Mathematica Slovaca

Weimin Li

Green's relations on the endomorphism monoid of a graph

Mathematica Slovaca, Vol. 45 (1995), No. 4, 335--347

Persistent URL: http://dml.cz/dmlcz/136654

Math. Slovaca, 45 (1995), No. 4, 335–347

Mathematica Slovaca

GREEN'S RELATIONS ON THE ENDOMORPHISM MONOID OF A GRAPH

WEIMIN LI

(*Communicated by Martin Škoviera*)

ABSTRACT. In this paper, Green's relations on the endomorphism monoid of a graph are explicitly described. In particular, it is revealed that for endomorphism monoids of some special classes of graphs, Green's relations may possess some distinct combinatorial features.

1. Introduction

At present there are quite a few research papers concentrating on the endomorphism monoid of a graph. The reference papers [3] and [4] can serve as a survey. These monoids can be considered not only as concrete semigroups, but also from an abstract point of view, i.e., up to an isomorphism. Both approaches are of much interest because they open vast possibilities for applications of the algebraic theory of semigroups to the theory of graphs. For a concrete semigroup, it seems always significant to be concerned with taking various concepts introduced for abstract semigroups and finding out what these things mean for this semigroup. It is no doubt that one of the most important concepts in semigroup theory is Green's relations. So, I thought that it would be appropriate to devote this paper to investigating the combinatorial characteristics of Green's relations on the endomorphism monoid of a graph.[1)]

The graphs we consider in this paper are finite undirected graphs without loops and multiple edges. If G is a graph, we denote by $V(G)$ (or simply G) and $E(G)$ its vertex set and edge set respectively. A graph H is called a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. As usual, by P_n and C_n denote a path and a circle with n vertices respectively. Let G and H be graphs. A *homomorphism* $f\colon G \to H$ is a vertex-mapping $V(G) \to V(H)$ which preserves adjacency, i.e., such that for any $a, b \in V(G)$, $\{a,b\} \in E(G)$ implies that $\{f(a), f(b)\} \in E(H)$.

AMS Subject Classification (1991): Primary 05C25.
Key words: Green's relations, endomorphism monoid, graph.

1) The characterization of Green's relations on the strong endomorphism monoid of a graph was given in [6].

Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say that G is *isomorphic* to H (under f), denoted by $G \cong H$. A homomorphism from G to itself is called an *endomorphism* of G. A bijective endomorphism of G is called an *automorphism* of G. By $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ denote the sets of endomorphisms and automorphisms of G respectively. Obviously, for any G, $\mathrm{Aut}(G) \subset \mathrm{End}(G)$. A graph G is said to be *unretractive* if $\mathrm{Aut}(G) = \mathrm{End}(G)$ (cf. [5]).

It is well known that $\mathrm{End}(G)$ is a monoid (a *monoid* is a semigroup with an identity element) and $\mathrm{Aut}(G)$ is a group with respect to the composition of mappings. We denote an endomorphism f (or a homomorphism f from one graph to another) in the obvious sense as $f = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$ and $f^{-1}(a) := \{b \in V(G) \mid f(b) = a\}$. If A is a subgraph of a graph G, and f is an endomorphism of G, we will denote by $f|_A$ the restriction of f on A.

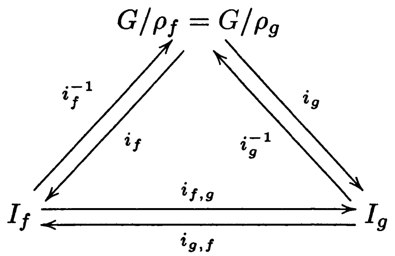
Let f be an endomorphism of a graph G. A subgraph of G is called the *endomorphic image* of G under f, denoted by I_f, if $V(I_f) = f(V(G))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$, where $a, b, c, d \in V(G)$. This definition seems to be natural since it ensures not only that a vertex in I_f must be an "image" of some vertex in G under f but also that an edge in I_f must be an "image" of some edge in G under f.

Let $G(V, E)$ be a graph. Let $\rho \subset V \times V$ be an equivalence relation on V. Denote by $[a]_\rho$ the equivalence class of $a \in V$ under ρ. A graph, denoted by G/ρ, is called the *factor graph* of G under ρ if $V(G/\rho) = V/\rho$ and $\{[a]_\rho, [b]_\rho\} \in E(G/\rho)$ if and only if there exist $c \in [a]_\rho$, $d \in [b]_\rho$ such that $\{c, d\} \in E(G)$. Let f be an endomorphism of G; by ρ_f denote the equivalence relation on $V(G)$ induced by f, i.e., for $a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. The graph G/ρ_f is simply called *the factor graph of* f. Define a mapping $i_f \colon V(G/\rho_f) \to V(I_f)$ with $i_f([x]_{\rho_f}) = f(x)$ for $x \in V(G)$. Obviously, i_f is well defined. We now have:

PROPOSITION 1.1. *Let G be a graph and let $f \in \mathrm{End}(G)$. Then the mapping i_f is an isomorphism from G/ρ_f to I_f.*

Proof. It is well known by the homomorphism theorem that i_f is bijective. We now show that i_f and i_f^{-1} are both homomorphisms. From the definition of the factor graph of f and the endomorphic image of G under f, it is easy to see the following: For $x, y \in G$, $\{[x]_{\rho_f}, [y]_{\rho_f}\} \in E(G/\rho_f) \iff$ there exist $c \in [x]_{\rho_f}$, $d \in [y]_{\rho_f}$ such that $\{c, d\} \in E(G) \iff$ there exist $c \in f^{-1}(f(x))$, $d \in f^{-1}(f(y))$ such that $\{c, d\} \in E(G) \iff \{f(x), f(y)\} \in E(I_f)$. This completes the proof. □

Remark 1.2. Let $f, g \in \mathrm{End}(G)$. If $\rho_f = \rho_g$, then $G/\rho_f = G/\rho_g$. By Proposition 1.1, $G/\rho_f \cong I_f$ under the isomorphism i_f, and $G/\rho_g \cong I_g$ under the isomorphism i_g. Thus $I_f \cong I_g$. We denote $i_{f,g} := i_g i_f^{-1}$ and $i_{g,f} := i_f i_g^{-1}$. It is easy to see that $i_{f,g}$ ($i_{g,f}$) is an isomorphism from I_f to I_g (from I_g to I_f) and $i_{f,g}^{-1} = i_{g,f}$. This can be shown in the following diagram:



Recall the definition of endomorphic image and notice that an endomorphism of a graph is an adjacency-preserving mapping. Then the following facts are almost trivial.

Remark 1.3. Let G be a graph. Let $f \in \mathrm{End}(G)$ and let $a, b \in G$.

(1) If G is connected, then I_f is connected.
(2) $d_{I_f}(f(a), f(b)) \leq d_G(a, b)$ (where $d_H(x, y)$ denotes the distance between the vertices x and y in the graph H).

The following definitions of Green's relations are based on the book [2].

Let S be a semigroup. Define a relation $\mathcal{L}$ on S such that $(a, b) \in \mathcal{L}$ if $S^1 a = S^1 b$ (S^1 is the semigroup obtained from S by adjoining an identity if necessary); similarly, define a relation $\mathcal{R}$ on S such that $(a, b) \in \mathcal{R}$ if $aS^1 = bS^1$. $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations on S. $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence. $\mathcal{L}$ and $\mathcal{R}$ commute with each other. Define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. Owing to the commutativity of $\mathcal{L}$ and $\mathcal{R}$, $\mathcal{D} = \mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. These equivalence relations are called *Green's relations* on the semigroup S.[2] The following proposition will be used in this paper.

PROPOSITION 1.4. ([2; Lemma II 1.1]) *Let a, b be elements of a semigroup S. Then $(a, b) \in \mathcal{L}$ if and only if there exist x, y in S^1 such that $xa = b$, $yb = a$. Also, $(a, b) \in \mathcal{R}$ if and only if there exist u, v in S^1 such that $au = b$, $bv = a$.*

For any graph and semigroup theoretic concepts needed which are not defined here, please refer to usual books on graph theory and semigroup theory, for example, [1] and [2].

---

[2] There is another Green's relation $\mathcal{J}$, which is defined as $(a, b) \in \mathcal{J}$ if $S^1 a S^1 = S^1 b S^1$ for $a, b \in S$. Since $\mathcal{D} = \mathcal{J}$ in any finite semigroup, we will not mention $\mathcal{J}$ in this paper.

2. Green's relations on End(G)

In this section, we will answer the question of what Green's relations mean on the endomorphism monoid of a graph. The main results of this section are Theorem 2.1 and Theorem 2.3. First, we give the characterization of Green's relation $\mathcal{L}$ on End(G).

THEOREM 2.1. *Let $f,g \in \mathrm{End}(G)$. Then $(f,g) \in \mathcal{L}$ if and only if $\rho_f = \rho_g$, and there exist $h,k \in \mathrm{End}(G)$ such that $h|_{I_g} = i_{g,f}$, $k|_{I_f} = i_{f,g}$.*

Proof.

Sufficiency. By Proposition 1.4, we only need to show that $f = hg$ and $g = kf$. Let $a \in G$. Then $g(a) \in I_g$ and $hg(a) = h|_{I_g}(g(a)) = i_{g,f}(g(a)) = i_f i_g^{-1}(g(a)) = i_f([a]_{\rho_g}) = i_f([a]_{\rho_f}) = f(a)$ by Proposition 1.1 and Remark 1.2. Thus, we have the first equality. The second one can be obtained in a similar manner.

Necessity. Let $(f,g) \in \mathcal{L}$. By Proposition 1.4, there exist $u,v \in \mathrm{End}(G)$ such that $f = ug$ and $g = vf$. Let $a,b \in V(G)$ with $f(a) = f(b)$. Then $g(a) = vf(a) = vf(b) = g(b)$. Similarly, we can see that $g(a) = g(b)$ implies $f(a) = f(b)$. Hence, we obtain that $\rho_f = \rho_g$. We now show that $u|_{I_g} = i_{g,f}$ and $v|_{I_f} = i_{f,g}$. Let $a \in V(I_g)$. Then there exists $x \in G$ with $g(x) = a$. It follows that $i_g^{-1}(a) = i_g^{-1}(g(x)) = [x]_{\rho_g}$. Hence $i_{g,f}(a) = i_f i_g^{-1}(a) = i_f([x]_{\rho_g}) = i_f([x]_{\rho_f}) = f(x)$. On the other hand, $u|_{I_g}(a) = u(a) = ug(x) = f(x)$. Therefore $u|_{I_g} = i_{g,f}$. The proof of $v|_{I_f} = i_{f,g}$ is very similar. □

Example 2.2. Let G be a graph as shown in Fig. 1 and let $g=\begin{pmatrix} 1&2&3&4&5&6 \\ 4&1&4&2&4&5 \end{pmatrix}$, $f = \begin{pmatrix} 1&2&3&4&5&6 \\ 4&2&4&3&4&5 \end{pmatrix}$, $f' = \begin{pmatrix} 1&2&3&4&5&6 \\ 4&1&4&2&4&3 \end{pmatrix}$. Then it is easy to see that $g, f, f' \in \mathrm{End}(G)$ and $\rho_g = \rho_f = \rho_{f'}$. One can readily check that $i_{g,f} = \begin{pmatrix} 1&2&4&5 \\ 2&3&4&5 \end{pmatrix}$, $i_{f,g} = \begin{pmatrix} 2&3&4&5 \\ 1&2&4&5 \end{pmatrix}$ and $i_{f',g} = \begin{pmatrix} 1&2&3&4 \\ 1&2&5&4 \end{pmatrix}$.

Let $h = \begin{pmatrix} 1&2&3&4&5&6 \\ 2&3&2&4&5&4 \end{pmatrix}$ and $k = \begin{pmatrix} 1&2&3&4&5&6 \\ 2&1&2&4&5&4 \end{pmatrix}$. Then we have $h,k \in \mathrm{End}(G)$ and $h|_{I_g} = i_{g,f}$ and $k|_{I_f} = i_{f,g}$. So, by Theorem 2.1, $(f,g) \in \mathcal{L}$. Since $\{2,3\} \in E(G)$ and $\{2,5\} \notin E(G)$, there does not exist $h \in \mathrm{End}(G)$ such that $h|_{I_{f'}} = i_{f',g}$. Thus, by Theorem 2.1, $(f',g) \notin \mathcal{L}$.

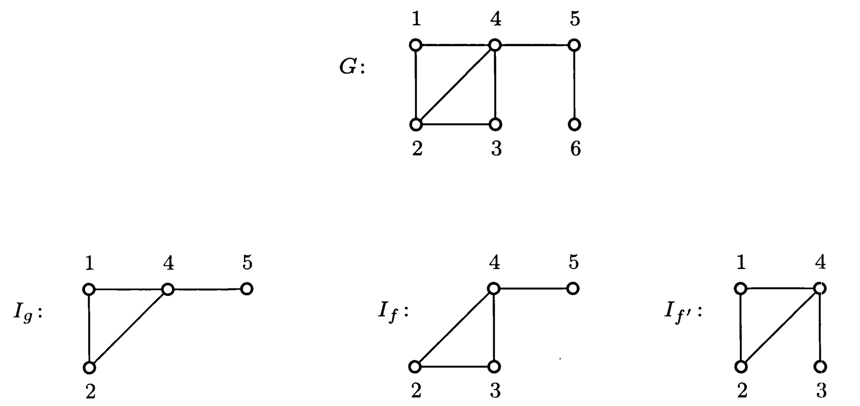


Figure 1.

Now, we turn to the characterization of Green's relation $\mathcal{R}$ on $\mathrm{End}(G)$.

THEOREM 2.3. *Let $f, g \in \mathrm{End}(G)$. Then $(f, g) \in \mathcal{R}$ if and only if $I_f = I_g$ and there exist $u, v \in \mathrm{End}(G)$ such that for any $a \in V(I_f)$ $(= V(I_g))$, $u(f^{-1}(a)) \subset g^{-1}(a)$, $v(g^{-1}(a)) \subset f^{-1}(a)$.*

Proof.

Necessity. Since $(f, g) \in \mathcal{R}$, by Proposition 1.4, there exist $h, k \in \mathrm{End}(G)$ with $f = gh$ and $g = fk$. Thus $g(V(G)) = fk(V(G)) \subset f(V(G))$, $f(V(G)) = gh(V(G)) \subset g(V(G))$. So we have $V(I_f) = V(I_g)$.

Now, let $a, b \in G$ with $\{a, b\} \in E(I_f)$. Then there exist $x, y \in G$ such that $\{x, y\} \in E(G)$ and $f(x) = a$, $f(y) = b$. Therefore, $gh(x) = a$, $gh(y) = b$. Thus, $h(x) \in g^{-1}(a)$, $h(y) \in g^{-1}(b)$. Since $\{h(x), h(y)\} \in E(G)$, $\{a, b\} \in E(I_g)$. Accordingly, we can prove that $\{a, b\} \in E(I_g)$ implies that $\{a, b\} \in E(I_f)$. So, we conclude that $I_f = I_g$.

Denote $I := I_f = I_g$. Take $u = h$ and $v = k$. If $x \in u(f^{-1}(a))$, then there exists $y \in f^{-1}(a)$ such that $u(y) = x$. Thus $a = f(y) = gh(y) = gu(y) = g(x)$, which means $x \in g^{-1}(a)$. Consequently, $u(f^{-1}(a)) \subset g^{-1}(a)$. By a similar argument, we can obtain $v(g^{-1}(a)) \subset f^{-1}(a)$.

Sufficiency. We show that $f = gu$ and $g = fv$. Let $x \in G$ and $f(x) = a$. Then $a \in I_f = I_g$. Notice the hypothesis $u(f^{-1}(a)) \subset g^{-1}(a)$ and $x \in f^{-1}(a)$; we have $u(x) \in g^{-1}(a)$. Thus, $gu(x) = a = f(x)$. So we have $f = gu$. Similarly, we can prove $g = fv$. Then by Proposition 1.4, $(f, g) \in \mathcal{R}$. □

Example 2.4. We still take the graph G in Fig. 1 as an example. Also let the endomorphism g be as shown in Example 2.2, i.e., $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 4 & 2 & 4 & 5 \end{pmatrix}$. Let

$f = \begin{pmatrix} 1&2&3&4&5&6 \\ 1&4&1&2&4&5 \end{pmatrix}$ and $f' = \begin{pmatrix} 1&2&3&4&5&6 \\ 2&4&2&1&4&5 \end{pmatrix}$. Then $g, f, f' \in \mathrm{End}(G)$. It is easy to see that $I_g = I_f = I_{f'}$ (cf. Fig. 2).

We have $g^{-1}(1) = \{2\}$; $g^{-1}(2) = \{4\}$; $g^{-1}(4) = \{1,3,5\}$; $g^{-1}(5) = \{6\}$; and $f^{-1}(1) = \{1,3\}$; $f^{-1}(2) = \{4\}$; $f^{-1}(4) = \{2,5\}$; $f^{-1}(5) = \{6\}$. Take $u = v = \begin{pmatrix} 1&2&3&4&5&6 \\ 2&3&2&4&5&6 \end{pmatrix} \in \mathrm{End}(G)$. Then for any $a \in V(I_g)$ $(= V(I_f) = \{1,2,4,5\})$, $u(f^{-1}(a)) \subset g^{-1}(a)$, $v(g^{-1}(a)) \subset f^{-1}(a)$. Thus, by Theorem 2.3, $(f,g) \in \mathcal{R}$. Now assume that there exists $w \in \mathrm{End}(G)$ such that $w(f'^{-1}(a)) \subset g^{-1}(a)$ for any $a \in V(I_{f'})$ $(= V(I_g))$. Since $f'^{-1}(5) = g^{-1}(5) = \{6\}$, $f'^{-1}(1) = \{4\}$ and $g^{-1}(1) = \{2\}$, then $w(6) = 6$ and $w(4) = 2$. Note that $\{5,4\}, \{5,6\} \in E(G)$, so $\{w(5), w(4)\} = \{w(5), 2\} \in E(G)$ and $\{w(5), w(6)\} = \{w(5), 6\} \in E(G)$. But there does not exist a vertex in G which is adjacent to both of the vertices 2 and 6, which means that such an endomorphism w does not exist. Hence, by Theorem 2.3, we have $(f', g) \notin \mathcal{R}$.

$I_g = I_f = I_{f'}$:

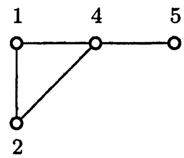


Figure 2.

The following two corollaries follow directly from the previous two theorems and the definitions of Green's relations $\mathcal{D}$ and $\mathcal{H}$.

COROLLARY 2.5. *Let G be a graph and $f, g \in \mathrm{End}(G)$. Then $(f,g) \in \mathcal{D}$ is equivalent to the following two conditions:*

(1) *There exists $h \in \mathrm{End}(G)$ such that $\rho_f = \rho_h$ and $I_h = I_g$.*

(2) *For the endomorphism h in (1), there exist $u, v \in \mathrm{End}(G)$ such that $u|_{I_h} = i_{h,f}$, $v|_{I_f} = i_{f,h}$, and there exist $u', v' \in \mathrm{End}(G)$ such that for any $a \in I_h$ $(= I_g)$, $u'(h^{-1}(a)) \subset g^{-1}(a)$, $v'(g^{-1}(a)) \subset h^{-1}(a)$.*

COROLLARY 2.6. *Let G be a graph and $f, g \in \mathrm{End}(G)$. Then $(f,g) \in \mathcal{H}$ is equivalent to the following two conditions:*

(1) *$I_f = I_g$ and $\rho_f = \rho_g$,*

(2) *there exist $u, v \in \mathrm{End}(G)$ such that $u|_I = i_{g,f}$, $v|_I = i_{f,g}$, and there exist $u', v' \in \mathrm{End}(G)$ such that for any $a \in V(I)$, $u'(f^{-1}(a)) \subset g^{-1}(a)$, $v'(g^{-1}(a)) \subset f^{-1}(a)$ (here, denote $I := I_f = I_g$ by condition (1)).*

Remark 2.7. We have also the following results concerning regular endomorphisms of a graph (An element a of a semigroup S is said to be *regular* if there exists an element b of S such that $aba = a$.):

Let G be a graph. Suppose $f, g \in \mathrm{End}(G)$ are regular. Then

$$(f,g) \in \mathcal{L} \iff \rho_f = \rho_g\,;$$
$$(f,g) \in \mathcal{R} \iff I_f = I_g\,;$$
$$(f,g) \in \mathcal{H} \iff \rho_f = \rho_g \text{ and } I_f = I_g\,;$$
$$(f,g) \in \mathcal{D} \iff I_f \cong I_g\,.$$

As one of the referees pointed out, they are not used for the next study of this paper and similar results concerning regular elements are well known for many transformation monoids (see, for example, [8]). So it would be appropriate just to mention them.

3. Green's relations on the endomorphism monoids of some special classes of graphs

For endomorphism monoids of some special classes of graphs, Green's relations may be more distinctive from the viewpoint of combinatorics. In this section, using the results of Section 2, we will show that for a tree T or a circle C_n, two endomorphisms are $\mathcal{L}$-equivalent if and only if they share the same factor graph (Theorems 3.3 and 3.8). The results regarding Green's relation $\mathcal{R}$ are also mentioned. First, we give a lemma concerning a tree.

LEMMA 3.1. *Let T be a tree and $f, g \in \mathrm{End}(T)$. If $\rho_f = \rho_g$, then there exist $k, h \in \mathrm{End}(T)$ such that $k\big|_{I_f} = i_{f,g}$ and $h\big|_{I_g} = i_{g,f}$.*

Proof. By symmetry, we only need to prove the existence of k. Since $\rho_f = \rho_g$, by Remark 1.2, $I_f \cong I_g$ and $i_{f,g}$ is an isomorphism from I_f to I_g. If $f \in \mathrm{Aut}(T)$, obviously, $I_f = I_g = T$, and so we only need to put $k = i_{f,g}$. Now, suppose that $f \notin \mathrm{Aut}(T)$. As $f \in \mathrm{End}(T)$, by Remark 1.3 (1), I_f is connected, i.e., I_f is a subtree of T. By $T - I_f$ denote a graph obtained from T by removing all the edges of I_f and all the vertices of I_f which are only incident to the edges of I_f (for convenience, to see this, we give an example in Fig. 3 and 4). Clearly, each component of $T - I_f$ contains exactly one vertex which is also a vertex of I_f. We write $T - I_f := \bigcup_{i=1}^{n} T_{x_i}$ for some $n \geq 1$, where T_{x_i} denotes a component of $T - I_f$ with a unique $x_i \in I_f$ (T_{x_i} is obviously a subtree of T). For each T_{x_i}, select (arbitrarily and fixed) a vertex in I_f, denoted by x_i', which is adjacent to x_i in I_f (noticing that $f \notin \mathrm{Aut}(T)$, such a vertex x_i' must exist). Define a mapping $k\colon V(T) \to V(T)$ by the following rule:

$$k(x) = \begin{cases} i_{f,g}(x) & \text{if } x \in V(I_f)\,, \\ i_{f,g}(x_i) & \text{if } x \in V(T_{x_i}) \setminus \{x_i\} \text{ and } d(x, x_i) \text{ is even,} \\ i_{f,g}(x_i') & \text{if } x \in V(T_{x_i}) \setminus \{x_i\} \text{ and } d(x, x_i) \text{ is odd,} \end{cases}$$

where $d(x, x_i)$ denotes the distance between the vertices x and x_i in T_{x_i} and $i = 1, 2, \ldots, n$.

Clearly, k, as a mapping, is well-defined. We now show that $k \in \mathrm{End}(T)$. Take $a, b \in V(T)$ such that $\{a, b\} \in E(T)$. Then there are the following three cases to be considered:

(1) $a, b \in V(I_f)$. Then $\{k(a), k(b)\} = \{i_{f,g}(a),\, i_{f,g}(b)\} \in E(I_g) \subset E(T)$.

(2) $a, b \in V(T_{x_i})$ for some x_i and one of them, say, $a = x_i$. In this case, $b \in V(T_{x_i}) \setminus \{x_i\}$ and $d(b, x_i) = d(b, a)$ $(= 1)$ is odd. Noticing that $\{x_i, x_i'\} \in E(I_f)$, we have $\{k(a), k(b)\} = \{k(x_i), k(b)\} = \{i_{f,g}(x_i),\, i_{f,g}(x_i')\} \in E(I_g) \subset E(T)$.

(3) $a, b \in V(T_{x_i}) \setminus \{x_i\}$ for some x_i. Without loss of generality, we may suppose that $d(a, x_i)$ is even and $d(b, x_i)$ is odd. It follows that $\{k(a), k(b)\} = \{i_{f,g}(x_i),\, i_{f,g}(x_i')\} \in E(I_g) \subset E(T)$. Consequently, $k \in \mathrm{End}(T)$.

That $k\big|_{I_f} = i_{f,g}$ is straightforward from the definition of k. The proof is completed. □

Example 3.2.

T:

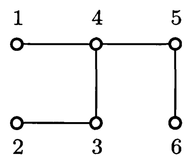


Figure 3.

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 4 & 1 & 4 \end{pmatrix}$. Then $f \in \mathrm{End}(T) \setminus \mathrm{Aut}(T)$ and

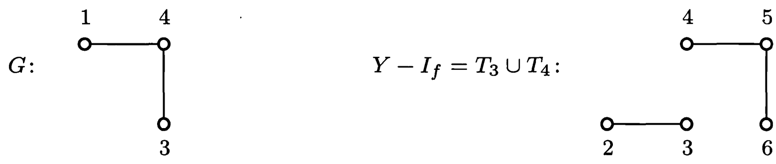


Figure 4.

Now, by Theorem 2.1 and the preceding lemma, we immediately have the following theorem.

THEOREM 3.3. *Let T be a tree and $f, g \in \mathrm{End}(T)$. Then $(f, g) \in \mathcal{L}$ if and only if $\rho_f = \rho_g$.*

Remark 3.4.

(i) For a tree T, there is no corresponding result for Green's relation $\mathcal{R}$. Namely, for $f,g \in \mathrm{End}(T)$, $I_f = I_g$ does not imply in general that $(f,g) \in \mathcal{R}$. We can see this in the following example.

T:

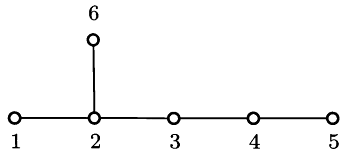


Figure 5.

Let T be the tree as shown in Fig. 5 and let $f = \begin{pmatrix} 1&2&3&4&5&6 \\ 1&2&1&2&3&6 \end{pmatrix}$, $g = \begin{pmatrix} 1&2&3&4&5&6 \\ 1&2&3&2&3&6 \end{pmatrix}$. Then $f,g \in \mathrm{End}(T)$. Clearly, $I_f = I_g$. Assume that there exists $v \in \mathrm{End}(T)$ such that $v\big(g^{-1}(a)\big) \subset f^{-1}(a)$ for any $a \in V(I_f)$ $(= V(I_g))$. On one hand, since $3 \in V(I_f)$ and $g^{-1}(3) = \{3,5\}$ and $f^{-1}(3) = \{5\}$, one has $v(3) = 5$. Thus $v(2) = 4$. But on the other hand, since $6 \in V(I_f)$ and $g^{-1}(6) = \{6\}$ and $f^{-1}(6) = \{6\}$, we have $v(6) = 6$, and so $v(2) = 2$. This yields a contradiction. Hence, such an endomorphism v does not exist. By Theorem 2.3, $(f,g) \notin \mathcal{R}$.

However, we can prove the following statement (in [7]):

Let T be a graph. Then the following two assertions are equivalent:

(1) *$d(T) \le 3$ or $T = P_5$ (where $d(T)$ denotes the diameter of the tree T).*
(2) *For any $f,g \in \mathrm{End}(T)$, $(f,g) \in \mathcal{R}$ if and only if $I_f = I_g$.*

Since the proof is somewhat tedious, we will not give the proof here.

(ii) Theorem 3.3 cannot be generalized to a forest F. Namely, for $f,g \in \mathrm{End}(F)$, $\rho_f = \rho_g$ does not imply in general $(f,g) \in \mathcal{L}$. The following example illustrates this.

F:

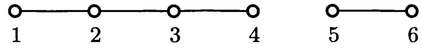


Figure 6.

Let F be the forest as shown in Fig. 6 and let $f = \begin{pmatrix} 1&2&3&4&5&6 \\ 1&2&1&2&3&4 \end{pmatrix}$, $g = \begin{pmatrix} 1&2&3&4&5&6 \\ 1&2&1&2&5&6 \end{pmatrix}$. It is easy to see that $f,g \in \mathrm{End}(F)$, $\rho_f = \rho_g$ and $i_{f,g} = \begin{pmatrix} 1&2&3&4 \\ 1&2&5&6 \end{pmatrix}$.

Since $\{2,3\} \in E(F)$ and $\{2,5\} \notin E(F)$, there does not exist an endomorphism $k \in \mathrm{End}(F)$ such that $k|_{I_f} = i_{f,g}$. Thus by Theorem 2.1, $(f,g) \notin \mathcal{L}$.

(iii) Theorem 3.3 cannot be generalized to a (connected) bipartite graph B. Namely, for $f,g \in \mathrm{End}(B)$, $\rho_f = \rho_g$ does not imply in general $(f,g) \in \mathcal{L}$. The following example provides a justification.

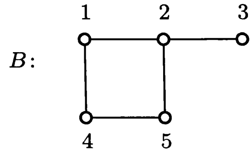


Figure 7.

Let B be the (connected) bipartite graph as shown in Fig. 7 and let $f = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,5\,4\,1 \end{pmatrix}$, $g = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,3\,4\,1 \end{pmatrix}$. It is easy to see that $f,g \in \mathrm{End}(B)$ such that $\rho_f = \rho_g$ and $i_{f,g} = \begin{pmatrix} 1\,2\,5\,4 \\ 1\,2\,3\,4 \end{pmatrix}$. Since $\{4,5\} \in E(B)$ and $\{4,3\} \notin E(B)$, there does not exist an endomorphism $k \in \mathrm{End}(B)$ such that $k|_{I_f} = i_{f,g}$. Thus, by Theorem 2.1, $(f,g) \notin \mathcal{L}$.

The remainder of this section will be devoted to the investigation of a circle C_n. The following facts will be used later.

LEMMA 3.5. ([5; Remark 1.2]) *The circle C_{2k+1} $(k \in \mathbb{N})$ is unretractive (where $\mathbb{N}$ denotes the set of natural numbers).*

LEMMA 3.6. *Let C_{2k} $(k \geq 2)$ be a circle and let $f \in \mathrm{End}(C_{2k})$. If $f \notin \mathrm{Aut}(C_{2k})$, then $I_f = P_m$ for some $m \in \{2,\ldots,k+1\}$.*

Proof. This follows directly from Remark 1.3 (1) and (2). □

Now, we are going to prove the following lemma, which is similar to Lemma 3.1 and crucial to the next theorem.

LEMMA 3.7. *Let C_n $(n \geq 3)$ be a circle and let $f,g \in \mathrm{End}(C_n)$. If $\rho_f = \rho_g$, then there exist $k,h \in \mathrm{End}(C_n)$ such that $k|_{I_f} = i_{f,g}$ and $k|_{I_g} = i_{g,f}$.*

Proof. By symmetry, we only need to prove the existence of k.

If $f \in \mathrm{Aut}(C_n)$, since $\rho_f = \rho_g$, obviously $g \in \mathrm{Aut}(C_n)$. In this case, $I_f = I_g = C_n$. We set $k = i_{f,g}$. It is easy to see that $k \in \mathrm{End}(C_n)$ and $k|_{I_f} = k = i_{f,g}$.

Thus, by virtue of Lemma 3.5, we only need to deal with the cases $n = 2k$ $(k \geq 2)$ and $f \in \mathrm{End}(C_n) \setminus \mathrm{Aut}(C_n)$. Using Lemma 3.6, we have $I_f = P_m$ for some $m \in \{2,\ldots,k+1\}$. Since $2 \leq m \leq k+1$, $2m \leq 2k+2$, and so

$n = 2k \geq 2m-2$. Thus we may let $n = (2m-2)+2t_0$, where $t_0 := k-m+1 \geq 0$. Since $\rho_f = \rho_g$, by Remark 1.2, we see that $I_f \cong I_g$ under $i_{f,g}$.

Without loss of generality, we may suppose that C_n and I_f are as shown in Fig. 8.

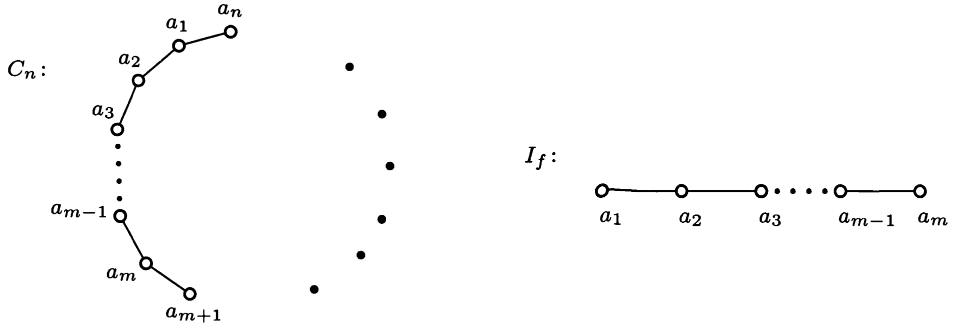


Figure 8.

We now define a mapping $k\colon V(C_n) \to V(C_n)$ by the following rule:

For any $x \in I_f$, i.e., for $x \in \{a_1, a_2, \dots, a_m\}$, set $k(x) = i_{f,g}(x)$.

For any $x \notin I_f$, i.e., for $x \in \{a_{m+1}, a_{m+2}, \dots, a_n\}$, define $k(x)$ in the following two cases:

Case 1. $t_0 = 0$. Set $k(a_{m+s}) = i_{f,g}(a_{m-s})$ $(= k(a_{m-s}))$ for $s = 1, 2, \dots, m-2$.
Case 2. $t_0 > 0$. Set $k(a_{m+s}) = i_{f,g}(a_{m-s})$ $(= k(a_{m-s}))$ for $s = 1, 2, \dots, m-2$;
set $k\big(a_{(2m-2)+(2t-1)}\big) = i_{f,g}(a_1)$ $(= k(a_1))$ and
set $k\big(a_{(2m-2)+2t)}\big) = i_{f,g}(a_2)$ $(= k(a_2))$ for $t = 1, 2, \dots, t_0$.

It is easy to see that the mapping k is well-defined and $k\big|_{I_f} = i_{f,g}$. It remains to prove that $k \in \mathrm{End}(C_n)$. Let $a, b \in C_n$ with $\{a, b\} \in E(C_n)$. We want to show that $\{k(a), k(b)\} \in E(C_n)$. As in the definition of the mapping k, we consider two cases correspondingly:

Case 1. $t_0 = 0$.

Suppose that $a, b \in I_f$. Then $\{a, b\} \in E(I_f)$, and, by the definition of k, we have $\{k(a), k(b)\} = \{i_{f,g}(a),\, i_{f,g}(b)\} \in E(I_g) \subset E(C_n)$.

Suppose that just one of a and b belongs to I_f. Without loss of generality, let $a \in I_f$ and $b \notin I_f$. There are two cases to be considered:

(1) $a = a_m$ and $b = a_{m+1}$. Then $k(a) = i_{f,g}(a_m)$ and $k(b) = k(a_{m+1}) = k(a_{m-1}) = i_{f,g}(a_{m-1})$. Since $\{a_{m-1}, a_m\} \in E(I_f)$, $\{k(b), k(a)\} = \{i_{f,g}(a_{m-1}),\, i_{f,g}(a_m)\} \in E(I_g) \subset E(C_n)$.

(2) $a = a_1$ and $b = a_n$. Then $b = a_{2m-2}$ and $k(b) = k(a_{2m-2}) = k\big(a_{m+(m-2)}\big) = k\big(a_{m-(m-2)}\big) = k(a_2)$. So, since $\{a_1, a_2\} \in E(I_f)$, $\{k(a), k(b)\} = \{k(a_1), k(a_2)\} = \{i_{f,g}(a_1),\, i_{f,g}(a_2)\} \in E(I_g) \subset E(C_n)$.

Suppose that $a, b \notin I_f$. Without loss of generality, we may let $a = a_{m+s_0}$ and $b = a_{m+s_0+1}$, where $1 \le s_0 \le m-3$. Then we have $k(a) = k(a_{m+s_0}) = k(a_{m-s_0})$ and $k(b) = k(a_{m+s_0+1}) = k(a_{m-s_0-1})$. It is easy to see that $a_{m-s_0-1}, a_{m-s_0} \in I_f$ with $\{a_{m-s_0-1}, a_{m-s_0}\} \in E(I_f)$. So $\{k(a), k(b)\} = \{k(a_{m-s_0}), k(a_{m-s_0-1})\} = \{i_{f,g}(a_{m-s_0}), i_{f,g}(a_{m-s_0-1})\} \in E(I_g) \subset E(C_n)$.

Case 2. $t_0 > 0$.

Suppose that $a, b \in I_f$. By the same argument as in the corresponding part of Case 1, we have $\{k(a), k(b)\} \in E(C_n)$.

Suppose that just one of a and b belongs to I_f. Without loss of generality, let $a \in I_f$ and $b \notin I_f$. There are two cases to be considered:

(1) $a = a_m$ and $b = a_{m+1}$. By the same argument as in Case 1, we have $\{k(a), k(b)\} \in E(C_n)$.

(2) $a = a_1$ and $b = a_n$. Then $b = a_{(2m-2)+2t_0}$ with $t_0 > 0$. Thus $k(b) = k(a_2)$, and we also have $\{k(a), k(b)\} \in E(C_n)$.

Suppose that $a, b \notin I_f$. If $a, b \in \{a_{m+s} \mid s = 1, 2, \ldots, m-2\}$, then in a similar manner as in Cases 1, we can obtain $\{k(a), k(b)\} \in E(C_n)$. If $a, b \notin \{a_{m+s} \mid s = 1, 2, \ldots, m-2\}$, then for some $t \in \{1, 2, \ldots, t_0\}$, $\{a, b\}$ $(= \{b, a\})$ $= \{a_{(2m-2)+(2t-1)}, a_{(2m-2)+2t}\}$. Thus by the definition of k, $\{k(a), k(b)\} = \{k(a_1), k(a_2)\} \in E(C_n)$. Now, without loss of generality, let $a \in \{a_{m+s} \mid s = 1, 2, \ldots, m-2\}$ and $b \notin \{a_{m+s} \mid s = 1, 2, \ldots, m-2\}$. Noticing that $a, b \notin I_f$, there is only one possibility, i.e., $a = a_{2m-2}$ and $b = a_{2m-1}$. So,

$$\begin{aligned}\{k(a), k(b)\} &= \Big\{k\big(a_{m+(m-2)}\big),\ k\big(a_{(2m-2)+1}\big)\Big\} \\ &= \Big\{k\big(a_{m-(m-2)}\big),\ k\big(a_{(2m-2)+(2\cdot 1-1)}\big)\Big\} \\ &= \big\{k(a_2), k(a_1)\big\} \in E(I_g) \subset E(C_n).\end{aligned}$$

The proof is now completed. □

Now, the following theorem follows directly from Theorem 2.1 and Lemma 3.7.

THEOREM 3.8. *Let C_n be a circle and $f, g \in \mathrm{End}(C_n)$. Then $(f, g) \in \mathcal{L}$ if and only if $\rho_f = \rho_g$.*

R e m a r k 3.9. Regarding Green's relation $\mathcal{R}$ on the endomorphism monoid of a circle, we have the following result (in [7]). Also because of the tediousness of the proof, we will not verify it here.

Let C_n be a circle with n vertices. Then the following two assertions are equivalent:

(1) $n = 2k + 1$ $(k \in \mathbb{N})$, *or* $n \in \{4, 6, 8\}$.

(2) *for any* $f, g \in \mathrm{End}(C_n)$, $(f, g) \in \mathcal{R}$ *if and only if* $I_f = I_g$.

Acknowledgement

The author would like to thank Professor Ulrich Knauer for his instruction and advice; and Professor Oto Strauch and the referees for their comments and suggestions.

REFERENCES

[1] HARARY, F.: *Graph Theory*, Addison-Wesley, Reading, 1969.

[2] HOWIE, J. M.: *An Introduction to Semigroup Theory*, Academic Press, New York-London, 1976.

[3] KNAUER, U.—NIEPORTE, M.: *Endomorphisms of graphs I*, Arch. Math. (Basel) **52** (1989), 607–614.

[4] KNAUER, U.: *Endomorphisms of graphs II*, Arch. Math. (Basel) **55** (1990), 193–203.

[5] KNAUER, U.: *Unretractive and s-unretractive joins and lexicographic products of graphs*, J. Graph Theory **11** (1987), 429–440.

[6] LI, W.-M.: *Green's relations on the strong endomorphism monoid of a graph*, Semigroup Forum **47** (1993), 209–214.

[7] LI, W.-M.: *The Structure of the Endomorphism Monoid of a Graph.* Ph.D. Thesis, Germany, Universität Oldenburg, 1993.

[8] MAGILL, K. D.: *A survey of semigroups of continuous selfmaps*, Semigroup Forum **11** (1975/76), 189–282.

Received June 3, 1993
Revised October 4, 1993

Department of Mathematics
Shanghai Jiao-Tong University
200 240 Shanghai
CHINA