Math. Slovaca, **44** (1994), No. 4, 413–425

Mathematica Slovaca

REGULARIZATION OF CLOSED–VALUED MULTIFUNCTIONS IN A NON–METRIC SETTING

DIEGO AVERNA[1]

(*Communicated by Ladislav Mišík*)

ABSTRACT. In this paper, the existence of a regularization of multifunctions $\Phi\colon T \to Z$ and $F\colon T \times X \to Y$ is stated if T is a topological measurable space, and X, Y and Z are topological spaces with a countable base (Theorems 1 and 3). Utilizing Sainte-Beuve's selection theorem ([6]), uniqueness theorems (Theorems 2 and 4) are also derived. The obtained results generalize those of Rzeżuchowski in [5].

1. Introduction

Scorza-Dragoni type theorems for multifunctions $F\colon T\times X \to Y$ of Carathéodory type are useful for the study of the set of solutions of Cauchy problems associated with the differential inclusion

$$\dot{x} \in F(t,x)\,. \tag{1}$$

This occurs because the separated regularity of F with respect to t and x (i.e. the Carathéodory type property) implies (through the Scorza-Dragoni type theorem) an almost regularity with respect to (t,x).

For example, if $T = [0,1]$, $X = Y = \mathbb{R}$, $F\colon T \times X \to Y$ has closed values, $F(\,\cdot\,,x)$ is weakly measurable, and $F(t,\cdot\,)$ is continuous, then for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset [0,1]$, whose Lebesgue measure is $> 1-\varepsilon$, such that $F\big|_{C_\varepsilon}$ is lower semicontinuous and has closed graph.

A tool for the study of the set of solutions of (1) when F is not of Carathéodory type can be the "regularization" of F.

AMS Subject Classification (1991): Primary 28B20, 54C60. Secondary 26E25.
Key words: Multifunctions, Regularization, Measurability, Semicontinuity.
[1] This research was supported by 60% MURST.

Roughly speaking, a regularization of F is a multifunction $G: T \times X \to Y$ which has the following properties:

i$_1$) $G(t,x) \subset F(t,x)$,
i$_2$) $q(t) \in F(t,p(t)) \implies q(t) \in G(t,p(t))$ whenever $p: T \to X$ and $q: T \to Y$ are measurable functions,

plus some Scorza-Dragoni type property.

In virtue of the properties i$_1$) and i$_2$), the set of solutions of (1) is the same as that of

$$\dot{x} \in G(t,x)\,. \tag{2}$$

Existence and uniqueness theorems for regularizations G of certain multifunctions F have been given in [1], [4], [5].

In particular, Rzeżuchowski proved in [5] an existence theorem for regularizations of a given closed-valued multifunction $F: T \times X \to Y$ such that $F(t,\cdot)$ has closed graph when T is a locally compact metric space endowed with a Borel, σ-finite, regular and complete measure μ, and X and Y are separable metric spaces ([5; Theorem 1]). A uniqueness theorem was also presented when X and Y are even complete ([5; Theorem 4]).

The aim of this paper is essentially to show that the Rzeżuchowski existence and uniqueness theorems above remain still valid without assuming that T, X and Y are metric and without strengthening of other hypotheses: that is, existence and uniqueness theorems for regularizations proved here extend in a more general framework the results of [5].

The main idea for doing this, in existence theorems, is to replace the point-set distance function in the range-space (which plays a key role in the Rzeżuchowski existence proof) by a function δ, taking only two values, which flags when two suitable sets, one coming from a basis of the topology of the range-space, the other from the values of the multifunction, intersect.

This idea has been tested also in a previous paper [3], concerning Lusin and Scorza-Dragoni type theorems; some results of [3] are also useful here.

The extension in uniqueness theorems essentially carries out in virtue of Sainte-Beuve's selection theorem.

Moreover, the results of this paper not only extend but also improve those of [5], establishing indeed further properties for the regularization G.

2. Preliminaries

Let S be a non-empty set and (Z, τ_Z) be a topological space. $\mathcal{P}(Z)$ (resp. $\mathrm{Cl}(Z)$) denotes the family of all subsets (resp. closed subsets) of Z, while $\mathcal{B}(Z)$ denotes the Borel σ-algebra on Z. Let $\Phi\colon S \to Z$ be a multifunction, i.e. a function from S into the family $\mathcal{P}(Z)$. If the values of Φ are closed subsets of Z, we write $\Phi\colon S \to \mathrm{Cl}(Z)$. $\mathrm{Gr}(\Phi)$ denotes the graph of Φ, i.e. the set $\{(s,z) \in S \times Z :\ z \in \Phi(s)\}$. If $E \subset S$, we call $\Phi|_E$ the restriction of Φ to E. If $W \subset Z$, we put $\Phi^-(W) = \{s \in S :\ \Phi(s) \cap W \neq \emptyset\}$ and $\Phi^+(W) = \{s \in S :\ \Phi(s) \subset W\}$. We have the fundamental relations $\Phi^-(W) = S - \Phi^+(Z - W) = \mathrm{proj}_S(\mathrm{Gr}(\Phi) \cap (S \times W))$, where proj_S denotes the projection map of $S \times Z$ onto S, and for each family $\{W_\alpha :\ \alpha \in \mathcal{A}\} \subset \mathcal{P}(Z)$, $\Phi^-\Big(\bigcup\limits_{\alpha \in \mathcal{A}} W_\alpha\Big) = \bigcup\limits_{\alpha \in \mathcal{A}} \Phi^-(W_\alpha)$. If $\Phi_1, \Phi_2\colon S \to Z$ are two multifunctions, we denote by $\Phi_1 \triangle \Phi_2$ the symmetric difference multifunction, that is the multifunction defined by $(\Phi_1 \triangle \Phi_2)(s) = \Phi_1(s) \triangle \Phi_2(s)$ for each $s \in S$.

If (S, τ_S) is a topological space, we say that Φ is lower (resp. upper) semicontinuous at $s_0 \in S$ if for each $W \in \tau_Z$ such that $s_0 \in \Phi^-(W)$ (resp. $s_0 \in \Phi^+(W)$) there exists an open neighbourhood I of s_0 such that $I \subset \Phi^-(W)$ (resp. $I \subset \Phi^+(W)$). We say that Φ is lower (resp. upper) semicontinuous if it is lower (resp. upper) semicontinuous at every $s \in S$, or equivalently, if for each $W \in \tau_Z$ the set $\Phi^-(W)$ (resp. $\Phi^+(W)$) is open in S. We say that Φ is continuous if it is simultaneously lower and upper semicontinuous.

If (S, Σ_S, μ) is a measure space, we denote by Σ_S^* the completion of Σ_S with respect to μ and with μ^* the completion measure. (S, Σ_S^*, μ^*) is a complete measure space. Recall that $E \in \Sigma_S^*$ if and only if there exist $E', E'' \in \Sigma_S$ such that $E' \subset E \subset E''$ and $\mu(E') = \mu^*(E) = \mu(E'')$.

If Σ is a σ-algebra of subsets of S, we say that Φ is Σ-weakly measurable (resp. Σ-measurable) [resp. Σ-$\mathcal{B}$ measurable] if for each $W \in \tau_Z$ (resp. $W \in \mathrm{Cl}(Z)$) [resp. $W \in \mathcal{B}(Z)$] $\Phi^-(W) \in \Sigma_S$.

The definitions of lower and upper semicontinuity for real valued functions and those of measurability and continuity for functions with values in a topological space are the usual ones.

If (S, τ) is a topological space, (S, Σ) is a measurable space, and $E \subset S$, then $\tau_E = \tau|_E$ and $\Sigma_E = \Sigma|_E$ denote respectively the induced topology and the induced σ-algebra on E. If $E \subset S$, and E has the induced topology, then $\mathcal{B}(S)|_E = \mathcal{B}(E)$. If $E \in \Sigma$, then $\Sigma|_E = \{A \in \Sigma :\ A \subset E\}$; so we speak of Σ-weak measurability (resp. Σ-measurability) of a multifunction (resp. function) instead of $\Sigma|_E$-weak measurability (resp. $\Sigma|_E$-measurability) whenever the multifunction (resp. function) is defined on E.

If S is a structured space, and we want a structure on $E \subset S$ when it is not specified, we refer to the induced structure; i.e., if S is a topological (resp. measurable) space, then E is a topological (resp. measurable) space with the induced topology (resp. σ-algebra).

If S and S' are two sets and $E \subset S \times S'$, then for $s \in S$, $E_s = \{s' \in S' : (s,s') \in E\}$ denotes the s-section of E, and for $s' \in S'$, $E_{s'} = \{s \in S : (s,s') \in E\}$ denotes the s'-section of E.

If S and S' are two structured spaces, and we want a structure on $S \times S'$ when it is not specified, we refer to the product structure; i.e., if S and S' are topological (resp. measurable) spaces, then $S \times S'$ is a topological (resp. measurable) space with the product topology (resp. product σ-algebra). We notice that if S and S' are topological spaces, in general, $\mathcal{B}(S) \times \mathcal{B}(S') \subset \mathcal{B}(S \times S')$, and the inclusion can be proper; $\mathcal{B}(S) \times \mathcal{B}(S') = \mathcal{B}(S \times S')$ if, for example, S and S' are second-countable topological spaces or Suslin spaces.

Moreover, when in the sequel we deal with the product of three sets S, S', and S'', we always identify $S \times (S' \times S'')$ with $S \times S' \times S''$, even if the structure is essentially that of $S \times (S' \times S'')$. So, for example, if Σ_S and $\Sigma_{S' \times S''}$ are σ-algebras on S and $S' \times S''$ respectively, when we say that $E \subset S \times S' \times S''$ lies in $\Sigma_S \times \Sigma_{S' \times S''}$, we mean that $\{(s,(s',s'')) \in S \times (S' \times S'') : (s,s',s'') \in E\} \in \Sigma_S \times \Sigma_{S' \times S''}$.

As in [7], we say that a topological space is Polish if it is separable and metrizable by a complete metric, Suslin if it is Hausdorff and a continuous image of a Polish space.

3. Regularization of closed-valued multifunctions

We begin with the following proposition in measure theory.

LEMMA 1. *Let (T, Σ_T) be a measurable space, and μ be a σ-finite measure on Σ_T.*

For each subset E of T there exists $M \in \Sigma_T$ such that:

α_1) *$M \subset E$;*

α_2) *for each $L \in \Sigma_T^*$ such that $L - E \in \Sigma_T^*$ and $\mu^*(L - E) = 0$, there holds $\mu^*(L - M) = 0$.*

Proof. We prove Lemma 1 for $\mu(T) < +\infty$ because it is obvious how to extend it to the σ-finite case.

Let $\alpha = \sup\{\mu(A) : A \in \Sigma_T,\ A \subset E\} < +\infty$. Then there exists a sequence $(A_n)_n$ of sets in Σ_T such that, for each $n \in \mathbb{N}$, $A_n \subset E$ and $\mu(A_n) > \alpha - 1/n$.

The set $M = \bigcup_n A_n$ is the requested set.

In fact, obviously, $M \in \Sigma_T$ and satisfies $\alpha_1)$.

Moreover, let $L \in \Sigma_T^*$ be such that $L - E \in \Sigma_T^*$ and $\mu^*(L - E) = 0$; then $(L - M) \cap E = (L - M) - (L - E) \in \Sigma_T^*$ and $\mu^*\big((L - M) \cap E\big) = \mu^*(L - M)$.

Obviously, $\big((L - M) \cap E\big) \cup M \in \Sigma_T^*$; so let $L' \in \Sigma_T$ be such that $L' \subset \big((L - M) \cap E\big) \cup M$ and $\mu(L') = \mu^*\Big(\big((L - M) \cap E\big) \cup M\Big)$. We have $L' \subset E$ and $\alpha \geq \mu(L') = \mu^*(L - M) + \mu(M) \geq \alpha$, from which $\mu^*(L - M) = 0$. Hence M verifies $\alpha_2)$. □

We need, for the sequel, to reformulate Lemma 1 in terms of functions with only two values.

COROLLARY 1. *Let (T, Σ_T) be a measurable space, μ be a σ-finite measure on Σ_T, and let $\{0,1\}$ be endowed with the discrete topology.*

If $\varphi \colon T \to \{0,1\}$ is a function, then there exists a Σ_T-measurable function $\psi \colon T \to \{0,1\}$ such that:

- $\beta_1)$ *$\psi(t) \leq \varphi(t)$ for each $t \in T$;*
- $\beta_2)$ *for each Σ_T^*-measurable function $\vartheta \colon T \to \{0,1\}$ such that $\vartheta(t) \leq \varphi(t)$ a.e. in T, there holds $\vartheta(t) \leq \psi(t)$ a.e. in T.*

From now on, unless otherwise stated, (T, τ_T) is a topological space, Σ_T is a σ-algebra of subsets of T such that $\tau_T \subset \Sigma_T$ (equivalently $\mathcal{B}(T) \subset \Sigma_T$), μ is a σ-finite measure on Σ_T such that for every $A \in \Sigma_T$ and every $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset A$ with $\mu(A - C_\varepsilon) < \varepsilon$. Obviously, Σ_T^* and μ^* have also these properties.

LEMMA 2. *Let Z be a topological space and $\mathcal{B}(T \times Z) = \mathcal{B}(T) \times \mathcal{B}(Z)$.*

Let $\Psi \colon T \to Z$ be a multifunction such that for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\operatorname{Gr}\big(\Psi|_{C_\varepsilon}\big)$ is closed in a $\Sigma_T^* \times \mathcal{B}(Z)$-measurable set Ω which contains $\operatorname{Gr}(\Psi)$.

Then there exists $T_0 \in \Sigma_T$ with $\mu(T_0) = 0$ such that $\operatorname{Gr}\big(\Psi|_{T - T_0}\big) \in \Sigma_T^* \times \mathcal{B}(Z)$.

P r o o f. For each $k \in \mathbb{N}$ there exists a closed set $C_k \subset T$ with $\mu(T - C_k) < 1/k$ such that $\operatorname{Gr}\big(\Psi|_{C_k}\big) = \overline{\operatorname{Gr}\big(\Psi|_{C_k}\big)} \cap \Omega$, where $\overline{\operatorname{Gr}\big(\Psi|_{C_k}\big)}$ denotes the closure of $\operatorname{Gr}\big(\Psi|_{C_k}\big)$ in $T \times Z$. But $\mathcal{B}(T \times Z) = \mathcal{B}(T) \times \mathcal{B}(Z)$, so $\operatorname{Gr}\big(\Psi|_{C_k}\big) \in \Sigma_T^* \times \mathcal{B}(Z)$. Put $T_0 = \bigcap_k (T - C_k)$; then $\mu(T_0) = 0$ and $\operatorname{Gr}\big(\Psi|_{T - T_0}\big) = \bigcup_k \operatorname{Gr}\big(\Psi|_{C_k}\big) \in \Sigma_T^* \times \mathcal{B}(Z)$. □

We need also the following proposition, whose proof we give for completeness.

LEMMA 3. *Let Z be a Suslin space. If $\Psi: T \to Z$ is a multifunction such that $\mathrm{Gr}(\Psi) \in \Sigma_T \times \mathcal{B}(Z)$, then for every $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\Psi|_{C_\varepsilon}$ is lower semicontinuous.*

Proof. Let g be a continuous function from a Polish space Z' onto Z. Define the multifunction $\Psi': T \to Z'$ by putting $\Psi'(t) = g^{-1}(\Psi(t))$ for all $t \in T$.

We claim that $\mathrm{Gr}(\Psi') \in \Sigma_T \times \mathcal{B}(Z')$. Indeed, it is easily seen that $(1_T, g): T \times Z' \to T \times Z$, defined by $(1_T, g)(t, z') = (t, g(z'))$ for all $(t, z') \in T \times Z'$, is continuous, and $(1_T, g)^{-1}(\Omega) \in \Sigma_T \times \mathcal{B}(Z')$ for each $\Omega \in \Sigma_T \times \mathcal{B}(Z)$. Thus the claim follows from the fact that $\mathrm{Gr}(\Psi') = (1_T, g)^{-1}(\mathrm{Gr}(\Psi))$.

By Sainte-Beuve's projection theorem [6; Theorem 4], Ψ' is Σ_T^*-$\mathcal{B}$-measurable since $\Psi'^{-}(W') = \mathrm{proj}_T(\mathrm{Gr}(\Psi') \cap (T \times W'))$ for each $W' \subset Z'$. Hence, a fortiori, Ψ' is Σ_T^*-weakly measurable.

Now, by Theorem 1 of [3], for every $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\Psi'|_{C_\varepsilon}$ is lower semicontinuous.

Since g is surjective, $\Psi(t) = g(\Psi'(t))$ for every $t \in T$, and so $\Psi^{-}(W) = \Psi'^{-}(g^{-1}(W))$ for each $W \subset Z$; thus $\Psi|_{C_\varepsilon}$ is lower semicontinuous. □

As in [3], if $B, B' \subset Z$, we define

$$\delta(B, B') = \begin{cases} 1 & \text{if } B \cap B' \neq \emptyset, \\ 0 & \text{if } B \cap B' = \emptyset. \end{cases}$$

The following theorem is the key result of this paper.

THEOREM 1. *Let Z be a second-countable topological space and $\Phi: T \to \mathrm{Cl}(Z)$ a multifunction.*

Then there exists a multifunction $\Psi: T \to \mathrm{Cl}(Z)$ such that:

γ_1) *$\Psi(t) \subset \Phi(t)$ for each $t \in T$;*

γ_2) *for each $\Delta \in \Sigma_T^*$ and for each Σ_T^*-weakly measurable multifunction $\Theta: \Delta \to Z$ such that $\Theta(t) \subset \Phi(t)$ a.e. in Δ, there holds $\Theta(t) \subset \Psi(t)$ a.e. in Δ;*

γ_3) *for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\mathrm{Gr}(\Psi|_{C_\varepsilon})$ is closed in $T \times Z$;*

γ_4) $\mathrm{Gr}(\Psi) \in \Sigma_T \times \mathcal{B}(Z)$.

If, moreover, we assume that Z is also a Suslin space, then:

γ_5) *for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\Psi|_{C_\varepsilon}$ is lower semicontinuous;*

γ_6) *for each* $\Delta \in \Sigma_T^*$ *and for each multifunction* $\Theta\colon \Delta \to Z$ *with* $\mathrm{Gr}(\Theta) \in \Sigma_T^* \times \mathcal{B}(Z)$ *such that* $\Theta(t) \subset \Phi(t)$ *a.e. in* Δ, *we have* $\Theta(t) \subset \Psi(t)$ *a.e. in* Δ.

Proof. Let $\mathfrak{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis for τ_Z. For each $n \in \mathbb{N}$ define $\varphi_n\colon T \to \{0,1\}$ by putting, for each $t \in T$, $\varphi_n(t) = \delta\big(B_n, \Phi(t)\big)$.

By Corollary 1, there exists a Σ_T-measurable function $\psi_n\colon T \to \{0,1\}$ such that:

1) $\psi_n(t) \le \varphi_n(t)$ for each $t \in T$;
2) for each Σ_T^*-measurable function $\vartheta\colon T \to \{0,1\}$ such that $\vartheta(t) \le \varphi_n(t)$ a.e. in T, we have $\vartheta(t) \le \psi_n(t)$ a.e. in T.

Let us define $\Psi\colon T \to \mathrm{Cl}(Z)$ by putting, for each $t \in T$,

$$\Psi(t) = \bigcap \big\{ Z - B_n : \psi_n(t) = 0 \big\}.$$

Ψ verifies γ_1). In fact, for each $z \in \Psi(t)$ and each $n \in \mathbb{N}$ such that $\varphi_n(t) = 0$, it follows that $z \in Z - B_n$ in virtue of 1). Thus we obtain $z \in \Phi(t)$, taking into account that, $\Phi(t)$ being closed, $\Phi(t) = \bigcap\{Z - B_n : \varphi_n(t) = 0\}$.

Ψ verifies γ_2). Let $\Delta \in \Sigma_T^*$ and $\Theta\colon \Delta \to Z$ be a Σ_T^*-weakly measurable multifunction such that $\Theta(t) \subset \Phi(t)$ a.e. in Δ. For each $n \in \mathbb{N}$ define $\vartheta_n\colon T \to \{0,1\}$ by putting:

$$\vartheta_n(t) = \begin{cases} \delta\big(B_n, \Theta(t)\big) & \text{if } t \in \Delta, \\ 0 & \text{if } t \notin \Delta. \end{cases}$$

Then, by using [3; Lemma 2. 3) $(\Rightarrow)$], it follows that ϑ_n is Σ_T^*-measurable; moreover, since $\Theta(t) \subset \Phi(t)$ a.e. in Δ, $\vartheta_n(t) \le \varphi_n(t)$ a.e. in T. Hence, by 2), $\vartheta_n(t) \le \psi_n(t)$ a.e. in T, and thus $\Theta(t) \subset \Psi(t)$ a.e. in Δ.

Ψ verifies γ_3). Fix $\varepsilon > 0$. By Lusin's theorem (see also [3; Lemma 1]) and by a standard argument which takes into account the countability of the family $\{\psi_n : n \in \mathbb{N}\}$, we can find a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\psi_n|_{C_\varepsilon}$ is continuous for each $n \in \mathbb{N}$. We claim that $\mathrm{Gr}\big(\Psi|_{C_\varepsilon}\big)$ is closed in $T \times Z$. Indeed, if $z_0 \notin \Psi(t_0)$, $t_0 \in C_\varepsilon$, then there is $\overline{n} \in \mathbb{N}$ such that $\psi_{\overline{n}}(t_0) = 0$ and $z_0 \in B_{\overline{n}}$. By the upper semicontinuity of $\psi_{\overline{n}}|_{C_\varepsilon}$ at t_0, there exists an open neighbourhood I of t_0 such that $\psi_{\overline{n}}(t) = 0$ for each $t \in I \cap C_\varepsilon$. Thus $\big((I \cap C_\varepsilon) \times B_{\overline{n}}\big) \cap \mathrm{Gr}\big(\Psi|_{C_\varepsilon}\big) = \emptyset$.

Ψ verifies γ_4). In fact, $(T \times Z) - \mathrm{Gr}(\Psi) = \bigcup_n \big(\psi_n^{-1}(\{0\}) \times B_n\big)$.

Finally, under the additional hypothesis on Z, γ_5) is a direct consequence of γ_4), by Lemma 3, while γ_6) is a consequence of γ_2) because, using the equality

$\Theta^-(W) = \mathrm{proj}_T\big(\mathrm{Gr}(\Theta)\cap(\Delta\times W)\big)$ for $W\in\tau_Z$, it follows that Θ is Σ_T^*-weakly measurable by Sainte-Beuve's projection theorem. □

R e m a r k 1. Obviously $\gamma_1)$ and $\gamma_2)$ of Theorem 1 imply respectively the following:

- $\gamma_1')$ $\Psi(t)\subset\Phi(t)$ a.e. in T;
- $\gamma_2')$ for each $\Delta\in\Sigma_T^*$ and for each Σ_T^*-measurable function $\theta\colon\Delta\to Z$ such that $\theta(t)\in\Phi(t)$ a.e. in Δ, there holds $\theta(t)\in\Psi(t)$ a.e. in Δ.

Hence Theorem 1 extends and improves [5; Theorem 2], in which $\gamma_1')$, $\gamma_2')$ and $\gamma_3)$ are proved when T is a locally compact metric space, μ is a Borel, σ-finite, regular and complete measure on T, and Z is a separable metric space.

Moreover, if Ω is a $\Sigma_T^*\times\mathcal{B}(Z)$-measurable set, with $\mathrm{Gr}(\Psi)\subset\Omega$, then $\gamma_3)$ of Theorem 1 implies the following:

- $\gamma_3')$ for each $\varepsilon>0$ there exists a closed set $C_\varepsilon\subset T$ with $\mu(T-C_\varepsilon)<\varepsilon$ such that $\mathrm{Gr}\big(\Psi|_{C_\varepsilon}\big)$ is closed in Ω.

Now we prove the following uniqueness result, whose part 1) extends [5; Theorem 5].

THEOREM 2. *Let Z be a Suslin space and $\Phi,\Psi_1,\Psi_2\colon T\to Z$ be three multifunctions.*

Let us consider the following properties for $i=1,2$:

- $\gamma_1')$ *$\Psi_i(t)\subset\Phi(t)$ a.e. in T;*
- $\gamma_2')$ *for each $\Delta\in\Sigma_T^*$ and for each Σ_T^*-measurable function $\theta\colon\Delta\to Z$ such that $\theta(t)\in\Phi(t)$ a.e. in Δ, we have $\theta(t)\in\Psi_i(t)$ a.e. in Δ;*
- $\gamma_3')$ *for some $\Omega\in\Sigma_T^*\times\mathcal{B}(Z)$ with $\mathrm{Gr}(\Phi)\subset\Omega$, and for each $\varepsilon>0$, there exists a closed set $C_\varepsilon\subset T$ with $\mu(T-C_\varepsilon)<\varepsilon$ such that $\mathrm{Gr}(\Psi_i|_{C_\varepsilon})$ is closed in Ω;*
- $\gamma_4')$ *there is $T_0\in\Sigma_T$ with $\mu(T_0)=0$ such that $\mathrm{Gr}\big(\Psi_i|_{T-T_0}\big)\in\Sigma_T^*\times\mathcal{B}(Z)$;*
- $\gamma_5)$ *for each $\varepsilon>0$ there exists a closed set $C_\varepsilon\subset T$ with $\mu(T-C_\varepsilon)<\varepsilon$ such that $\Psi_i|_{C_\varepsilon}$ is lower semicontinuous.*

Then:

1. *$\gamma_1')$, $\gamma_2')$ and $\gamma_4')$ imply that $\Psi_1(t)=\Psi_2(t)$ a.e. in T.*
2. *If $\mathcal{B}(T\times Z)=\mathcal{B}(T)\times\mathcal{B}(Z)$, then $\gamma_1')$, $\gamma_2')$, and $\gamma_3')$ imply that $\Psi_1(t)=\Psi_2(t)$ a.e. in T.*
3. *If Z is also a second-countable topological space and Ψ_1 and Ψ_2 are closed-valued, then $\gamma_1')$, $\gamma_2')$ and $\gamma_5)$ imply that $\Psi_1(t)=\Psi_2(t)$ a.e. in T.*

P r o o f.

1) $\mathrm{Gr}\big((\Psi_1 \triangle \Psi_2)|_{T-T_0}\big) = \mathrm{Gr}\big(\Psi_1|_{T-T_0}\big) \triangle \mathrm{Gr}\big(\Psi_2|_{T-T_0}\big) \in \Sigma_T^* \times \mathcal{B}(Z)$. Put $\Delta = \mathrm{proj}_T\big(\mathrm{Gr}\big((\Psi_1 \triangle \Psi_2)|_{T-T_0}\big)\big)$; thus $\Delta \in \Sigma_T^*$ by Sainte-Beuve's projection theorem, and $\Delta \subset T - T_0$.

Define $\Gamma\colon T \to Z$ by putting

$$\Gamma(t) = \begin{cases} (\Psi_1 \triangle \Psi_2)(t) & \text{if } t \in \Delta\,, \\ Z & \text{if } t \notin \Delta\,. \end{cases}$$

$\mathrm{Gr}(\Gamma) = \mathrm{Gr}\big((\Psi_1 \triangle \Psi_2)|_{T-T_0}\big) \cup \big((T-\Delta) \times Z\big) \in \Sigma_T^* \times \mathcal{B}(Z)$; thus, by Saint-Beuve's selection theorem [6; Theorem 3], there exists a Σ_T^*-measurable selection θ of Γ. By γ_1'), $\theta(t) \in \Phi(t)$ a.e. in Δ, thus by γ_2'), $\theta(t) \in \Psi_1(t) \cap \Psi_2(t)$ a.e. in Δ. It follows that $\mu^*(\Delta) = 0$.

2) By Lemma 2, γ_3') implies γ_4'). So the conclusion follows by 1).

3) Ψ_1 and Ψ_2 are Σ_T^*-weakly measurable. In fact, for $i = 1, 2$ and for each $k \in \mathbb{N}$ there exists a closed set $C_k \subset T$ with $\mu(T - C_k) < 1/k$ such that $\Psi_i|_{C_k}$ is lower semicontinuous, thus Σ_{C_k}-weakly measurable ($\Sigma_{C_k} = \{A \in \Sigma_T : A \subset C_k\}$). Hence, for $W \in \tau_Z$, we have $\Psi_i^-(W) = \bigcup_k \big(\Psi_i|_{C_k}^-(W)\big) \cup N$, where $\mu^*(N) = 0$, so $\Psi_i^-(W) \in \Sigma_T^*$.

Then, thanks to [2; Theorem 2.4] (see also [2; Remarks 2.1 and 2.4]), we have that $\mathrm{Gr}(\Psi_1), \mathrm{Gr}(\Psi_2) \in \Sigma_T^* \times \mathcal{B}(Z)$; thus the conclusion follows again by 1). □

The following Theorem 3 is the two-variables version of Theorem 1.

THEOREM 3. *Let X and Y be two second-countable topological spaces and $D \subset T \times X$.*

If $F\colon D \to \mathrm{Cl}(Y)$ is a multifunction such that there is a $T_0 \in \Sigma_T$ with $\mu(T_0) = 0$ such that $\mathrm{Gr}\big(F(t,\cdot)\big)$ is closed in $D_t \times Y$ for each $t \in \mathrm{proj}_T(D) - T_0$, then there exists a multifunction $G\colon D \to \mathrm{Cl}(Y)$ such that:

- i_0) $\mathrm{Gr}\big(G(t,\cdot)\big)$ *is closed in $D_t \times Y$ for each $t \in \mathrm{proj}_T(D)$;*
- i_1) $G(t,x) \subset F(t,x)$ *for each $(t,x) \in D$;*
- i_2) *for each $\Delta \in \Sigma_T^*$, for each Σ_T^*-weakly measurable multifunction $Q\colon \Delta \to Y$, and for each Σ_T^*-measurable function $p\colon \Delta \to X$ such that $\big(t, p(t)\big) \in D$ and $Q(t) \subset F\big(t, p(t)\big)$ a.e. in Δ, there holds $Q(t) \subset G\big(t, p(t)\big)$ a.e. in Δ;*
- i_3) *for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\mathrm{Gr}\Big(G|_{D \cap (C_\varepsilon \times X)}\Big)$ is closed in $D \times Y$;*
- i_4) $\mathrm{Gr}(G) \in \big(\Sigma_T \times \mathcal{B}(X \times Y)\big)|_{D \times Y}$.

Moreover, if we assume that Y *is also a Suslin space and that* $\mathcal{B}(T \times Y) = \mathcal{B}(T) \times \mathcal{B}(Y)$, *then*

i$_5$) *for each* $\Delta \in \Sigma_T^*$, *for each* Σ_T^**-measurable function* $p\colon \Delta \to X$ *with* $(t, p(t)) \in D$ *a.e. in* Δ, *and for each* $\varepsilon > 0$ *there exists a closed set* $C_\varepsilon \subset T$ *with* $\mu(T - C_\varepsilon) < \varepsilon$ *such that* $G(\cdot, p(\cdot))|_{\Delta \cap C_\varepsilon}$ *is lower semicontinuous.*

Finally, if X *and* Y *are also two Suslin spaces and* $D \in \Sigma_T^* \times \mathcal{B}(X)$, *then*

i_6) *for each* $\Delta \in \Sigma_T^*$ *and for each multifunction* $H\colon D \cap (\Delta \times X) \to Y$ *with* $\mathrm{Gr}(H) \in \Sigma_T^* \times \mathcal{B}(X \times Y)$ *such that* $H(t,x) \subset F(t,x)$ *for almost all* $t \in \mathrm{proj}_T(D) \cap \Delta$ *and for each* $x \in D_t$, *there holds* $H(t,x) \subset G(t,x)$ *for almost all* $t \in \mathrm{proj}_T(D) \cap \Delta$ *and for each* $x \in D_t$.

P r o o f. First suppose $D = T \times X$. Consider the multifunction $\Phi\colon T \to \mathrm{Cl}(X \times Y)$ defined by

$$\Phi(t) = \begin{cases} \mathrm{Gr}\big(F(t,\cdot)\big) & \text{if } t \in T - T_0, \\ \emptyset & \text{if } t \in T_0. \end{cases}$$

By Theorem 1, there exists a multifunction $\Psi\colon T \to \mathrm{Cl}(X \times Y)$ satisfying γ_1), γ_2), γ_3), γ_4) and γ_6). We claim that the multifunction $G\colon T \times X \to \mathrm{Cl}(Y)$ defined by $G(t,x) = \big(\Psi(t)\big)_x$ is the required multifunction.

In fact, it is easily seen that G verifies i$_0$), i$_1$), i$_3$) and i$_4$).

G verifies i$_2$). Let $\Delta \in \Sigma_T^*$, $Q\colon \Delta \to Y$ be a Σ_T^*-weakly measurable multifunction and $p\colon \Delta \to X$ be a Σ_T^*-measurable function such that $Q(t) \subset F(t, p(t))$ a.e. in Δ. The multifunction $\Theta\colon \Delta \to X \times Y$ defined by $\Theta(t) = \{(p(t), y) : y \in Q(t)\}$ is Σ_T^*-weakly measurable because for $U \in \tau_X$ and $V \in \tau_Y$, $\Theta^-(U \times V) = p^{-1}(U) \cap Q^-(V) \in \Sigma_T^*$. Moreover, $\Theta(t) \subset \Phi(t)$ a.e. in Δ, thus by γ_2), $\Theta(t) \subset \Psi(t)$ a.e. in Δ, from which it follows that $Q(t) \subset G(t, p(t))$ a.e. in Δ.

G verifies i$_5$). Let $\Delta \in \Sigma_T^*$ and $p\colon \Delta \to X$ be a Σ_T^*-measurable function. Extend p to the Σ_T^*-measurable function $\hat{p}\colon T \to X$ defined by putting

$$\hat{p}(t) = \begin{cases} p(t) & \text{if } t \in \Delta, \\ \text{constant} & \text{if } t \notin \Delta. \end{cases}$$

If we show that $G(\cdot, \hat{p}(\cdot))$ is Σ_T^*-weakly measurable, then by [3; Theorem 1], it follows that for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $G(\cdot, \hat{p}(\cdot))|_{C_\varepsilon}$, and thus also $G(\cdot, p(\cdot))|_{\Delta \cap C_\varepsilon}$, is lower semicontinuous.

To prove this, it suffices to show that for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\mathrm{Gr}\big(G(\,\cdot\,,\hat{p}(\cdot))|_{C_\varepsilon}\big)$ is closed in $T \times Y$. In fact, this last condition being verified, by Lemma 2 there exists $T_0 \in \Sigma_T$ with $\mu(T_0) = 0$ such that $\mathrm{Gr}\big(G(\,\cdot\,,\hat{p}(\cdot))|_{T - T_0}\big) \in \Sigma_T^* \times \mathcal{B}(Y)$; so, from the equality $G(\,\cdot\,,\hat{p}(\cdot))^-(V) = \mathrm{proj}_T\Big(\mathrm{Gr}\Big(G(\,\cdot\,,\hat{p}(\cdot))|_{T - T_0}\Big) \cap \big((T - T_0) \times V\big)\Big) \cup N$, where $N \subset T_0$, and by Sainte-Beuve's projection theorem, it follows that $G(\,\cdot\,,\hat{p}(\cdot))$ is Σ_T^*-weakly measurable.

So fix $\varepsilon > 0$. By i$_3$) and using [3; Theorem 1], there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\mathrm{Gr}\big(G|_{C_\varepsilon \times X}\big)$ is closed in $T \times X \times Y$, and $\hat{p}|_{C_\varepsilon}$ is continuous. $\mathrm{Gr}\big(G(\,\cdot\,,\hat{p}(\cdot))|_{C_\varepsilon}\big)$ is closed in $T \times Y$. Indeed, if we take $(t_0, y_0) \notin \mathrm{Gr}\big(G(\,\cdot\,,\hat{p}(\cdot))|_{C_\varepsilon}\big)$, $t_0 \in C_\varepsilon$, then $\big(t_0, \hat{p}(t_0), y_0\big) \notin \mathrm{Gr}\big(G|_{C_\varepsilon \times X}\big)$; hence, by this and by the continuity of $\hat{p}|_{C_\varepsilon}$, there are two open neighbourhoods I and V of t_0 and y_0 respectively such that, for $t \in I \cap C_\varepsilon$ and $y \in V$, $(t, y) \notin \mathrm{Gr}\big(G(\,\cdot\,,\hat{p}(\cdot))|_{C_\varepsilon}\big)$.

Finally, we prove i$_6$). Define $\Theta\colon \Delta \to X \times Y$ by putting, for each $t \in \Delta$, $\Theta(t) = \mathrm{Gr}\big(H(t, \cdot\,)\big)$. $\mathrm{Gr}(\Theta) = \mathrm{Gr}(H) \in \Sigma_T^* \times \mathcal{B}(X \times Y)$. Moreover, $\Theta(t) \subset \Phi(t)$ a.e. in Δ; then, by γ_6) ($X \times Y$ is Suslin), $\Theta(t) \subset \Psi(t)$ a.e. in Δ, hence $H(t, x) \subset G(t, x)$ for almost all $t \in \Delta$ and for each $x \in X$.

Now we sketch the proof when $D \subset T \times X$.

Define $\hat{F}\colon T \times X \to \mathrm{Cl}(Y)$ by putting

$$\hat{F}(t, x) = \begin{cases} \big(\overline{\mathrm{Gr}(F(t, \cdot\,))}\big)_x & \text{if } t \in T - T_0\,, \\ \emptyset & \text{if } t \in T_0\,, \end{cases}$$

where the closure is taken in $X \times Y$.

$\mathrm{Gr}\big(\hat{F}(t, \cdot\,)\big) = \overline{\mathrm{Gr}(F(t, \cdot\,))}$ for each $t \in T - T_0$, and taking into account that $\mathrm{Gr}\big(F(t, \cdot\,)\big) = \overline{\mathrm{Gr}(F(t, \cdot\,))} \cap (D_t \times Y)$ for each $t \in \mathrm{proj}_T(D) - T_0$, then we obtain that $\hat{F}(t, x) = F(t, x)$ for all $(t, x) \in D - (T_0 \times X)$.

Let $\hat{G}\colon T \times X \to Y$ be as in the first part of the proof with respect to $\hat{F}$; it is not difficult to verify that $G = \hat{G}|_D$ is the required multifunction. □

R e m a r k 2. Obviously i$_1$) and i$_2$) of Theorem 3 imply respectively the following

- i$'_1$) $G(t, x) \subset F(t, x)$ for almost every $t \in \mathrm{proj}_T(D)$ and for each $x \in D_t$;
- i$'_2$) for each $\Delta \in \Sigma_T^*$ and for all Σ_T^*-measurable functions $q\colon \Delta \to Y$ and $p\colon \Delta \to X$ such that $\big(t, p(t)\big) \in D$ and $q(t) \in F\big(t, p(t)\big)$ a.e. in Δ, we have $q(t) \in G\big(t, p(t)\big)$ a.e. in Δ.

Hence Theorem 3 extends and improves [5; Theorem 1], in which i_1'), i_2') and i_3) are proved when T is a locally compact metric space, μ is a Borel, σ-finite, regular and complete measure on T, X and Y are two separable metric spaces.

Moreover, if Ω is a $\Sigma_T^* \times \mathcal{B}(X \times Y)$-measurable set with $\mathrm{Gr}(G) \subset \Omega$, then i_3) in Theorem 3 implies the following:

i_3') for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\mathrm{Gr}\Big(G|_{D \cap (C_\varepsilon \times X)}\Big)$ is closed in $(D \times Y) \cap \Omega$.

The following is a uniqueness theorem for the two-variables case; its part 2) extends [5; Theorem 4].

THEOREM 4. *Let X be a topological space, Y be a Suslin space, $D \in \Sigma_T^* \times \mathcal{B}(X)$, and $F, G_1, G_2 \colon D \to Y$ be three multifunctions.*

Let us consider the following properties for $i = 1, 2$:

- i_1') *$G_i(t,x) \subset F(t,x)$ for almost every $t \in \mathrm{proj}_T(D)$ and for each $x \in D_t$;*
- i_2') *for each $\Delta \in \Sigma_T^*$, for all Σ_T^*-measurable functions $q \colon \Delta \to Y$ and $p \colon \Delta \to X$ such that $\big(t, p(t)\big) \in D$ and $q(t) \in F\big(t, p(t)\big)$ a.e. in Δ, we have $q(t) \in G_i\big(t, p(t)\big)$ a.e. in Δ;*
- i_3') *for some $\Sigma_T^* \times \mathcal{B}(X \times Y)$-measurable set Ω with $\mathrm{Gr}(F) \subset \Omega$, and for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\mathrm{Gr}\Big(G_i|_{D \cap (C_\varepsilon \times X)}\Big)$ is closed in $(D \times Y) \cap \Omega$;*
- i_4') *there is $T_0 \in \Sigma_T$ with $\mu(T_0) = 0$ such that $\mathrm{Gr}\Big(G_i|_{D \cap ((T - T_0) \times X)}\Big) \in \Sigma_T^* \times \mathcal{B}(X \times Y)$;*
- i_5) *for each $\Delta \in \Sigma_T^*$, for each Σ_T^*-measurable function $p \colon \Delta \to X$ with $\big(t, p(t)\big) \in D$ a.e. in Δ, and for each $\varepsilon > 0$, there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $G_i\big(\cdot, p(\cdot)\big)|_{\Delta \cap C_\varepsilon}$ is lower semicontinuous.*

Then:

1) *If X is a Suslin space, then i_1'), i_2') and i_4') imply that $G_1(t,x) = G_2(t,x)$ for almost every $t \in \mathrm{proj}_T(D)$ and for each $x \in D_t$.*
2) *If X is a Suslin space and $\mathcal{B}(T \times X \times Y) = \mathcal{B}(T) \times \mathcal{B}(X \times Y)$, then i_1'), i_2') and i_3') imply that $G_1(t,x) = G_2(t,x)$ for almost every $t \in \mathrm{proj}_T(D)$ and for each $x \in D_t$.*
3) *If Y is also a second-countable topological space, and G_1 and G_2 are closed-valued, then i_1'), i_2') and i_5) imply that for each $\Delta \in \Sigma_T^*$ and for each Σ_T^*-measurable function $p \colon \Delta \to X$ with $\big(t, p(t)\big) \in D$ a.e. in Δ, it is $G_1\big(t, p(t)\big) = G_2\big(t, p(t)\big)$ a.e. in Δ.*

Sketch of the proof. First we prove the assertion 1) for $D = T \times X$. The multifunctions $\Phi, \Psi_1, \Psi_2 \colon T \to X \times Y$ defined respectively by $\Phi(t) = \mathrm{Gr}\big(F(t,\cdot)\big)$, $\Psi_1(t) = \mathrm{Gr}\big(G_1(t,\cdot)\big)$, and $\Psi_2(t) = \mathrm{Gr}\big(G_2(t,\cdot)\big)$ satisfy 1) of Theorem 2; then $\Psi_1(t) = \Psi_2(t)$ a.e. in T, from which $G_1(t,x) = G_2(t,x)$ for almost every $t \in T$ and for each $x \in X$.

The assertion 2) can be proved as above, taking into account 2) of Theorem 2.

To prove 3) when $D = T \times X$, extend p to all of T by putting $p(t) =$ constant outside of Δ. Then apply 3) of Theorem 2 to $\Phi(\cdot) = F\big(\cdot, p(\cdot)\big)$, $\Psi_1(\cdot) = G_1\big(\cdot, p(\cdot)\big)$, and $\Psi_2(\cdot) = G_2\big(\cdot, p(\cdot)\big)$; so we obtain $\Psi_1(t) = \Psi_2(t)$ a.e. in T. Now return to the original p defined in Δ, so we obtain $G_1\big(t, p(t)\big) = G_2\big(t, p(t)\big)$ a.e. in Δ.

For the general case $D \subset T \times X$, extend F, G_1 and G_2 to all of $T \times X$ by putting their values empty outside of D; then apply the already proved uniqueness theorem for the case $D = T \times X$ and finally return to D. □

REFERENCES

[1] ARTSTEIN, Z.: *Weak convergence of set-valued functions and control*, SIAM J. Control Optim. **13** (1975), 865–878.

[2] AVERNA, D.: *Separation properties in X and 2^X. Measurable multifunctions and graphs*, Math. Slovaca **41** (1991), 51–60.

[3] AVERNA, D.: *Lusin type theorems for multifunctions, Scorza Dragoni's property and Carathéodory selections*, Boll. Un. Mat. Ital. (7) (To appear).

[4] JARNÍK, J.—KURZWEIL, J.: *On conditions on right-hand sides of differential relations*, Časopis Pěst. Mat. **102** (1977), 334–349.

[5] RZEŻUCHOWSKI, T.: *Scorza-Dragoni type theorem for upper semicontinuous multivalued functions*, Bull. Polish Acad. Sci. Math. **28** (1980), 61–66.

[6] SAINTE-BEUVE, M. F.: *On the extension of Von Neumann-Aumann's theorem*, J. Funct. Anal. **17** (1974), 112-129.

[7] SCHWARTZ, L.: *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Oxford U.P., Oxford, 1973.

Received February 3, 1993

Dipartimento di Matematica ed Applicazioni
Facoltà di Ingegneria
Viale delle Scienze
I-90128 Palermo
Italy
E-mail: AVERNA@IPAMAT.CRES.IT