $(abc) = c$ *implies* $(bac) = c = (cab)$,
(10) $(bac) = (cab)$ *implies* $(abc) = (bac)$,
(11) $\big(a(dbc)(abc)\big) = (abc)$.

Recall from [6; Remark 1.1] that $\mathcal{M}$ is a congruence distributive variety since (1), (3) and (5) give the majority term.

Let $A \in \mathcal{M}$, $x, y, z \in A$. We say that y *is between* x *and* z, and write xyz, if $(xyz) = y$. By (9) and (4), xyz implies zyx.

LEMMA B. ([6; Lemma 1.2, Lemma 1.3, Lemma 2.1]) *The following identities and implications hold in each $A \in \mathcal{M}$.*

(12) $\big((abc)(bac)(cab)\big) = (abc)$.
(13) $\big((acd)cb\big) = \big(ac(dcb)\big) = \big(ac(bcd)\big) = \big((acb)cd\big)$.
(14) $\big(ab(cda)\big) = \big(a(bda)(cda)\big) = \big(ac(bda)\big)$.
(15) *axb and ayb imply $(xay) = (axy) = (yax)$.*
(16) *An algebra $A \in \mathcal{T}$ is subdirectly irreducible if and only if for every $x, y, z \in A$ $(xyz) = x$ if $y \neq z$ and $(xyz) = y$ if $y = z$.*
(17) *Let $\theta \in \operatorname{Con} A$, $A \in \mathcal{M}$, $x, y, z, u \in A$. If xyz, yzu and $x\theta u$, then $y\theta z$. In particular, xyz, yzu and $x = u$ imply $y = z$.*

Denote by T_2 the two element algebra from $\mathcal{M}$. If $A \in \mathcal{M}$, a, b, c are pairwise different elements of A, and $a = (abc)$, $b = (bac)$ and $c = (cab)$ hold, then we say that the elements a, b, c form a *triangle*, and we use the notation T_3 for it. For each cardinal $n \geq 3$ denote by T_n the algebra of order n in which any three elements form a subalgebra isomorphic to the triangle T_3. The algebras T_n are the only subdirectly irreducible algebras in the variety $\mathcal{T}$ (see, e.g., (16)). Let $A \in \mathcal{M}$, $a, b, c, d \in A$. A quadruple (a, b, c, d) is said to be *cyclic* whenever abc, bcd, cda and dab hold.

Results

The following Theorem is due to J. Hedlíková (oral communication).

THEOREM 1. *Let $A \in \mathcal{M}$, $x, y, z, u, s \in A$, $y \neq u$, $(\{x, y, z\}; ()) \cong T_3$ and (y, z, s, u) be a cyclic quadruple. Then the elements x, y, z, s, u generate a subalgebra B of A, where $B = (\{x, y, z, t = (xsu), s, u\}; ())$, which is isomorphic to the direct product $T_3 \times T_2$. Moreover, $B \in \mathcal{T}$.*

Proof. Note that $y \neq s$ because of (3) in Lemma A. Using (17) of Lemma B, from $y \neq u$, we get $s \neq z$. Similarly, $y \neq z$ implies $u \neq s$. Hence, $y \neq u \neq s \neq z$ hold. We shall prove that the following relations follow from our assumptions:

(1.1) $x = (xys)$ and $x = (xzu)$,
(1.2) $y = (xyu)$ and $z = (xzs)$,
(1.3) $y = (yxs)$ and $z = (zxu)$,
(1.4) $z = (sxy)$ and $y = (uxz)$,
(1.5) $u = (usx)$ and $s = (sux)$.

From the cyclic quadruple (y, z, s, u), we get yzs, hence, by (9),

(1.6) $(zys) = z$.

Then $(xys) = ((xyz)ys) \overset{(13)}{=} (xy(zys)) \overset{(1.6)}{=} (xyz) = x$. Symmetrically, $(xzu) = x$ can be proved and (1.1) holds. $(xyu) = ((xyz)yu) \overset{(13)}{=} (xy(zyu)) = (xyy) \overset{(1)}{=} y$ (zyu holds since (y, z, s, u) is a cyclic quadruple). Symmetrically, $z = (xzs)$ and (1.2) holds. $(yxs) = ((yxz)xs) \overset{(13)}{=} (yx(zxs)) \overset{(1.2)(9)}{=} (yxz) = y$. Symmetrically, $(zxu) = z$ and (1.3) holds. $(sxy) = (sx(yxz)) \overset{(13)}{=} ((sxz)xy) \overset{(1.2)(9)}{=} (zxy) = z$. Symmetrically, $(uxz) = y$ and (1.4) holds. $(usx) = ((ysu)sx) \overset{(13)}{=} ((ysx)su) \overset{(1.3)}{=} (ysu) = u$. Symmetrically, $(sux) = s$ and (1.5) holds.

Take $t = (xsu)$. According to (1.5), $(usx) = u \neq s = (sux)$, we get $u \neq t \neq s$ by (10) of Lemma A. In view of (12),

(1.7) $(\{t, u, s\}; ()) \cong T_3$.

Since (u,s,z,y) is a cyclic quadruple, too, and $u \neq y \neq z \neq s$ hold, we get that the analogous relations to (1.1) – (1.5) hold:

(1.8) $t = (tuz)$ and $t = (tsy)$,
(1.9) $u = (tuy)$ and $s = (tsz)$,
(1.10) $u = (utz)$ and $s = (sty)$,
(1.11) $s = (zut)$ and $u = (yts)$,
(1.12) $y = (yzt)$ and $z = (zyt)$.

Now we shall show that

(1.13) (y,x,t,u) is a cyclic quadruple.

According to (4) and (9), we get

(1.14) xtu, hence, utx.

With respect to (1.2), (4), and (9), we get

(1.15) xyu, hence, uyx.

In view of (15), (1.14), and (1.15), we get

(1.16) $(xty) = (txy) = (yxt)$.

Then $(xty) \overset{(4)}{=} (xyt) = ((xyz)yt) \overset{(13)}{=} (xy(zyt)) \overset{(1.12)}{=} (xyz) = x$. It implies $(yxt) = x$ by (1.16), hence,

(1.17) yxt.

Now (1.13) follows from (1.14), (1.17), (1.15) and (1.9). Analogously, it can be proved that

(1.18) (z,x,t,s) is a cyclic quadruple, in particular, txz,

hence,

(1.19) $(tzx) = x$.

(1.20) $(tyz) = x$:

$(tyz) \overset{(4)}{=} (tzy) \overset{(1.4)(4)}{=} (tz(uzx)) \overset{(13)}{=} ((tzu)zx) \overset{(1.7)(4)}{=} (tzx) \overset{(1.19)}{=} x$.

(1.21) $t \neq x$:

In view of (1.13), tuy and uyx. If $t = x$, then according to (17), $y = u$, a contradiction.

(1.22) $t \neq y$:

Let $t = y$. Then $t \overset{(1.18)}{=} (xts) = (xys) \overset{(1.1)}{=} x$, hence, $y = x$, a contradiction. Analogously, it can be proved

(1.23) $t \neq z$.

We have proved that all elements from B are pairwise different. Denote $\alpha = \theta(x,y)$, $\beta = \theta(x,t)$. According to (1.13), (1.18), (1.7), and (17), we get $B/\alpha \cong T_2$ and $B/\beta \cong T_3$. It is easy to see that $B \cong B/\alpha \times B/\beta$. Hence, $B \cong T_2 \times T_3$. Finally, $B \in \mathcal{T}$ by (16). □

THEOREM 2. *Let $A \in \mathcal{T}$, $a, b, c, d \in A$, $(\{a, b, c\}; ()) \cong T_3$, $c \neq d \neq a$, and cda hold. Then the subalgebra B of A generated by the elements a, b, c, d is isomorphic to the direct product $T_3 \times T_3$.*

Proof. Let $B \subseteq \Pi(A_i : i \in I)$ be a subdirect decomposition of subdirectly irreducible algebras A_i, $A_i \in \mathcal{T}$, $i \in I$. Without loss of generality, we can suppose that for each $i \in I$ the algebra A_i has more than one element, and that all projections p_i from B onto A_i have pairwise different kernels $\operatorname{Ker} p_i$. For arbitrary element $x \in B$ denote by x_i the ith component of the element x, hence, $x = (x_i : i \in I)$. The elements a, b, c form a triangle, hence, for each $i \in I$ either $a_i = b_i = c_i$ or $a_i \neq b_i \neq c_i \neq a_i$ holds. The element d_i has to be between the elements a_i and c_i in $A_i \cong T_n$, which is possible only if $d_i \in \{a_i, c_i\}$ by (16) of Lemma B. In the case $a_i = b_i = c_i$, the algebra $A_i = p_i(B)$ has only one element. Hence, for each $i \in I$, $a_i \neq b_i \neq c_i \neq a_i$ holds and $A_i \cong (\{a_i, b_i, c_i\}; ()) \cong T_3$. According to $a \neq d \neq c$, the elements $i, j \in I$ must exist such that $d_i = a_i$ and $d_j = c_j$. We shall show that $I = \{i, j\}$. Let $k \in I$. Without loss of generality, suppose $d_k = a_k$. Then the mapping $f\colon A_k \to A_i$ given by $f(a_k) = a_i$, $f(b_k) = b_i$, $f(c_k) = c_i$ ($f(d_k) = d_i$ holds, too) is an isomorphism, and $p_i = f \circ p_k$ holds (since these homomorphisms coincide on the set $\{a, b, c, d\}$ of generators of the algebra B). It implies $\operatorname{Ker} p_i = \operatorname{Ker} p_k$, hence, $i = k$ (for we have supposed that different projections have different kernels). It was shown that $B \subseteq A_i \times A_j \cong T_3 \times T_3$. It is easy to verify that the elements $a = (a_i, a_j)$, $b = (b_i, b_j)$, $c = (c_i, c_j)$, $d = (a_i, c_j)$ generate the whole algebra $A_i \times A_j$. Really, for the elements $e = (bad)$, $f = (cbe)$, $g = (acf)$, $h = (bag)$, $l = (cbh)$ the following equalities hold: $e = (a_i, b_j)$, $f = (c_i, b_j)$, $g = (c_i, a_j)$, $h = (b_i, a_j)$, $l = (b_i, c_j)$. □

THEOREM 3. *Let $A \in \mathcal{T}$, $a, b, c, c' \in A$, $c \neq c'$, and $(\{a, b, c\}; ()) \cong T_3 \cong (\{a, b, c'\}; ())$. Then the subalgebra B of A generated by the elements a, b, c, c' is isomorphic either to T_4 or to the direct product $T_4 \times T_3$.*

Proof. Similarly as in the proof of Theorem 2, let $B \subseteq \Pi(A_i : i \in I)$ be a subdirect decomposition of subdirectly irreducible algebras A_i, $A_i \in \mathcal{T}$, $A_i > 1$, $i \in I$, and all projections p_i of B onto A_i have pairwise different kernels $\operatorname{Ker} p_i$ ($i \in I$). For each $i \in I$ either $a_i = b_i = c_i$ or $a_i \neq b_i \neq c_i \neq a_i$ holds. In the case $a_i = b_i = c_i$, we get $a_i = c'_i$ and $A_i = 1$. Hence, $a_i \neq b_i \neq c_i \neq a_i$, and analogously, $b_i \neq c'_i \neq a_i$. According to $c \neq c'$, there exists $i \in I$ such that the elements a_i, b_i, c_i, c'_i are pairwise different, hence, $A_i \cong T_4$. Now we have two possibilities:

a) There does not exist $j \in I$ with the property $c_j = c'_j$. Then for each $k \in I$ the elements a_k, b_k, c_k, c'_k are pairwise different. Similarly as in the proof of Theorem 2, the mapping $f\colon A_k \to A_i$ given by $f(a_k) = a_i$, $f(b_k) = b_i$,

$f(c_k) = c_i$, $f(c'_k) = c'_i$ is an isomorphism such that $p_i = f \circ p_k$ holds. Then $\operatorname{Ker} p_i = \operatorname{Ker} p_k$, and $k = i$, $I = \{i\}$, $B = A_i$.

b) There exists $j \in I$ such that $c_j = c'_j$. Then $A_j = \big(\{a_j, b_j, c_j\}; ()\big) \cong T_3$. We shall show that $I = \{i, j\}$. If $k \in I$, then we have either $c_k \neq c'_k$ and then we get $\operatorname{Ker} p_k = \operatorname{Ker} p_i$ and $k = i$, or $c_k = c'_k$ and then we get $\operatorname{Ker} p_k = \operatorname{Ker} p_j$ and $k = j$. It implies that $B \subseteq A_i \times A_j \cong T_4 \times T_3$. It is easy to verify that the elements $a = (a_i, a_j)$, $b = (b_i, b_j)$, $c = (c_i, c_j)$, $c' = (c'_i, c_j)$ generate the whole algebra $A_i \times A_j$. Recall that $\mathcal{T}$ is locally finite variety by [13; 5.14]. If k and m are infinite cardinals, then the algebras T_k and T_m generate the same variety $\mathcal{T}_\omega$ since they all have the same finitely generated subalgebras. For n finite let $\mathcal{T}_n$ be the subvariety of $\mathcal{T}$ generated by the subdirectly irreducible algebra T_n (or equivalently, by all subdirectly irreducible algebras $A \in \mathcal{T}$ with $\operatorname{card} A \leq n$). The varieties $\mathcal{T}_n$, $n = 1, 2, \dots, \omega$, form a strictly increasing sequence (a chain) and $\mathcal{T} = \mathcal{T}_\omega$ (cf. [13; 5.16]). □

In the paper [9], it was found a finite equational base for a finite algebra in a dual discriminator variety using results of [2] and [16]. Recall from [6] that $\mathcal{M}$ (hence, $\mathcal{T}$, too) is a congruence distributive variety. The next Theorem will give a different finite base of such identities.

THEOREM 4. *The subvariety $\mathcal{T}_n$ of the variety $\mathcal{T}$, $1 < n < \omega$, has the following finite base of identities:* (1), (2), (T), *and*

(T_n) $d_n = d_n^*$,

where

$$d_2 = (x_0 x_1 x_2)\,, \qquad d_2^* = (x_1 x_0 x_2)\,,$$

and for $i > 2$ define inductively

$$d_3 = \big(((d_2 x_3 x_0) x_3 x_1) x_3 x_2\big)\,, \qquad d_3^* = \big(((d_2^* x_3 x_0) x_3 x_1) x_3 x_2\big)\,,$$

$$\vdots$$

$$d_n = \big(\dots\big(((d_{n-1} x_n x_0) x_n x_1) x_n x_2\big) \dots x_n x_{n-1}\big)\,,$$
$$d_n^* = \big(\dots\big(((d_{n-1}^* x_n x_0) x_n x_1) x_n x_2\big) \dots x_n x_{n-1}\big)\,.$$

Proof. According to (16) of Lemma B, it is easy to see that in T_n, the identity (T_n) is satisfied whenever at least two of the elements $x_0, x_1, \dots, x_n$ are equal, but fails whenever all $n+1$ elements are pairwise different. Hence, it holds in $\mathcal{T}$, but fails in T_{n+1}. □

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