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DIOPHANTINE INEQUALITIES IN IMAGINARY QUADRATIC NUMBER FIELDS

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ABSTRACT. Some elementary diophantine approximation results in imaginary quadratic number fields are proved. This generalizes results of J. F. Koksma, K. Mahler, and E. Hlawka.

1. Introduction

J. F. Koksma [2] and K. Mahler [4] have proved some diophantine approximation results for square roots of real positive integers. In [1] E. Hlawka extended these results to Gaussian integers. For instance, one theorem says that a complex rational number $\eta \in \mathbb{Q}(\mathrm{i})$ can be approximated by square roots $\sqrt{z}$ of Gaussian integers $z \in \mathbb{Z}(\mathrm{i})$ very badly, or $\eta - \sqrt{z} \in \mathbb{Z}(\mathrm{i})$. In the following we prove a generalization of this property to nth roots of numbers in an imaginary quadratic number field $\mathbb{Q}(\mathrm{i}\sqrt{d})$. We use the notation $\sqrt[n]{z}$ for the principle value of the nth root, i.e. $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi}{n} + \mathrm{i}\sin\frac{\varphi}{n}\right)$ with $\varphi = \arg z$, $r = |z|$ and $-\pi < \arg z \le \pi$. Furthermore for every $\eta \in \mathbb{C}$ we define $\|\eta\| = \min\{|\eta - z| : z \in \mathbb{Z}(\mathrm{i}\sqrt{d})\}$, where $\mathbb{Z}(\mathrm{i}\sqrt{d})$ denotes the ring of integers in $\mathbb{Q}(\mathrm{i}\sqrt{d})$.

2. An elementary lower bound

THEOREM 1. *Let $\eta \in \mathbb{Q}(\mathrm{i}\sqrt{d})$, and $n \ge 2$ be a positive integer. Then there exists a positive constant $C = C(n,d,\eta)$ such that either $\eta - \sqrt[n]{z} \in \mathbb{Z}(\mathrm{i}\sqrt{d})$ or*

$$\left\|\eta - \sqrt[n]{z}\right\| > \frac{C}{|z|^{\frac{n-1}{n}}}$$

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for all $z \in \mathbb{Z}(\mathrm{i}\sqrt{d}) \setminus \{0\}$.

P r o o f. Trivially, the result is true if $\eta - \sqrt[n]{z} \in \mathbb{Z}(\mathrm{i}\sqrt{d})$ or if there exists a constant $c_1 > 0$ such that $\|\eta - \sqrt[n]{z}\| > c_1$ for all $z \in \mathbb{Z}(\mathrm{i}\sqrt{d})$ with the exception of at most finitely many z. Hence we can assume that there is a sequence $\{z_i\}$ in $\mathbb{Z}(\mathrm{i}\sqrt{d})$ ($|z_i| \to \infty$) with the property:

For every $\varepsilon > 0$ there exists a $Z > 1$ and a sequence $x_i \in \mathbb{Z}(\mathrm{i}\sqrt{d})$ such that for all z_i with $|z_i| > Z$

$$0 < \left|\eta - \sqrt[n]{z_i} + x_i\right| < \varepsilon . \tag{1}$$

Let ξ denote a primitive nth root of unity. We will show for $k = 1, \dots, n-1$

$$\eta + x_i - \xi^k \sqrt[n]{z_i} \neq 0 . \tag{2}$$

We suppose that $\eta + x_i - \xi^k \sqrt[n]{z_i} = 0$ for some $k \in \{1, \dots, n-1\}$. Hence we obtain from (1)

$$|1 - \xi^{-k}||\eta + x_i| < \varepsilon .$$

Setting $\eta = \dfrac{p}{q}$ with $p, q \in \mathbb{Z}(\mathrm{i}\sqrt{d})$ and $(p, q) = 1$, we derive

$$|p + qx_i| < \frac{\varepsilon|q|}{|1 - \xi|} . \tag{3}$$

Choosing $\varepsilon = \dfrac{|1-\xi|}{2|q|}$ yields $p + qx_i = 0$, since $p, q, x_i \in \mathbb{Z}(\mathrm{i}\sqrt{d})$. Thus $\eta + x_i = 0$, and from (1) we have

$$\left|\sqrt[n]{z_i}\right| < \frac{|1-\xi|}{2|q|} < 1 , \tag{4}$$

a contradiction to $|z_i| > 1$. Hence (2) is proved.

Now we set again $\eta = \dfrac{p}{q}$ with $p, q \in \mathbb{Z}(\mathrm{i}\sqrt{d})$ and $(p, q) = 1$. We obtain

$$|q|^n \prod_{k=0}^{n-1} \left(\eta + x_i - \xi^k \sqrt[n]{z_i}\right) = |(p + qx_i)^n - z_i q^n| . \tag{5}$$

Since $\omega = (p + qx_i)^n - z_i q^n$ is a non-zero integer in the quadratic number field $\mathbb{Q}(\mathrm{i}\sqrt{d})$, we have

$$|\omega| \geq C_0 = \sqrt{d+1} . \tag{6}$$

Furthermore we get

$$\left|\eta - \xi^k \sqrt[n]{z_i} + x_i\right| \leq \left|\eta + x_i - \sqrt[n]{z_i}\right| + \left|\sqrt[n]{z_i}\right|\left|1 - \xi^k\right| < \varepsilon + \sqrt[n]{|z_i|}\left|1 - \xi^k\right| . \quad (7)$$

Combining (5), (6) and (7) yields

$$\begin{aligned}\left|\eta + x_i - \sqrt[n]{z_i}\right| &> \frac{C_0}{|q|^n \prod_{k=1}^{n-1}\left(\varepsilon + \sqrt[n]{|z_i|}\,|1-\xi^k|\right)} > \frac{1-\varepsilon_1}{|q|^n |z_i|^{\frac{n-1}{n}} \prod_{k=1}^{n-1}|1-\xi^k|} \\ &= \frac{1-\varepsilon_1}{|q|^n n |z_i|^{\frac{n-1}{n}}} ,\end{aligned} \quad (8)$$

where ε_1 is a suitable positive number. (Note that the formula $\prod_{k=1}^{n-1}(1-\xi^k) = n$ has been used here.) Choosing $C(n,d,\eta) = \min\left\{C_2, \frac{1-\varepsilon_1}{|q|^n n}\right\}$, where

$$|\eta - \sqrt[n]{z} + x| > C_2$$

for all z not contained in the sequence $\{z_i\}$ with $|z_i| > Z$, proves the theorem.

3. Concluding remarks

We establish a converse inequality to Theorem 1 for square roots. We need the following lemma (cf. [3]).

Lemma 1. *Let $\mathbb{Q}(\mathrm{i}\sqrt{d})$ be an imaginary quadratic field and $\theta \notin \mathbb{Q}(\mathrm{i}\sqrt{d})$. Then there exist infinitely many pairs (p,q) of integers in $\mathbb{Q}(\mathrm{i}\sqrt{d})$ with $q \neq 0$ and $|q| \to \infty$ such that*

$$\left|\theta - \frac{p}{q}\right| < \frac{c_1}{|q|^2}$$

for some positive constant $c_1 = c_1(d)$.

From this lemma one can deduce the following inhomogeneous diophantine approximation result by standard arguments:

Proposition 1. *Let $\mathbb{Q}(\mathrm{i}\sqrt{d})$ be an imaginary quadratic field and $\theta \notin \mathbb{Q}(\mathrm{i}\sqrt{d})$, η arbitrary. Then there exist infinitely many pairs (x,y) of integers in $\mathbb{Q}(\mathrm{i}\sqrt{d})$ with $x \neq 0$, $\operatorname{Re} x \geq 0$, $\operatorname{Im} x \geq 0$, and $|x| \to \infty$ such that*

$$|\theta x - y - \eta| \leq \frac{c_2}{|x|}$$

for some positive constant $c_2 = c_2(d)$.

Proof. Set $\theta = \frac{p}{q} + \frac{\tilde{\delta}}{q^2}$ with p and q relatively prime and with $|\tilde{\delta}| < c_1$ in Lemma 1 and choose $q_1 \in \mathbb{Z}(\mathrm{i}\sqrt{d})$ such that

$$|\eta q - q_1| \le \frac{\sqrt{d+1}}{2}\,. \tag{9}$$

Set $\xi = \frac{\sqrt{d+1}}{2}(1+\mathrm{i})$, and let $|x_0, y_0|$ be a solution of the diophantine equation $px - qy = q_1$. Then all solutions are of the form $x = x_0 + q\lambda$, $y = y_0 + p\lambda$ with $\lambda \in \mathbb{Z}(\mathrm{i}\sqrt{d})$.

Take an arbitrary $\delta > 0$, and choose λ such that

$$\left|\frac{1}{q}\big(x_0 - \xi(1+\delta)|q|\big) + \lambda\right| \le \frac{\sqrt{d+1}}{2}\,. \tag{10}$$

With $x = x_0 + q\lambda$ we obtain

$$\begin{aligned} \big|\mathrm{Re}\big(x - \xi(1+\delta)|q|\big)\big| &\le \frac{\sqrt{d+1}}{2}\,|q|\,, \\ \big|\mathrm{Im}\big(x - \xi(1+\delta)|q|\big)\big| &\le \frac{\sqrt{d+1}}{2}\,|q|\,. \end{aligned} \tag{11}$$

Thus we get $0 < |q|\frac{\sqrt{d+1}}{2}\delta \le \mathrm{Re}\,x \le |q|\left((1+\delta)\,\mathrm{Re}\,\xi + \frac{\sqrt{d+1}}{2}\right)$ and similarly for the imaginary parts. From this we immediately derive

$$|q||\delta| < |x| \le |q|\sqrt{\frac{d+1}{2}}(2+\delta)\,. \tag{12}$$

Hence we have

$$|\theta x - y - \eta| = \left|\frac{px - qy}{q} + \frac{\delta x}{q^2} - \eta\right| = \left|\frac{q_1 - \eta q}{q} + \frac{\delta x}{q^2}\right| \le \frac{\sqrt{d+1}}{2}\,\frac{1}{|q|} + c_1\frac{|x|}{|q|^2}\,,$$

and inserting (12) yields Proposition 1.

Using Proposition 1 and following the lines of Hlawka [1; Satz 1], one can show

PROPOSITION 2. *Let* $\mathbb{Q}(\mathrm{i}\sqrt{d})$ *be an imaginary quadratic number field, and* $\theta \notin \mathbb{Q}(\mathrm{i}\sqrt{d})$. *Then there exist infinitely many integers* z *of* $\mathbb{Q}(\mathrm{i}\sqrt{d})$ *with* $\mathrm{Re}\,z > 0$, $\mathrm{Im}\,z \ge 0$, $|z| \to \infty$ *such that*

$$\|\theta - \sqrt{z}\| < \frac{c_3}{|z|}$$

for some positive constant $c_3 = c_3(d)$.

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