Jozef Džurina

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PROPERTY (A) OF THE THIRD ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

JOZEF DŽURINA

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ABSTRACT. The aim of this paper is to derive sufficient conditions for property (A) of delay and advanced differential equations of the form

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{u(t)}{r_0(t)}\right)'\right)'\right)' + p(t)u(\tau(t)) = 0\,.$$

We consider the differential equations of the form

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{u(t)}{r_0(t)}\right)'\right)'\right)' + p(t)u(\tau(t)) = 0\,. \tag{1}$$

We always assume that

(i) $r_i(t)$, $0 \leqslant i \leqslant 2$, $\tau(t)$ and $p(t)$ are continuous on $[t_0, \infty)$, $r_i(t) > 0$, $p(t) > 0$, and $\tau(t) \to \infty$ as $t \to \infty$.

(ii) $R_i(t) = \int\limits_{t_0}^{t} \frac{\mathrm{d}s}{r_i(s)} \to \infty$ as $t \to \infty$, $i = 1, 2$.

We say that the operator

$$L_3 u(t) = \left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{u(t)}{r_0(t)}\right)'\right)'\right)' \tag{2}$$

is in the *canonical form* if (ii) holds. It is well known that any differential operator of the form (2) can always be represented in a canonical form in an essentially unique way (see Trench [9]).

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For convenience, we introduce the following functions:

$$L_0u(t) = \frac{u(t)}{r_0(t)}\,,$$
$$L_iu(t) = \frac{1}{r_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}L_{i-1}u(t)\,, \qquad i = 1,2\,,$$
$$L_3u(t) = \frac{\mathrm{d}}{\mathrm{d}t}L_2u(t)\,.$$

The *domain* $\mathcal{D}(L_3)$ of L_3 is defined to be the set of all functions $u\colon [T_u,\infty) \to \mathbb{R}$ such that $L_iu(t)$, $0 \leqslant i \leqslant 3$, exist and are continuous on $[T_u,\infty)$. A nontrivial solution of (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory.*

The asymptotic behavior of the nonoscillatory solutions of (1) is described in the following lemma, which is a generalization of a lemma of Kiguradze [4; Lemma 3].

Lemma 1. *Let $u(t)$ be a nonoscillatory solution of* (1). *Then there exist an integer ℓ, $\ell \in \{0,2\}$, and $t_1 \geqslant t_0$ such that*

$$\begin{aligned} u(t)L_iu(t) > 0\,, &\quad 0 \leqslant i \leqslant \ell\,, \\ (-1)^{i-\ell}u(t)L_iu(t) > 0\,, &\quad \ell \leqslant i \leqslant 3\,, \end{aligned} \qquad \textit{for all} \quad t \geqslant t_1\,. \tag{3}$$

A function $u(t)$ satisfying (3) is said to be a *function of degree ℓ*. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then, by Lemma 1,

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2\,.$$

Following Kusano and Naito [5], we say that equation (1) has *property* (A) if $\mathcal{N} = \mathcal{N}_0$.

In the recent papers (see, e.g., [1], [2], [5], [6] and [8]), various kinds of sufficient conditions for equation (1) to have property (A) appear. In this paper, we investigate delay and advanced equations of the form (1), and we present some criteria for property (A) of (1) which generalize and extend those in [1], [5] and [6].

Define the function

$$I_2(t,t_0;r_1,r_2) = \int_{t_0}^{t} r_1(s) \int_{t_0}^{s} r_2(x)\,\mathrm{d}x\,\mathrm{d}s\,.$$

Theorem 1. *Assume that $\tau(t) < t$ and $\tau(t)$ is increasing. Let $g(t)$ be a continuous function satisfying*

$$g(t) > t\,, \qquad \tau\big(g(t)\big) \leqslant t\,. \tag{4}$$

Let

$$\liminf_{t\to\infty} \int_t^{g(t)} p(s) r_0\big(\tau(s)\big) I_2\big(\tau(s), t_0; r_1, r_2\big)\, \mathrm{d}s > 1\,. \tag{5}$$

Then equation (1) *has property* (A).

P r o o f . For the sake of contradiction, we assume that (1) has a positive nonoscillatory solution $u(t)$ such that $u \in \mathcal{N}_2$. Then (3) reduces to

$$L_0 u(t) > 0\,, \quad L_1 u(t) > 0\,, \quad L_2 u(t) > 0 \quad \text{and} \quad L_3 u(t) < 0\,, \qquad \text{for} \quad t \geqslant t_1\,. \tag{6}$$

An integration of (1) from t ($\geqslant t_1$) to ∞ yields

$$L_2 u(t) \geqslant \int_t^{\infty} p(s) u\big(\tau(s)\big)\, \mathrm{d}s\,, \qquad t \geqslant t_1\,. \tag{7}$$

On the other hand, integrating twice the identity $L_2u(t) = L_2u(t)$ from t_1 to t, we have in view of (6)

$$L_0 u(t) \geqslant \int_{t_1}^{t} r_1(s) \int_{t_1}^{s} r_2(x) L_2 u(x)\, \mathrm{d}x\, \mathrm{d}s \geqslant L_2 u(t) I_2(t, t_1; r_1, r_2)\,. \tag{8}$$

Combining (7) with (8) and taking (i) and (6) into account we get

$$\begin{aligned} L_2 u(t) &\geqslant \int_t^{\infty} p(s) r_0\big(\tau(s)\big) L_2 u\big(\tau(s)\big) I_2\big(\tau(s), t_1; r_1, r_2\big)\, \mathrm{d}s \\ &\geqslant \int_t^{g(t)} p(s) r_0\big(\tau(s)\big) L_2 u\big(\tau(s)\big) I_2\big(\tau(s), t_1; r_1, r_2\big)\, \mathrm{d}s \\ &\geqslant L_2 u\big(\tau\big(g(t)\big)\big) \int_t^{g(t)} p(s) r_0\big(\tau(s)\big) I_2\big(\tau(s), t_1; r_1, r_2\big)\, \mathrm{d}s\,, \qquad t \geqslant t_2\,, \end{aligned}$$

where $t_2 \geqslant t_1$ is large enough. Since $\tau\big(g(t)\big) \leqslant t$ and $L_2u(t)$ is decreasing, the above inequalities yield

$$1 \geqslant \int_t^{g(t)} p(s) r_0\big(\tau(s)\big) I_2\big(\tau(s), t_1; r_1, r_2\big)\, \mathrm{d}s\,, \qquad t \geqslant t_2\,,$$

which contradicts (5). The proof is complete. □

For a special choice of the function $g(t)$ we have the following corollary:

COROLLARY 1. *Assume that* $\tau(t) < t$ *and* $\tau(t)$ *is increasing. Let* $\tau^{-1}(t)$ *be the inverse function to* $\tau(t)$. *Let*

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(t)} p(s) r_0\big(\tau(s)\big) I_2\big(\tau(s), t_0; r_1, r_2\big) \,\mathrm{d}s > 1 .$$

Then equation (1) *has property* (A) .

P r o o f . Since the function $g(t) = \tau^{-1}(t)$ satisfies (4), the assertion of this corollary follows immediately from Theorem 1. □

For a special case of (1) we have the following criteria for property (A) (see [5] and [1]):

THEOREM A. *Let* $p(t) > 0$, $\tau(t) \leqslant t$ *and* $\tau'(t) > 0$. *Let*

$$\liminf_{t\to\infty} \tau(t) \int_t^{\infty} \big(\tau(s) - \tau(t)\big) p(s) \,\mathrm{d}s > \frac{1}{4} .$$

Then the delay equation

$$y'''(t) + p(t) y\big(\tau(t)\big) = 0 \tag{9}$$

has property (A) .

THEOREM B. *Let* $p(t) > 0$, $\tau(t) \leqslant t$ *and* $\tau'(t) > 0$. *Let*

$$\liminf_{t\to\infty} \tau^2(t) \int_t^{\infty} p(s) \,\mathrm{d}s > \frac{1}{3\sqrt{3}} .$$

Then equation (9) *has property* (A).

In the following illustrative examples, we compare criteria from Theorems A and B with that in Corollary 1.

E x a m p l e 1. Let us consider the third order delay equation

$$y'''(t) + \frac{a}{t^2} y\big(\sqrt{t}\,\big) = 0 , \qquad a > 0 \text{ and } t \geqslant 1 . \tag{10}$$

By Corollary 1, equation (10) has property (A) for all $a > 0$. On the other hand, by Theorem A (or Theorem B), equation (10) has property (A) if $a > \frac{1}{4}$ (or $a > \frac{1}{3\sqrt{3}}$).

E x a m p l e 2. Let us consider the third order delay equation

$$y'''(t) + \frac{2.1}{(0.01)^2 \ln 100} \frac{1}{t^3} y(0.01t) = 0 , \qquad t \geqslant 1 . \tag{11}$$

By Corollary 1, equation (11) has property (A). On the other hand, by the above-mentioned result of K u s a n o and N a i t o , Theorem A fails for (11).

Now we turn to advanced equations of the form (1).

THEOREM 2. *Assume that* $\tau(t) \geqslant t$. *Let*

$$\liminf_{t\to\infty} R_2(t) \int_t^\infty p(s) r_0\big(\tau(s)\big)\big(R_1\big(\tau(s)\big) - R_1(t)\big)\, \mathrm{d}s > 1\,. \tag{12}$$

Then equation (1) *has property* (A).

P r o o f. For the sake of contradiction, we assume that $u(t)$ is a positive nonoscillatory solution of (1) such that $u \in \mathcal{N}_2$. Then (6) and (7) hold for $t \geqslant t_1$. By integrating the identity $L_1u(t) = L_1u(t)$ from t_1 to t, we get, in view of (6),

$$L_0u(t) \geqslant \int_{t_1}^t r_1(s) L_1u(s)\, \mathrm{d}s\,. \tag{13}$$

Collecting (7) and (13) and using that $L_1u(t)$ is increasing we obtain

$$\begin{aligned} L_2u(t) &\geqslant \int_t^\infty p(s) r_0\big(\tau(s)\big) \int_{t_1}^{\tau(s)} r_1(x) L_1u(x)\, \mathrm{d}x\, \mathrm{d}s \\ &\geqslant \int_t^\infty p(s) r_0\big(\tau(s)\big) \int_t^{\tau(s)} r_1(x) L_1u(x)\, \mathrm{d}x\, \mathrm{d}s \\ &\geqslant L_1u(t) \int_t^\infty p(s) r_0\big(\tau(s)\big)\big(R_1\big(\tau(s)\big) - R_1(t)\big)\, \mathrm{d}s\,, \qquad t \geqslant t_2\,, \end{aligned} \tag{14}$$

where $t_2 \geqslant t_1$ is large enough. E l i a s in [3] has generalized the third lemma of K i g u r a d z e. From his result (see also [7]), we have the following relation between $L_1u(t)$ and $L_2u(t)$:

$$L_1u(t) \geqslant L_2u(t) R_2(t)\,, \qquad t \geqslant t_1\,. \tag{15}$$

Then (14) and (15) yield

$$1 \geqslant R_2(t) \int_t^\infty p(s) r_0\big(\tau(s)\big)\big(R_1\big(\tau(s)\big) - R_1(t)\big)\, \mathrm{d}s\,, \qquad t \geqslant t_2\,,$$

which contradicts (12). The proof is complete. □

The following result for advanced equation (9) is due to O l á h [6].

THEOREM C. *Let* $p(t)$ *and* $\tau(t)$ *satisfy* (i). *Assume that* $\tau(t) \geqslant t$. *Let*

$$\liminf_{t\to\infty} \int_t^{\tau(t)} (s-t)^2 p(s)\, \mathrm{d}s > 2\,.$$

Then equation (9) *has property* (A).

Example 3. Let us consider the third order advanced equation

$$y'''(t) + \frac{a}{t^3} y(\lambda t) = 0\,, \qquad a > 0\,,\ \ \lambda \geqslant 1 \ \text{ and } \ t \geqslant 1\,, \tag{16}$$

By Theorem 2, equation (16) has property (A) if $a(\lambda - 0.5) > 1$. On the other hand, Theorem C guarantees property (A) of (16) if a stronger condition holds, namely

$$a\Big(\ln\lambda + \frac{2}{\lambda} - \frac{1}{2\lambda^2} - \frac{3}{2}\Big) > 2\,.$$

Remark 1. In the above examples, we have compared our results concerning (9) with those in [1], [5] and [6]. Note that Theorems 1 and 2 can also be applied to more general equation (1).

Let us consider the mixed differential equation

$$L_3 u(t) + p_1(t) u\big(\tau_1(t)\big) + p_2(t) u\big(\tau_2(t)\big) = 0\,, \tag{17}$$

where L_3 is defined as for (1) and

(iii) $p_i(t)$ and $\tau_i(t)$ are continuous on $[t_0,\infty)$, $p_i(t) > 0$, $1 \leqslant i \leqslant 2$, $\tau_1(t)$ is increasing, $\tau_1(t) < t$, $\tau_2(t) \geqslant t$, and $\tau_1(t) \to \infty$ as $t \to \infty$.

As Lemma 1 holds also for (17), we can seek criteria for property (A) of (17).

THEOREM 3. *Assume that* (iii) *holds. Let* $g(t)$ *be a continuous function satisfying*

$$g(t) > t\,, \qquad \tau_1\big(g(t)\big) \leqslant t\,.$$

Let

$$\liminf_{t\to\infty} \Bigg\{ \int_t^{g(t)} p_1(s) r_0\big(\tau(s)\big) I_2\big(\tau_1(s), t_0; r_1, r_2\big)\, \mathrm{d}s + R_2(t) \int_t^{\infty} p_2(s) r_0\big(\tau(s)\big)\big(R_1\big(\tau_2(s)\big) - R_1(t)\big)\, \mathrm{d}s \Bigg\} > 1\,. \tag{18}$$

Then equation (17) *has property* (A).

P r o o f. Assume that (17) has a positive solution $u(t) \in \mathcal{N}_2$. Proceeding exactly as in the proofs of Theorems 1 and 2 we can verify that

$$1 \geqslant \left\{ \int_t^{g(t)} p_1(s) r_0\bigl(\tau(s)\bigr) I_2\bigl(\tau_1(s), t_0; r_1, r_2\bigr)\, \mathrm{d}s \right.$$
$$\left. + R_2(t) \int_t^{\infty} p_2(s) r_0\bigl(\tau(s)\bigr)\bigl(R_1\bigl(\tau_2(s)\bigr) - R_1(t)\bigr)\, \mathrm{d}s \right\},$$

which contradicts (18). □

E x a m p l e 4. Let us consider the third order mixed equation

$$y'''(t) + \frac{a}{t^3}\Bigl[y(\lambda t) + y\Bigl(\frac{t}{\lambda}\Bigr)\Bigr] = 0\,, \qquad a > 0\,,\ \ 0 < \lambda < 1 \ \text{ and } \ t \geqslant 1\,. \tag{19}$$

We put $g(t) = \dfrac{t}{\lambda}$. Then, by Theorem 3, equation (19) has property (A) if

$$a\Bigl(\frac{1}{\lambda} - \frac{1}{2} - \frac{\lambda^2}{2}\ln\lambda\Bigr) > 1\,.$$

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Department of Mathematical Analysis
Faculty of Science
Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA

E-mail: dzurina@turing.upjs.sk