Oto Strauch

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Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

A NEW MOMENT PROBLEM OF DISTRIBUTION FUNCTIONS IN THE UNIT INTERVAL [1]

OTO STRAUCH [2]

(*Communicated by Robert F. Tichy*)

ABSTRACT. Given a triple of numbers $X_1, X_2, X_3 \in [0,1]$, one may ask under which circumstances it is possible to determine a distribution function $g\colon [0,1] \to [0,1]$ such that

$$X_1 = \int_0^1 g(x)\,\mathrm{d}x\,, \qquad X_2 = \int_0^1 xg(x)\,\mathrm{d}x\,, \qquad X_3 = \int_0^1 g^2(x)\,\mathrm{d}x\,.$$

Necessary and sufficient conditions for existence and uniqueness are established. As an application we find conditions that a given sequence have a linear or one-step limiting distribution.

1. Introduction

The purpose of this paper is to illustrate an application of two abstract methods, called *Method A* and *Method B* in the sequel, deriving necessary and sufficient conditions that a sequence from the unit interval have a linear or one-step limiting distribution. Method A can be used to find a new limit law of sequences, and Method B to determine a solution of a suitable moment problem.

AMS Subject Classification (1991): Primary 11K06. Secondary 60E05.
Key words: Sequences, Limiting distribution, Moment problem.

[1] Parts of this paper were delivered at the 9th Czechoslovak Colloquium on Number Theory held in Račkova dolina, September 11–15, 1989, Czechoslovakia [6].

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Method A. Applying a general criterion for limit law of sequences to solutions of a moment problem with finitely many solutions, we can select a corresponding solution, which leads to a special law. The idea is to use the fact [7; Satz 5] that if a sequence has more than one distribution function, then it has infinitely many (different in a common point of continuity).

Method B. In many cases we can make the solution of a given moment problem dependent on the nature of points of a body "corresponding" to the moment problem. Defining an operation on the set of points, it can be proved that every interior point corresponds to infinitely many, and every boundary point to finitely many solutions of the given moment problem. To do this, a related operation on the set of neighbourhoods of points can be used.

We present here two results concerning distribution functions. We introduce a new type of criterion (Theorem 3), which characterizes sequences having a limit law from a class of distribution functions (linear and one-step). This result can be extended to a large class possessing some polynomial functions, provided that we use a corresponding moment problem in a higher dimension.

The second result given here concerns the solutions of our three-dimensional moment problem. For a classical moment problem [5; p. vii] the solution may be unique, or there may be more than one solution in which case there are, of necessity, infinitely many solutions. It is interesting to note that (Theorem 2) our moment problem has, for some values, exactly two different solutions.

The paper is organized as follows. In Section 2 we give the basic definitions. In Section 3 we state our criterion and moment problem. As a consequence, we obtain our limit law for sequences. In Section 4 we prove the main result: the solution of our moment problem. The proof is elementary, but atypical. An explicit construction of the solution is given.

2. Basic definitions and notations

A *distribution function* in $[0,1]$ is any $g\colon [0,1] \to [0,1]$ such that

(i) $g(0) = 0$, $g(1) = 1$,
(ii) g is nondecreasing,
(iii) g is left continuous on $(0,1)$.

In the following, let

$\omega = (x_n)_{n=1}^{\infty}$ be a sequence of real numbers from the interval $[0,1]$, and
$\omega_N = (x_n)_{n=1}^{N}$ be the initial segment formed by the first N terms of ω.
For a given ω_N, the distribution function $F_N(x)$ is

$$F_N(x) = \frac{\#\{n \le N:\ x_n < x\}}{N} \qquad \text{for all} \quad x \in [0,1) \quad \text{and} \quad F_N(1) = 1\,.$$

If there exists an increasing sequence of natural numbers $N_1, N_2, \dots$ such that

$$\lim_{j\to\infty} F_{N_j}(x) = g(x)$$

for every continuity point x of a given distribution function g, then g is called a distribution function of ω.

The sequence ω is said to have a *limiting distribution* $g\colon [0,1] \to [0,1]$ if the limiting relation

$$\lim_{N\to\infty} F_N(x) = g(x)$$

holds at every point x at which $g(x)$ is continuous.[1)]

Note that throughout this paper we shall always denote both the row vector (X_1, X_2, X_3) and the column vector $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ by the same letter X.

Define, for every nondecreasing $g\colon [0,1] \to [0,1]$, the following operator

$$\mathcal{F}(g) = \left(\int_0^1 g(x)\,\mathrm{d}x,\ \int_0^1 xg(x)\,\mathrm{d}x,\ \int_0^1 g^2(x)\,\mathrm{d}x \right).$$

For $\mathcal{F}$, we introduce its body

$$\Omega = \{\mathcal{F}(g);\ g\colon [0,1] \to [0,1],\ g \text{ nondecreasing}\},$$

and $\partial\Omega$ denote the *boundary* of Ω.

The equation $X = \mathcal{F}(g)$ is referred to as a moment problem.

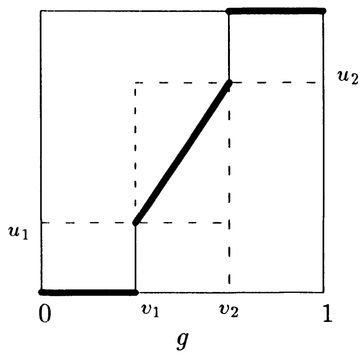


Figure 1.

[1)] Sometimes g is referred to as the *limit law*, and is otherwise known as the *asymptotic distribution function* of ω, see Schoenberg [4]. Our notation is different from that of the book [3; p. 53].

Let (see Fig. 1) $g(u_1, v_1, u_2, v_2)$ denote the distribution function $h(x)$ defined by

$$h(x) = \begin{cases} 0 & \text{for } 0 \le x \le v_1 , \\ \frac{u_2 - u_1}{v_2 - v_1} x + u_1 - v_1 \frac{u_2 - u_1}{v_2 - v_1} & \text{for } v_1 < x \le v_2 , \\ 1 & \text{for } v_2 < x \le 1 . \end{cases} \tag{1}$$

(in any case $h(1) = 1$). Then

$$\mathcal{F}\big(g(u_1, v_1, u_2, v_2)\big) = \begin{pmatrix} 1 - v_2 + \frac{1}{2}(v_2 - v_1)(u_1 + u_2) \\ \frac{1}{2} - \frac{1}{6}v_2^2(3 - u_1 - 2u_2) - \frac{1}{6}v_1^2(2u_1 + u_2) - \frac{1}{6}v_1 v_2(u_2 - u_1) \\ 1 - v_2 + \frac{1}{3}(v_2 - v_1)(u_1 u_2 + u_1^2 + u_2^2) \end{pmatrix} .$$

For varying parameters u_1, v_1, u_2, and v_2, the point $\mathcal{F}\big(g(u_1, v_1, u_2, v_2)\big)$ ($= \mathcal{F}(g)$) describes surfaces $\Pi_1 - \Pi_6$ or the curve Π_7 specified by the following list of formulas:

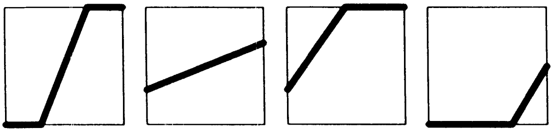
$$\begin{aligned} \Pi_1 &= \big\{\mathcal{F}(g) ;\ 0 \le v_1 \le v_2 \le 1 ,\ u_1 = 0 ,\ u_2 = 1\big\} , \\ \Pi_2 &= \big\{\mathcal{F}(g) ;\ v_1 = 0 ,\ v_2 = 1 ,\ 0 \le u_1 \le u_2 \le 1\big\} , \\ \Pi_3 &= \big\{\mathcal{F}(g) ;\ v_1 = 0 ,\ 0 \le v_2 \le 1 ,\ u_2 = 1 ,\ 0 \le u_1 \le 1\big\} , \\ \Pi_4 &= \big\{\mathcal{F}(g) ;\ 0 \le v_1 \le 1 ,\ v_2 = 1 ,\ u_1 = 0 ,\ 0 \le u_2 \le 1\big\} , \\ \Pi_5 &= \left\{\mathcal{F}(g) ;\ v_1 = 0 ,\ 0 \le v_2 \le 1 ,\ 0 \le u_1 = u_2 \le 1 ,\ v_2(1 - u_2) > \frac{1}{2}\right\} , \\ \Pi_6 &= \left\{\mathcal{F}(g) ;\ 0 \le v_1 \le 1 ,\ v_2 = 1 ,\ 0 \le u_1 = u_2 \le 1 ,\ u_1(1 - v_1) > \frac{1}{2}\right\} , \\ \Pi_7 &= \left\{\mathcal{F}(g) ;\ v_1 = 0 ,\ \frac{1}{2} < v_2 < 1 ,\ u_1 = u_2 = 1 - \frac{1}{2v_2}\right\} \\ &= \left\{\mathcal{F}(g) ;\ 0 < v_1 < \frac{1}{2} ,\ v_2 = 1 ,\ u_1 = u_2 = \frac{1}{2(1 - v_1)}\right\} . \end{aligned} \tag{2}$$

We shall specify the following eight types of distribution functions in this paper: Let u_1, v_1, u_2, and v_2 be fixed parameters which satisfy the inequalities given in the above definition of Π_i. Then the distribution function $g(u_1, v_1, u_2, v_2)$ is said to be of type $g^{(i)}$. The graphs[2)] of $g^{(i)}$, $i = 1, \dots, 6$, have a form depicted in Fig. 2 and 3. In Fig. 3 the areas of rectangles bounded

[2)] We write $g^{(i)}$ instead of $g(u_1, v_1, u_2, v_2)$.

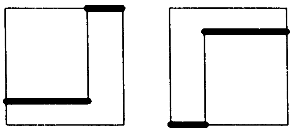
by the graphs of $g^{(5)}$ and $g^{(6)}$ are $\geq 1/2$. We write $g^{(5)} = g^{(7)}$ and $g^{(6)} = g^{(7^*)}$ in the case $= 1/2$.

Note that a given distribution function g can sometimes belong to various types.



$g^{(1)}$ $g^{(2)}$ $g^{(3)}$ $g^{(4)}$

Figure 2.



$g^{(5)}$ $g^{(6)}$

Figure 3.

Eliminating the parameters u_1, v_1, u_2, v_2, we arrive at the following canonical expressions:

$$
\begin{aligned}
g^{(1)} &= g(0,\, (1-X_1) - 3(X_1 - X_3),\, 1,\, (1-X_1) + 3(X_1 - X_3))\,,\\
g^{(2)} &= g\left(X_1 - \sqrt{3(X_3 - X_1^2)},\, 0,\, X_1 + \sqrt{3(X_3 - X_1^2)},\, 1\right),\\
g^{(3)} &= g\left(1 - \frac{3}{2}\frac{1 + X_3 - 2X_1}{1 - X_1},\, 0,\, 1,\, \frac{4}{3}\frac{(1-X_1)^2}{(1 + X_3 - 2X_1)}\right),\\
g^{(4)} &= g\left(0,\, 1 - \frac{4X_1^2}{3X_3},\, \frac{3X_3}{2X_1},\, 1\right),\\
g^{(5)} &= g\left(\frac{X_1 - X_3}{1 - X_1},\, 0,\, \frac{X_1 - X_3}{1 - X_1},\, \frac{(1-X_1)^2}{1 + X_3 - 2X_1}\right),\\
g^{(6)} &= g\left(\frac{X_3}{X_1},\, 1 - \frac{X_1^2}{X_3},\, \frac{X_3}{X_1},\, 1\right),\\
g^{(7)} &= g\left(1 - 2X_3,\, 0,\, 1 - 2X_3,\, \frac{1}{4X_3}\right),\\
g^{(7^*)} &= g\left(2X_3,\, 1 - \frac{1}{4X_3},\, 2X_3,\, 1\right).
\end{aligned}
\tag{3}
$$

$$\Pi_1 = \Big\{(X_1, X_2, X_3)\,;\;\; X_2 = \frac{1}{2} - \frac{1}{2}(1 - X_1)^2 - \frac{3}{2}(X_1 - X_3)^2,$$
$$\max\Big(\frac{4}{3}X_1 - \frac{1}{3}, \frac{2}{3}X_1\Big) \le X_3 \le X_1\,,\;\; 0 \le X_1 \le 1\Big\}.$$
$$\Pi_2 = \Big\{(X_1, X_2, X_3)\,;\;\; X_2 = \frac{1}{2}X_1 + \frac{1}{2}\sqrt{\frac{1}{3}(X_3 - X_1^2)}\,,$$
$$X_1^2 \le X_3 \le \min\Big(\frac{4}{3}X_1^2, \frac{4}{3}X_1^2 - \frac{2}{3}X_1 + \frac{1}{3}\Big),\;\; 0 \le X_1 \le 1\Big\}.$$
$$\Pi_3 = \Big\{(X_1, X_2, X_3)\,;\;\; X_2 = \frac{1}{2} - \frac{4}{9}\frac{(1 - X_1)^3}{(1 + X_3 - 2X_1)}\,,$$
$$\frac{4}{3}X_1^2 - \frac{2}{3}X_1 + \frac{1}{3} \le X_3 \le \frac{4}{3}X_1 - \frac{1}{3},\;\; \frac{1}{2} \le X_1 \le 1\Big\}.$$
$$\Pi_4 = \Big\{(X_1, X_2, X_3)\,;\;\; X_2 = X_1 - \frac{4}{9}\frac{X_1^3}{X_3}\,,\;\; \frac{4}{3}X_1^2 \le X_3 \le \frac{2}{3}X_1\,,\;\; 0 \le X_1 \le \frac{1}{2}\Big\}.$$
$$\Pi_5 = \Big\{(X_1, X_2, X_3)\,;\;\; X_2 = \frac{1}{2} - \frac{1}{2}\frac{(1 - X_1)^3}{(1 + X_3 - 2X_1)}\,,$$
$$X_1^2 \le X_3 \le X_1\,,\;\; 0 \le X_1 < \frac{1}{2}\Big\}.$$
$$\Pi_6 = \Big\{(X_1, X_2, X_3)\,;\;\; X_2 = X_1 - \frac{1}{2}\frac{X_1^3}{X_3}\,,\;\; X_1^2 \le X_3 \le X_1\,,\;\; \frac{1}{2} < X_1 \le 1\Big\}.$$
$$\Pi_7 = \Big\{\Big(\frac{1}{2}, \frac{1}{2} - \frac{1}{16X_3}, X_3\Big)\,;\;\; \frac{1}{4} < X_3 < \frac{1}{2}\Big\}. \tag{4}$$

3. The main results

We now state our main theorems. A starting point of this paper is the following criterion.

THEOREM 1. *Let $g\colon [0,1] \to [0,1]$ be a given distribution function. A sequence $\omega = (x_n)_{n=1}^{\infty} \subset [0,1]$ has the limiting distribution g if and only if*

(i) $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = \int_0^1 x \,\mathrm{d}g(x)\,,$

(ii) $\displaystyle\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = \int_0^1\!\!\int_0^1 |x - y| \,\mathrm{d}g(x)\,\mathrm{d}g(y)\,,$

(iii) $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_0^{x_n} g(x) \,\mathrm{d}x = \int_0^1 \Big(\int_0^x g(t) \,\mathrm{d}t\Big) \,\mathrm{d}g(x).$

Applying this to the moment problem, we derive a criterion in Theorem 3.

THEOREM 2. *For the moment problem $X = \mathcal{F}(g)$, to have only a finite number of solutions in distribution functions g it is necessary and sufficient that $X \in \partial\Omega$, where $\partial\Omega$ denotes the boundary of Ω. We can express the boundary $\partial\Omega$ as*

$$\partial\Omega = \bigcup_{1\le i\le 7} \Pi_i .$$

In addition, for $X \in \Pi_i$, $i = 1,2,\dots,6$, the moment problem $X = \mathcal{F}(g)$ is uniquely solvable as $g = g^{(i)}$, and for $X \in \Pi_7$ has precisely two solutions of types $g^{(7)}$ and $g^{(7^*)}$.

THEOREM 3. *Let $\omega = (x_n)_{n=1}^{\infty} \subset [0,1]$ be a sequence with the limits*

$$X_1 = 1 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n ,$$

$$X_2 = \frac{1}{2} - \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n^2 ,$$

$$X_3 = 1 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \frac{1}{2} \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| .$$

If $X = (X_1, X_2, X_3) \in \bigcup_{1\le i\le 7} \Pi_i$, then the sequence ω has a limit law. These limiting distributions are given by formulae (3). Moreover, if $X \in \Pi_i$, $i = 1,\dots,6$, then ω has a limiting distribution $g^{(i)}$, and if $X \in \Pi_7$, then ω has a limiting distribution either $g^{(7)}$ or $g^{(7^*)}$, depending on whether

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_0^{x_n} g^{(7)}(t)\, \mathrm{d}t = X_1 - X_3$$

or

$$\lim_{n\to\infty} \frac{1}{N} \sum_{0}^{N} \int_0^{x_n} g^{(7^*)}(t)\, \mathrm{d}t = X_1 - X_3 .$$

Furthermore, $X \in \bigcup_{1\le i\le 7} \Pi_i$ is testable by (4).

We begin proving Theorems 1 and 3, and then we shall turn to the proof of Theorem 2, since that proof is rather long and difficult, although of an elementary nature.

P r o o f   o f   T h e o r e m 1. The necessity of (i) – (iii) can be verified using the Helly-Bray theorem. In order to show the sufficiency of (i) – (iii), we apply in the usual way the notion of L^2-discrepancy $\int_0^1 \big(F_N(x) - g(x)\big)^2 \,\mathrm{d}x$ of the initial segment ω_N with regard to g.

Integration by parts gives

$$\int_0^1 F_N^2(x)\,\mathrm{d}x = 1 - \int_0^1 x\,\mathrm{d}F_N(x) - \frac{1}{2}\int_0^1\int_0^1 |x-y|\,\mathrm{d}F_N(x)\,\mathrm{d}F_N(y).$$

$$\int_0^1 F_N(x)g(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x - \int_0^1\left(\int_0^x g(t)\,\mathrm{d}t\right)\mathrm{d}F_N(x).$$

It follows that the L^2-discrepancy can be written as

$$\int_0^1 \big(F_N(x) - g(x)\big)^2\,\mathrm{d}x = 1 + \int_0^1 g^2(x)\,\mathrm{d}x - 2\int_0^1 g(x)\,\mathrm{d}x + \frac{2}{N}\sum_{n=1}^{N}\int_0^{x_n} g(t)\,\mathrm{d}t - \frac{1}{N}\sum_{n=1}^{N} x_n - \frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m - x_n|. \tag{5}$$

Limiting (5) as $N \to \infty$, and applying (i) – (iii) and the integral identities

$$\begin{aligned}\int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y) &= 2\int_0^1\left(\int_0^x g(t)\,\mathrm{d}t\right)\mathrm{d}g(x),\\ &= 2\left(\int_0^1 g(x)\,\mathrm{d}x - \int_0^1 g^2(x)\,\mathrm{d}x\right)\end{aligned} \tag{6}$$

to the right, we find that $\lim_{N\to\infty} F_N(x) = g(x)$ for all the points of continuity of g. □

P r o o f   o f   T h e o r e m 3. The proof can be developed along the lines of Method A. According to (6) and applying the rule of integration by parts, it can be shown that the moment problem $X = \mathcal{F}(g)$ can be rewritten as

$$
\begin{aligned}
1 - X_1 &= \int_0^1 x \, \mathrm{d}g(x)\,, \\
1 - 2X_2 &= \int_0^1 x^2 \, \mathrm{d}g(x)\,, \\
2(X_1 - X_3) &= \int_0^1\!\!\int_0^1 |x-y| \, \mathrm{d}g(x)\,\mathrm{d}g(y)\,.
\end{aligned}
\tag{7}
$$

Under the assumption of the existence of limits X_1, X_2, and X_3, all distribution functions of ω belong to the solution of (7). Since, by hypothesis, $X \in \partial\Omega$, we obtain from Theorem 2 that the moment problem (7) has only a finite number of solutions in g. Obviously, the sequence ω has a limit law. By Theorem 1, the problem of determining a distribution function that is the limiting distribution of a given sequence ω is reduced to the problem of calculating (i) – (iii) for the solutions of (7). Since (i) and (ii) are satisfied automatically, we need only verify condition (iii). $\square$

4. Proof of Theorem 2

The technique used here to solve our moment problem in Theorem 2 is different from the one used in [1], [5]. We shall work with neighbourhoods of points in the related body Ω defined in Section 2. For the sake of more clarity, we give an outline of the proof. We shall prove the theorem in six paragraphs.

In §1 we collected a couple of elementary facts about the body Ω needed for the proof. We mention (Lemma 1) two affine transformations Φ and Λ, which we shall define a little later, leaving the body Ω fixed; (Lemma 2) Ω is convex in the directions of co-ordinate axes X_2 and X_3.[3] We shall find (Lemma 3) projections of the body Ω to the planes $X_1 \times X_3$ and $X_1 \times X_2$, and (Lemma 4) two curve-edges of Ω.

The principal technical result is given in §2, which establishes (Lemma 5) the closedness of Ω under a linear law of composition as follows:

For any finite set of elements $X^{(0)}, \dots, X^{(N)}$ in Ω, every sum $\sum_{i=0}^{N} a_i + B_i X^{(i)}$ with vectors $a_i = a(u_i, v_i, u_{i+1}, v_{i+1})$ and matrices

[3] i.e. contains the whole line-segment $tX + (1-t)Y$, $0 \le t \le 1$ when it contains X, Y and $X_1 = Y_1$, $X_3 = Y_3$ or $X_1 = Y_1$, $X_2 = Y_2$.

$B_i = B(u_i, v_i, u_{i+1}, v_{i+1})$ (where u_i, v_i, u_{i+1}, v_{i+1} are parameters; a and B will be defined later) also belongs to Ω, and each point $X \in \Omega$, for every N, can be decomposed as $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$ into corresponding terms $X^{(0)}, \dots, X^{(N)} \in \Omega$.

As a consequence, the following is proved (Lemma 6): If $X \in \operatorname{int} \Omega$, then the moment problem $X = \mathcal{F}(g)$ has infinitely many solutions.

Further, with the help of this decomposition (as u_i, v_i, u_{i+1}, v_{i+1} vary continuously), we find new surfaces and bodies which are lying in Ω and contain X. In this way, we shall form in §3 planar, half-spherical or spherical neighbourhoods of X in Ω.

In §4 we make a general observation about neighbourhoods in Ω. We shall discuss the transformation $O = \sum_{i=0}^{N} a_i + B_i O_i$ mapping a sequence $O_0, \dots, O_N$ of neighbourhoods O_i of $X^{(i)}$ into the neighbourhood O of X, and its limitation which maps neighbourhoods O_i of $X_0^{(i)}$, where $X^{(i)} \to X_0^{(i)}$, into the neighbourhood O of X. But every point X of the boundary $\partial\Omega$ of Ω would not possess a spherical neighbourhood in Ω, and hence in this way we have a link between the decomposition $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$, limits $X^{(i)} \to X_0^{(i)}$, neighbourhoods O_i of $X_0^{(i)}$, and the $X \in \partial\Omega$.

In §5 we shall derive that (Lemma 15) every $X \in \Omega$ can be decomposed as $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$, where $X^{(i)} \to X_0^{(i)}$, and $X_0^{(i)}$, for specified i, assumes values from a set of five points. In accordance with this decomposition, the boundary $\partial\Omega$ can be expressed (Lemma 16) as the $\mathcal{F}$-image of linear and step distribution functions. In order to remove the restriction on nonsingular functions, we shall apply the theory of Dini derivatives.

In §6 we complete the proof of Theorem 2 by reduction to the case that $\partial\Omega$ is the union of $\mathcal{F}$-images of functions of types from Fig. 2 and 3. It remains then to show that the set of solutions of the moment problem $X = \mathcal{F}(g)$, $X \in \partial\Omega$, is finite.

Let us turn to the details.

§1. The basic property.

Lemma 1. *Ω and $\partial\Omega$ are invariant with respect to the transformation group*

$$\{\mathrm{Identity}, \Phi, \Lambda, \Phi \circ \Lambda\},$$

where

$$\Phi(X_1, X_2, X_3) = \Big(1 - X_1,\ X_2 - X_1 + \frac{1}{2},\ 1 + X_3 - 2X_1\Big),$$

$$\Lambda(X_1, X_2, X_3) = \Big(1 - X_1,\ \frac{1 - X_3}{2},\ 1 - 2X_2\Big).$$

Proof. For any monotone $g\colon [0,1] \to [0,1]$, let $\tilde{g}\colon [0,1] \to [0,1]$ be defined as

$$\tilde{g}(x) = \lambda\big(\{y \in [0,1];\ g(y) < x\}\big),$$

where λ is the Lebesgue measure.

Given $\mathcal{F}(g) = X$, we can compute $\mathcal{F}\big(\tilde{g}(x)\big) = \Lambda(X)$ by using

$$\int_0^1 x \,\mathrm{d}\tilde{g}(x) = \int_0^1 g(x)\,\mathrm{d}x, \qquad \int_0^1 x^2 \,\mathrm{d}\tilde{g}(x) = \int_0^1 g^2(x)\,\mathrm{d}x,$$

$$\int_0^1\int_0^1 |x-y| \,\mathrm{d}\tilde{g}(x)\,\mathrm{d}\tilde{g}(y) = \int_0^1\int_0^1 |g(x)-g(y)| \,\mathrm{d}x\,\mathrm{d}y = 4\int_0^1 xg(x)\,\mathrm{d}x - 2\int_0^1 g(x)\,\mathrm{d}x.$$

Since the map Λ is affine and the matrix of Λ is unimodular, it is obvious that $\Lambda(\Omega) = \Omega$ and $\Lambda(\partial\Omega) = \partial\Omega$. Using $\tilde{g}(x) = 1 - g(1-x)$, we can obtain analogous results for Φ.[4] □

Lemma 2. *Any straight line parallel to the X_2 or X_3 axis meets Ω at a segment.*[5]

Proof. Assuming $X_1 = Y_1$ and $X_2 = Y_2$ for $X = \mathcal{F}(g)$ and $Y = \mathcal{F}(f)$, where $f, g\colon [0,1] \to [0,1]$ are nondecreasing, we arrive at

$$tX + (1-t)Y = \Big(X_1, X_2, \int_0^1 \big(tg^2(x) + (1-t)f^2(x)\big)\,\mathrm{d}x\Big).$$

[4] Λ can be considered as a transformation $g \to \tilde{g}$. Similarly for Φ.

[5] Lemma 2 is referred to as a convexity of Ω in the X_2 and X_3 directions.

Since

$$\int_0^1 \big(tg(x) + (1-t)f(x)\big)^2 \,\mathrm{d}x \le \int_0^1 \big(tg^2(x) + (1-t)f^2(x)\big)\,\mathrm{d}x$$

$$\le \max\left(\int_0^1 g^2(x)\,\mathrm{d}x, \int_0^1 f^2(x)\,\mathrm{d}x\right).$$

from continuity one obtains the existence of $t_0 \in [0,1]$ so that

$$\int_0^1 \big(t_0 g(x) + (1-t_0)f(x)\big)^2 \,\mathrm{d}x = \int_0^1 \big(tg^2(x) + (1-t)f^2(x)\big)\,\mathrm{d}x\,.$$

Hence the $\mathcal{F}$-image for nondecreasing $h(x) = t_0 g(x) + (1-t_0)f(x)$ takes the form $\mathcal{F}(h) = tX + (1-t)Y$.

The rest follows from the properties of the transformation Λ. □

Lemma 3. *The orthogonal projections of the body Ω onto the $X_1 \times X_3$ and $X_1 \times X_2$-planes are equal to*

$$\big\{(X_1, X_3)\,;\ X_1^2 \le X_3 \le X_1\,,\ 0 \le X_1 \le 1\big\}$$

and

$$\Big\{(X_1, X_2)\,;\ \frac{1}{2}X_1 \le X_2 \le X_1 - \frac{1}{2}X_1^2\,,\ 0 \le X_1 \le 1\Big\}\,,$$

respectively.

Proof. Here we consider ω_N as an N-dimensional vector $\omega_N = (x_1, \dots, x_N)$, where $x_1, \dots, x_N \in [0,1]$ are ordered according to their magnitude, that is $0 \le x_1 \le x_2 \le \dots \le x_N \le 1$. Defining F_N and $\mathcal{F}$ as in the introduction, it is not difficult to verify that

$$\mathcal{F}(F_N) = \left(1 - \frac{1}{N}\sum_{n=1}^N x_n,\ \frac{1}{2} - \frac{1}{2N}\sum_{n=1}^N x_n^2,\ 1 - \frac{1}{N^2}\sum_{n=1}^N (2n-1)x_n\right).$$

Since there is no risk of confusion, we shall write $\mathcal{F}(\omega_N)$ instead of $\mathcal{F}(F_N)$. For fixed N, consider the set Δ_N of all ω_N and the set Ω_N of all $\mathcal{F}$-images of ω_N. Representing Ω and Δ_N as

$$\Omega = \text{closure} \bigcup_{1 \le N \le \infty} \Omega_N\,,$$

$$\Delta_N = \text{convex hull}\Big\{\omega_N^{(i)} = (\underbrace{0,\dots,0}_{i\ \text{times}},\ \underbrace{1,\dots,1}_{N-i\ \text{times}})\,;\ i = 0,1,\dots,N\Big\},$$

we find that

$$\text{projection}\,\Omega = \text{closure} \bigcup_{1\le N\le\infty} \text{projection}\,\mathcal{F}\Big(\text{convex hull}\Big\{\omega_N^{(i)}\,;\ i=0,1,\dots,N\Big\}\Big).$$

Since $X = \mathcal{F}(\omega_N)$ consists of linear X_1 and X_3 over ω_N, and

$$(X_1, X_3) = \left(\frac{i}{N}, \left(\frac{i}{N}\right)^2\right)$$

for $\omega_N = \omega_N^{(i)}$, then the orthogonal projection of Ω onto the plane $X_1 \times X_3$ is

$$\text{closure} \bigcup_{1\le N\le\infty} \text{convex hull}\left\{\left(\frac{i}{N}, \left(\frac{i}{N}\right)^2\right);\ i = 0,1,\dots,N\right\}.$$

Finally, applying transformation Λ to the above projection, we find the orthogonal projection of Ω onto $X_1 \times X_2$. $\square$

Lemma 4. *The following curves*

$$\Big\{(X_1, X_2, X_3)\,;\ X_2 = \tfrac{1}{2}X_1(2 - X_1),\ X_3 = X_1,\ 0 \le X_1 \le 1\Big\},$$
$$\Big\{(X_1, X_2, X_3)\,;\ X_2 = \tfrac{1}{2}X_1,\ X_3 = X_1^2,\ 0 \le X_1 \le 1\Big\},$$

intersect in the points $(0,0,0)$ *and* $\left(1, \frac{1}{2}, 1\right)$, *and they belong to* Ω. *Moreover, their orthogonal projections coincide with the boundary of the above projections of* Ω.[6]

P r o o f. Putting $X = \mathcal{F}(g)$ and assuming that $X_3 = X_1$, i.e. $\int_0^1 g^2(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x$, we have $g(x) = 0$ or 1 for all $x \in [0,1]$; g is a one-jump function, and then we obtain $X_2 = \frac{1}{2}X_1(2 - X_1)$. Applying the transformation Λ to g, we obtain a constant function $\tilde{g}$, and then we have $\tilde{X}_2 = \frac{1}{2}\tilde{X}_1$ and $\tilde{X}_3 = \tilde{X}_1^2$ for $\tilde{X} = \mathcal{F}(\tilde{g})$. $\square$

Let us now introduce a new operation on the sequence $\left(X^{(i)}\right)_{i=0}^N \subset \Omega$.

[6] The projections show that the curves are lying in the boundary of Ω. Furthermore, each planar section of Ω by a plane parallel to $X_2 \times X_3$ and passing through a point from the curves is lying in a quarter-plane. Thus it seems natural to call these curves curve-edges of Ω.

§2. The law of composition.

Let $\left((v_i, u_i)\right)_{i=0}^{N+1}$ be a finite sequence of points in $[0,1]^2$. Suppose that

(i) $(v_0, u_0) = (0,0)$, $(v_{N+1}, u_{N+1}) = (1,1)$,
(ii) $v_i \le v_{i+1}$ and $u_i \le u_{i+1}$ for all $i = 0, 1, \dots, N$.

Defining a_i and B_i as

$$a_i = a(u_i, v_i, u_{i+1}, v_{i+1}) = \begin{pmatrix} (v_{i+1}-v_i)u_i \\ (v_{i+1}-v_i)u_i v_i + \frac{1}{2}(v_{i+1}-v_i)^2 u_i \\ (v_{i+1}-v_i)u_i^2 \end{pmatrix},$$

$$\begin{aligned} B_i &= B(u_i, v_i, u_{i+1}, v_{i+1}) \\ &= \begin{pmatrix} (u_{i+1}-u_i)(v_{i+1}-v_i) & 0 & 0 \\ v_i(u_{i+1}-u_i)(v_{i+1}-v_i) & (u_{i+1}-u_i)(v_{i+1}-v_i)^2 & 0 \\ 2u_i(u_{i+1}-u_i)(v_{i+1}-v_i) & 0 & (u_{i+1}-u_i)^2(v_{i+1}-v_i) \end{pmatrix}. \end{aligned}$$

we form the sum

$$X = \sum_{i=0}^{N} a_i + B_i X^{(i)}. \tag{8}$$

Lemma 5. *Let $\left((v_i, u_i)\right)_{i=0}^{N+1}$ be a given sequence which satisfies the above assumptions* (i) *and* (ii)*. Then, for every $\left(X^{(i)}\right)_{i=0}^{N} \subset \Omega$ and X determined by* (8)*, we have $X \in \Omega$, and vice versa: For any $X \in \Omega$ (i.e. $X = \mathcal{F}(g)$, g-nondecreasing) a sequence $\left(X^{(i)}\right)_{i=0}^{N} \subset \Omega$, which satisfies* (8)*, can be found. In the last assertion we need to add the following assumption*

(iii) $g(v_i - 0) \le u_i \le g(v_i + 0)$ *for all* $i = 1, \dots, N$.

P r o o f. We shall first demonstrate the second statement of the lemma. We have $X = \mathcal{F}(g)$, where g is nondecreasing and $g|_{(v_i,\, v_{i+1})} : (v_i, v_{i+1}) \to [u_i, u_{i+1}]$ denotes a restriction of g. Then by elementary reasoning one shows that the graph of $g|_{(v_i,\, v_{i+1})}$ has the linear expansion on $[0,1]^2$ given by

$$g_i(x) = \begin{cases} \dfrac{g(x(v_{i+1}-v_i)+v_i)}{u_{i+1}-u_i} - \dfrac{u_i}{u_{i+1}-u_i} & \text{if } u_i < u_{i+1}, \\ 0 & \text{if } u_i = u_{i+1}. \end{cases} \tag{9}$$

for all $x \in (0,1)$ (see Fig. 4).

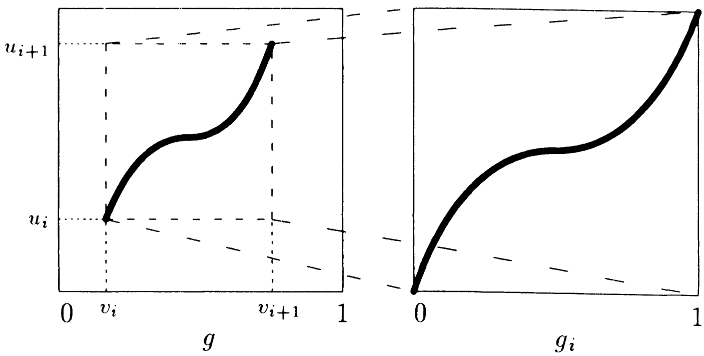


Figure 4.

Putting $X^{(i)} = \mathcal{F}(g_i)$ and assuming $v_i < v_{i+1}$, $u_i < u_{i+1}$, we find that

$$X_1^{(i)} = \int_0^1 g_i(x)\,\mathrm{d}x = \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)} \int_{v_i}^{v_{i+1}} g(x)\,\mathrm{d}x - \frac{u_i}{u_{i+1}-u_i}\,,$$

$$\begin{aligned} X_2^{(i)} &= \int_0^1 x g_i(x)\,\mathrm{d}x \\ &= \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)^2}\left(\int_{v_i}^{v_{i+1}} x g(x)\,\mathrm{d}x - v_i \int_{v_i}^{v_{i+1}} g(x)\,\mathrm{d}x\right) \\ &\qquad - \frac{1}{2}\frac{u_i}{u_{i+1}-u_i}\,, \end{aligned} \tag{10}$$

$$\begin{aligned} X_3^{(i)} &= \int_0^1 g_i^2(x)\,\mathrm{d}x \\ &= \frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}\left(\int_{v_i}^{v_{i+1}} g^2(x)\,\mathrm{d}x - 2u_i \int_{v_i}^{v_{i+1}} g(x)\,\mathrm{d}x\right) \\ &\qquad + \left(\frac{u_i}{u_{i+1}-u_i}\right)^2. \end{aligned}$$

As a result of these equalities, we have

$$\left(\int_{v_i}^{v_{i+1}} g(x)\,\mathrm{d}x,\ \int_{v_i}^{v_{i+1}} xg(x)\,\mathrm{d}x,\ \int_{v_i}^{v_{i+1}} g^2(x)\,\mathrm{d}x\right) = a_i + B_i X^{(i)}\,.$$

which holds also for $v_i = v_{i+1}$ and $u_i = u_{i+1}$. The equality (8) is shown.

In order to prove the first statement of Lemma 5, one observes that, for a given $\left(X^{(i)}\right)_{i=0}^{N} \subset \Omega$ and $\left((v_i, u_i)\right)_{i=0}^{N+1} \subset [0,1]^2$, where $X^{(i)} = \mathcal{F}(g_i)$, and g_i is nondecreasing, one can guarantee the existence of a nondecreasing $g\colon [0,1] \to [0,1]$ having g_i as in (9). Indeed, putting

$$g\big|_{(v_i, v_{i+1})} = \begin{cases} g_i\Big(\dfrac{x - v_i}{v_{i+1} - v_i}\Big)(u_{i+1} - u_i) + u_i & \text{if } v_i < v_{i+1}\,, \\ u_i & \text{if } v_i = v_{i+1}\,. \end{cases} \tag{11}$$

we have (9) for all $i = 0, 1, \dots, N$, and hence the second statement of the lemma can be applied to g. □

Finally we shall prove the following

LEMMA 6. *If $X \in \operatorname{int}\Omega$, then the moment problem $X = \mathcal{F}(g)$ has infinitely many solutions in distribution functions g.*

Proof. We start from (8). Assuming $X \in \operatorname{int}\Omega$, with the help of §1, Lemmas 3 and 4, we can find an index i such that $B_i \neq 0$. The expression for $X^{(i)}$ takes the form

$$X^{(i)} = B_i^{-1}\Big(X - a_i - \sum_{j=0,\, j\neq i}^{N} a_j + B_j X^{(j)}\Big), \tag{12}$$

where

$$\begin{aligned} B_i^{-1} &= B^{-1}(u_i, v_i, u_{i+1}, v_{i+1}) \\ &= \begin{pmatrix} \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)} & 0 & 0 \\ \frac{-v_i}{(u_{i+1}-u_i)(v_{i+1}-v_i)^2} & \frac{1}{(u_{i+1}-u_i)(v_{i+1}-v_i)^2} & 0 \\ \frac{-2u_i}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)} & 0 & \frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)} \end{pmatrix}. \end{aligned}$$

Here we consider the right hand side of (12) as a vector-valued function of the variables $\left(X^{(j)}\right)_{j=0, j\neq i}^{N} \subset \Omega$ and $\left((v_j, u_j)\right)_{j=0}^{N+1} \subset [0,1]^2$. Calculating the limits of B_i^{-1}, a_j, B_j as $(v_i, u_i) \to (0,0)$ and $(v_{i+1}, u_{i+1}) \to (1,1)$, we obtain [7]

$$B_i^{-1} \to 1\,, \qquad a_j \to 0 \quad \text{for all} \quad j\,, \qquad B_j \to 0 \quad \text{for all} \quad j \neq i\,.$$

[7] Here we use the symbols 1 and 0 to denote the unit and zero matrices or vectors.

Consequently,

$$B_i^{-1}\left(X - a_i - \sum_{j=0,\, j\neq i}^{N} a_j + B_j\Omega\right) \subset \operatorname{int}\Omega \tag{13}$$

for any (v_i, u_i) and (v_{i+1}, u_{i+1}) sufficiently near to $(0,0)$ and $(1,1)$, respectively. Thus, we have shown that for $\left(X^{(j)}\right)_{j=0,\, j\neq i}^{N} \subset \Omega$ it is possible to compute $X^{(i)} \in \Omega$ so that (8) is valid. Now, we can represent each $X^{(j)}$, $j = 0, 1, \dots, N$ in the form $X^{(j)} = \mathcal{F}(g_j)$, and finally, applying construction (11), we infer that $X = \mathcal{F}(g)$, where for different $X^{(j)}$ we get different g. □

For the sake of completeness, it is easy to establish the result that for any $X \in \operatorname{int}\Omega$, there exists an $\tilde{X} \in \operatorname{int}\Omega$ such that

$$X = \sum_{i=0}^{N} a_i + B_i\tilde{X}\,,$$

because in (13) the mapping is contractive. This will prove the existence of a solution g of $X = \mathcal{F}(g)$ which is obtained from the composition (11), where $g_i = \tilde{g}$ for all i and $\tilde{g}$ is a suitable distribution function.

The purpose of the following two sections is to study neighbourhoods O of X in the body Ω. Here we shall use the notation

$$O = X + V = \{X + Y\,;\ Y \in V\}\,,$$

where V denotes a suitable set of three-dimensional vectors.

§3. Linear neighbourhoods; Definition and construction.

With the help of operation (8) (as u_i, v_i, u_{i+1}, v_{i+1} vary continuously), we can find new surfaces and bodies lying in Ω, containing X, and forming neighbourhoods of X in Ω. Using local coordinates, we can give the following classification:

Let a, b, c be non-complanar vectors. Let

$$[a]\,,\quad [\pm a]\,,\quad [a,b]\,,\quad [\pm a, \pm b]\,,\quad [\pm a, \pm b, c]$$

denote the following sets of three-dimensional vectors:

$$[a] = \big\{au + \omega u\,;\ u \in [0,\varepsilon]\big\}\,,$$

$$[\pm a] = \big\{au + \omega u\,;\ u \in [-\varepsilon,\varepsilon]\big\}\,,$$

$$\dots$$

$$[\pm a, \pm b, c] = \Big\{au + bv + cw + \omega\sqrt{u^2+v^2+w^2}\,;\ u,v \in [-\varepsilon,\varepsilon]\,,\ w \in [0,\varepsilon]\Big\}\,,$$

where u, v, w are variables, ω is a continuously differentiable vector-valued function such that $\omega \to 0$ as $u \to 0$, $v \to 0$, $w \to 0$, and ε is a sufficiently small positive number. The sets

$$X + [a], \quad X + [\pm a], \quad X + [a, b], \quad X + [\pm a, \pm b], \quad X + [\pm a, \pm b, c] \subset \Omega$$

are called *linear neighbourhoods* [8] of X in Ω. For the sake of brevity, we call these neighbourhoods *half-line, line, angular, planar* and *half-spherical* neighbourhoods of X in Ω, respectively.

We collect several results in our construction of neighbourhoods. We shall make use of them in subsequent sections.

Lemma 7. *All the following sets*

$$\begin{aligned}
(0,0,0) &+ [(1,1,1)]\,, \\
(1, \tfrac{1}{2}, 1) &+ [(-1,0,-2)]\,, \\
(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}) &+ [\pm(3,1,2), \pm(3,2,4)]\,, \\
(1 - v, \tfrac{1}{2}(1 - v^2), 1 - v) &+ [\pm(1, v, 1)]\,, \qquad 0 < v < 1\,, \\
(u, \tfrac{1}{2}u, u^2) &+ [\pm(1, \tfrac{1}{2}, 2u)]\,, \qquad 0 < u < 1\,,
\end{aligned}$$

are linear neighbourhoods in Ω.[9] *Here* $(0,0,0)$, $(1, \frac{1}{2}, 1)$, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$, $(1 - v, \frac{1}{2}(1 - v^2), 1 - v)$, $(u, \frac{1}{2}u, u^2)$ *are* $\mathcal{F}$*-images of the functions from* Fig. 5, *respectively.*

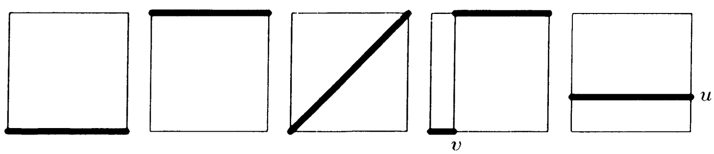


Figure 5.

Proof. We shall not give the details of the proof of Lemma 7; we show only $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}) + [\pm(3,1,2), \pm(3,2,4)] \subset \Omega$. Put $X = \mathcal{F}(g(u_1, v_1, u_2, v_2))$. Fig. 1

[8] The sets $[a], \dots, [\pm a, \pm b, c]$ depend on given ω and ε, and the formula $X + [a], \dots$ $\dots, X + [\pm a, \pm b, c] \subset \Omega$ indicates that the inclusion is valid for a sufficiently small $\varepsilon > 0$.

[9] It should be noted that any neighbourhood listed above can be extended to a large neighbourhood, e.g. in a particular case $(0,0,0) + [(3,1,0), (3,2,0), (1,1,1)] \subset \Omega$.

indicates that

$$\begin{aligned} X &= a(0,0,u_1,v_1) + B(0,0,u_1,v_1)(0,0,0) \\ &\quad + a(u_1,v_1,u_2,v_2) + B(u_1,v_1,u_2,v_2)\left(\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{3}\right) \\ &\quad + a(u_2,v_2,1,1) + B(u_2,v_2,1,1)\left(1,\tfrac{1}{2},1\right) \\ &= X(u_1,v_1,u_2,v_2) \\ &= \begin{pmatrix} 1 - v_2 + \frac{1}{2}(v_2 - v_1)(u_1 + u_2) \\ \frac{1}{2} - \frac{1}{6}v_2^2(3 - u_1 - 2u_2) - \frac{1}{6}v_1^2(2u_1 + u_2) - \frac{1}{6}v_1v_2(u_2 - u_1) \\ 1 - v_2 + \frac{1}{3}(v_2 - v_1)(u_1u_2 + u_1^2 + u_2^2) \end{pmatrix}. \end{aligned} \tag{14}$$

Since $X = \left(\frac{1}{2},\frac{1}{3},\frac{1}{3}\right)$ for $u_1 = v_1 = 0$, $u_2 = v_2 = 1$ and

$$\frac{\partial X}{\partial v_1} = \left(-\tfrac{1}{2},-\tfrac{1}{6},-\tfrac{1}{3}\right), \qquad \frac{\partial X}{\partial v_2} = \left(-\tfrac{1}{2},-\tfrac{1}{3},-\tfrac{2}{3}\right),$$
$$\frac{\partial X}{\partial u_1} = \left(\tfrac{1}{2},\tfrac{1}{6},\tfrac{1}{3}\right), \qquad \frac{\partial X}{\partial u_2} = \left(\tfrac{1}{2},\tfrac{1}{3},\tfrac{2}{3}\right),$$

we have

$$\begin{aligned} X &= \left(\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{3}\right) + \left(-\tfrac{1}{2},-\tfrac{1}{6},-\tfrac{1}{3}\right)(v_1 - 0) + \left(-\tfrac{1}{2},-\tfrac{1}{3},-\tfrac{2}{3}\right)(v_2 - 1) \\ &\quad + \left(\tfrac{1}{2},\tfrac{1}{6},\tfrac{1}{3}\right)(u_1 - 0) + \left(\tfrac{1}{2},\tfrac{1}{3},\tfrac{2}{3}\right)(u_2 - 1) + \omega\sqrt{u_1^2 + v_1^2 + (u_2 - 1)^2 + (v_2 - 1)^2}, \end{aligned}$$

where $\omega \to 0$ as $v_1, u_1 \to 0$ and $u_2, v_2 \to 1$. This completes the proof. □

§4. Linear neighbourhoods; An operation.

We start with the expression $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$. A basic problem is to understand how geometric properties of linear neighbourhoods O_i of $X^{(i)}$, $i = 0, 1, \ldots, N$, are "reflected" in a related neighbourhood O of X. Since no element of the boundary of Ω admits a spherical neighbourhood in Ω then, when investigating the existence of a reflection, we incidentally answer the question of characterization of $X \in \partial\Omega$ by means of a suitable decomposition $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$. The method used to solve this problem is based on operation (8) and convexity in Lemma 2. Here is the precise formulation:

Lemma 8. *If* $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$, *and* $X^{(i)} + V_i \subset \Omega$ *for all* $i = 0, 1, \dots, N$, *then*

$$X + \sum_{i=0}^{N} B_i V_i \subset \Omega .$$

Lemma 9. *If* $X + V \subset \Omega$, *and* $\tilde{V}$ *denotes the convex hull of* V *in the* X_2 *and* X_3 *directions (i.e.* $\tilde{V}$ *contains the whole line-segment* XY *whenever* $X, Y \in V$ *and* $X_1 = Y_1$, $X_3 = Y_3$ *or* $X_1 = Y_1$, $X_2 = Y_2$ *), then* $X + \tilde{V} \subset \Omega$.

Applying Lemmas 8 and 9 to the following situations, we shall now state explicitly some criteria for $X \in \operatorname{int} \Omega$ which will be used in the next section.

Lemma 10. *Suppose there are two planar neighbourhoods of* X *in* Ω *such that*

(i) $X + [\pm a_1, \pm b_1]$, $X + [\pm a_2, \pm b_2] \subset \Omega$, *and*

(ii) *the vector product* $(a_1 \times b_1) \times (a_2 \times b_2)$ *has a nonzero first co-ordinate.*

Then $X \in \operatorname{int} \Omega$.

First we give a definition which we shall need below.

For $X + [\pm a, \pm b, c]$, consider a vector n satisfying $n \times (a \times b) = 0$ and $n \cdot c < 0$ (scalar product). n is termed a *normal vector* of the given half-spherical neighbourhood of X .

Lemma 11. *A sufficient condition that* $X \in \operatorname{int} \Omega$ *is that the following three conditions be satisfied:*

(i) *There is a half-spherical neighbourhood of* X *in* Ω *such that* $X + [\pm a, \pm b, c] \subset \Omega$ *has a normal vector* n *with second co-ordinate different from zero.*

(ii) *There is a point* $\hat{X} \in \Omega$ *with a planar neighbourhood* $\hat{X} + [\pm \hat{a}, \pm \hat{b}] \subset \Omega$ *such that* $\hat{X}_1 = X_1$, $\hat{X}_3 = X_3$, $\hat{X}_2 \neq X_2$, *and the vector product* $\hat{a} \times \hat{b}$ *has a nonzero second co-ordinate.*

(iii) *The second co-ordinates of* n *and* $X_2 - \hat{X}_2$ *have mutually opposite signs.*

Lemma 12. *Assume that*

(i) X *can be decomposed as* $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$,
where $X^{(i)} \in \Omega$ *for* $i = 0, 1, \dots, N$,

(ii) *there is an index* i *such that* $X^{(i)} \in \operatorname{int} \Omega$.

Then $X \in \operatorname{int} \Omega$.

LEMMA 13. *Assume that*

(i) X *can be decomposed as* $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$,
where $X^{(i)} \in \Omega$ *for* $i = 0, 1, \ldots, N$,

(ii) *there are three indexes* $i \neq j \neq k \neq i$ *such that* $X^{(i)}$, $X^{(j)}$, $X^{(k)}$ *have line neighbourhoods* $X^{(i)} + [\pm a]$, $X^{(j)} + [\pm b]$, $X^{(k)} + [\pm c] \subset \Omega$,

(iii) *the vectors* $B_i a$, $B_j b$, $B_k c$ *are non-complanar.*

Then $X \in \operatorname{int} \Omega$.

P r o o f . For the sake of brevity, we shall not give the details of the proof of Lemmas 10 – 13, we only note that Lemmas 10 and 11 follow from Lemma 9; Lemmas 12 and 13 from Lemma 8; and to prove Lemma 13, we make use of the inclusion

$$\begin{aligned} &\{au + bv + cw\,;\ u, v, w \in [-\varepsilon_0, \varepsilon_0]\} \\ &\qquad \subset \{au + bv + cw + \omega_a u + \omega_b v + \omega_c w\,;\ u, v, w \in [-\varepsilon, \varepsilon]\} . \end{aligned} \tag{15}$$

Here a , b , c are non-complanar vectors, $\omega_a = \omega_a(u)$, $\omega_b = \omega_b(v)$, $\omega_c = \omega_c(w)$ are vector-valued continuously differentiable functions on $[-\varepsilon, \varepsilon]$, and

$$\varepsilon_0 = \varepsilon - 3\tilde{\varepsilon} ,$$

where

$$\tilde{\varepsilon} = \frac{\max\{|\omega_a u|, |\omega_b v|, |\omega_c w|\,;\ u, v, w \in [-\varepsilon, \varepsilon]\}}{\min\{|a|, |b|, |c|\}} .$$

□

Lemmas 12 and 13 admit generalizations which involve also a limiting process. We state explicitly only a version analogous to Lemma 13.

LEMMA 14. *Let us suppose that*

(i) *a given point* $X \in \Omega$ *can be decomposed as* $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$,
where $X^{(i)} \in \Omega$ *for* $i = 0, 1, \ldots, N$,[10)]

(ii) *there are three indexes* $0 \leq i, j, k \leq N$, *with* $|i - j|, |i - k|, |j - k| > 1$, *such that the limits*

$$X^{(i)} \to X_0^{(i)} , \qquad X^{(j)} \to X_0^{(j)} , \qquad X^{(k)} \to X_0^{(k)}$$

10) Here $X^{(i)} = X^{(i)}(u_i, v_i, u_{i+1}, v_{i+1}) = \mathcal{F}(g_i)$, g_i is defined by (9) , and the variables $(u_i, v_i, u_{i+1}, v_{i+1})$, $i = 0, 1, \ldots, N$, must satisfy the assumptions (i), (ii), and (iii) of Lemma 5.

exist as

$$\begin{aligned}
(u_i, v_i, u_{i+1}, v_{i+1}) &\to \left(u_i^{(0)}, v_i^{(0)}, u_{i+1}^{(0)}, v_{i+1}^{(0)}\right), \\
(u_j, v_j, u_{j+1}, v_{j+1}) &\to \left(u_j^{(0)}, v_j^{(0)}, u_{j+1}^{(0)}, v_{j+1}^{(0)}\right), \\
(u_k, v_k, u_{k+1}, v_{k+1}) &\to \left(u_k^{(0)}, v_k^{(0)}, u_{k+1}^{(0)}, v_{k+1}^{(0)}\right),
\end{aligned}$$

respectively.

Further, assume that

(iii) *the points* $X_0^{(i)}$, $X_0^{(j)}$, $X_0^{(k)}$ *have line neighbourhoods*

$$X_0^{(i)} + [\pm a]\,,\ X_0^{(j)} + [\pm b]\,,\ X_0^{(k)} + [\pm c] \subset \Omega\,,$$

(iv) *the limits*

$$\frac{B_i a}{(u_{i+1} - u_i)(v_{i+1} - v_i)} \to \tilde{a}\,, \qquad \frac{B_j b}{(u_{j+1} - u_j)(v_{j+1} - v_j)} \to \tilde{b}\,.$$
$$\frac{B_k c}{(u_{k+1} - u_k)(v_{k+1} - v_k)} \to \tilde{c}\,,$$

exist, where
$(u_i, v_i, u_{i+1}, v_{i+1})$, $(u_j, v_j, u_{j+1}, v_{j+1})$, $(u_k, v_k, u_{k+1}, v_{k+1})$
converge as in (ii).

Moreover, suppose that

(v) *the vectors* $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ *are non-complanar.*

Then the above assumptions (i) – (v) *imply that* $X \in \operatorname{int}\Omega$.

P r o o f. If we set

$$X_0 = a_i + B_i X_0^{(i)} + a_j + B_j X_0^{(j)} + a_k + B_k X_0^{(k)} + \sum_{\substack{n=0 \\ n \neq i,j,k}}^{N} a_n + B_n X^{(n)}\,.$$

then $X - X_0$ has the form

$$X - X_0 = B_i \left(X^{(i)} - X_0^{(i)}\right) + B_j \left(X^{(j)} - X_0^{(j)}\right) + B_k \left(X^{(k)} - X_0^{(k)}\right).$$

According to the assumptions (i) – (v), we shall obtain the existence of a spherical neighbourhood

$$O_0 = O_0(u_i, v_i, u_{i+1}, v_{i+1}, u_j, v_j, u_{j+1}, v_{j+1}, u_k, v_k, u_{k+1}, v_{k+1})$$

of X_0 in Ω.

As a further step, we shall determine, in (16'), that the order of convergence of $X - X_0 \to 0$ is higher than the order of convergence of diameters of O_0 to zero, and consequently, we obtain $X \in O_0$ for $(u_i, \dots, v_{k+1})$ sufficiently near to $(u_i^{(0)}, \dots, v_{k+1}^{(0)})$.

For a detailed proof, we consider the assumption (iii) in the form

$$X_0^{(i)} + \{au + \omega_a u;\ u \in [-\varepsilon, \varepsilon]\}, \qquad X_0^{(j)} + \{bu + \omega_b u;\ u \in [-\varepsilon, \varepsilon]\},$$
$$X_0^{(k)} + \{cu + \omega_c u;\ u \in [-\varepsilon, \varepsilon]\} \subset \Omega.$$

By means of Lemma 8 we can write

$$X_0 + \{B_i au + B_i\omega_a u + B_j bv + B_j\omega_b v + B_k cw + B_k\omega_c w;\ u, v, w \in [-\varepsilon, \varepsilon]\} \subset \Omega. \tag{16}$$

But, according to the limits (iv), this neighbourhood of X_0 can be represented as

$$\begin{aligned} X_0 + \{&(u_{i+1} - u_i)(v_{i+1} - v_i)\tilde{a}u + (u_{j+1} - u_j)(v_{j+1} - v_j)\tilde{b}v \\ &+ (u_{k+1} - u_k)(v_{k+1} - v_k)\tilde{c}w + (u_{i+1} - u_i)(v_{i+1} - v_i)\omega_{\tilde{a}}u \\ &+ (u_{j+1} - u_j)(v_{j+1} - v_j)\omega_{\tilde{b}}v + (u_{k+1} - u_k)(v_{k+1} - v_k)\omega_{\tilde{c}}w; \\ &\qquad\qquad u, v, w \in [-\varepsilon, \varepsilon]\} \subset \Omega, \end{aligned}$$

where

$$\omega_{\tilde{a}} = \frac{B_i\omega_a}{(u_{i+1} - u_i)(v_{i+1} - v_i)} + \left(\frac{B_i a}{(u_{i+1} - u_i)(v_{i+1} - v_i)} - \tilde{a}\right) \to 0,$$

as $(u_i, v_i, u_{i+1}, v_{i+1}) \to (u_i^{(0)}, v_i^{(0)}, u_{i+1}^{(0)}, v_{i+1}^{(0)})$ and $u \to 0$. Similarly, we find $\omega_{\tilde{b}} \to 0$ and $\omega_{\tilde{c}} \to 0$.

Since the variables indexed with i, j, k are independent, the $u_i, \dots, v_{k+1}$ can be chosen [11] in such a way that they satisfy

$$(u_{i+1} - u_i)(v_{i+1} - v_i) = (u_{j+1} - u_j)(v_{j+1} - v_j) = (u_{k+1} - u_k)(v_{k+1} - v_k) = t.$$

[11] We may suppose that $(u_{i+1} - u_i)(v_{i+1} - v_i) \to 0$, otherwise replace $X^{(i)}$ by $X_0^{(i)}$ in the decomposition of X, and then we have $X - X_0 = B_j(X^{(j)} - X_0^{(j)}) + B_k(X^{(k)} - X_0^{(k)})$. Similarly, for j and k.

Applying (15) to the set

$$\left\{\tilde{a}u+\tilde{b}v+\tilde{c}w+\omega_{\tilde{a}}u+\omega_{\tilde{b}}v+\omega_{\tilde{c}}w\,;\;\; u,v,w\in[-\varepsilon,\varepsilon]\right\},$$

we obtain

$$O_0 = X_0 + t\left\{\tilde{a}u+\tilde{b}v+\tilde{c}w\,;\;\; u,v,w\in[-\varepsilon_0,\varepsilon_0]\right\}\subset\Omega\,,$$

where $\varepsilon_0 = \varepsilon - 3\tilde{\varepsilon}$, and $\frac{\tilde{\varepsilon}}{\varepsilon}$ tends to zero as $\left(u_i,\dots,v_{k+1}\right)\to\left(u_i^{(0)},\dots,v_{k+1}^{(0)}\right)$ and $\varepsilon\to 0$. Therefore, after applying

$$\begin{aligned}&\frac{X-X_0}{t}\\ &=\frac{B_i}{t}\left(X^{(i)}-X_0^{(i)}\right)+\frac{B_j}{t}\left(X^{(j)}-X_0^{(j)}\right)+\frac{B_k}{t}\left(X^{(k)}-X_0^{(k)}\right)\to 0\end{aligned}\tag{16'}$$

as $t\to 0$, for sufficiently small t one obtains $X\in O_0$. This completes the proof of Lemma 14. □

It should be noted that the last argument works also in the following case:

LEMMA 14'. *In the above lemma assume the following changes:*

First, in (ii) *replace* $X^{(i)}\to X_0^{(i)}$ *by two limits*

$$X^{(i)}\to X_0^{(i)}\,,\qquad \textit{or}\qquad X^{(i)}\to X_{0^*}^{(i)}$$

as

$$\begin{aligned}&(u_i,v_i,u_{i+1},v_{i+1})\to\left(u_i^{(0)},v_i^{(0)},u_{i+1}^{(0)},v_{i+1}^{(0)}\right), &&\textit{or}\\ &(u_i,v_i,u_{i+1},v_{i+1})\to\left(u_i^{(0^*)},v_i^{(0^*)},u_{i+1}^{(0^*)},v_{i+1}^{(0^*)}\right), &&\textit{respectively.}\end{aligned}\tag{17}$$

Second, in (iii) *replace* $X_0^{(i)}+[\pm a]$ *by two half-line neighbourhoods*

$$X_0^{(i)}+[a]\,,\; X_{0^*}^{(i)}+[a^*]\subset\Omega\,.$$

Furthermore, in (iv) *assume that there exist limits*

$$\frac{B_i a}{(u_{i+1}-u_i)(v_{i+1}-v_i)}\to\tilde{a}\,,\qquad \frac{B_i a^*}{(u_{i+1}-u_i)(v_{i+1}-v_i)}\to-\tilde{a}\,,$$

as the variables $(u_i,v_i,u_{i+1},v_{i+1})$ *converge in the way described by* (17)*.*

Finally, in (v) *add the assumption that* $\tilde{b}\times\tilde{c}$ *is not parallel to the* X_1*-axis.*

Then the modified assumptions (i) – (v) *again imply* $X \in \operatorname{int}\Omega$.

P r o o f. Since we can find a spherical neighbourhood of X applying the convexity assertion (Lemma 9) to the half-spherical neighbourhoods of X_0 and

$$X_{0^*} = a_i + B_i X_{0^*}^{(i)} + a_j + B_j X_0^{(j)} + a_k + B_k X_0^{(k)} + \sum_{\substack{n=0 \\ n \neq i,j,k}}^{N} a_n + B_n X^{(n)},$$

which are constructed as the previous spherical neighbourhood of X_0 (repeating the arguments used in the proof), we need finally to add in (v) the assumption about parallelity of $\tilde{b} \times \tilde{c}$. □

We will use results of the present paragraph to establish certain necessary conditions for $\mathcal{F}(g) \in \partial\Omega$.

§5. Criteria.

Let $g\colon [0,1] \to [0,1]$ be a given distribution function. Define[12] the following four sets depending on the mapping g:

$A =$ the set of points $v^{(1)}$ in which g has a one-side derivative with $0 < g'(v^{(1)}) < +\infty$;

$B =$ the set of points $v^{(2)}$ in which g has a jump discontinuity and $0 < v^{(2)} < 1$;

$C =$ the set of constancy intervals $(v^{(3)}, v^{(4)})$ of g in which g has a value with $0 < g < 1$;

$D =$ the set of continuity points $v^{(5)}$ of g in which g has Dini derivatives such that $D^+ g = D^- g = +\infty$, and $D_+ g = D_- g = 0$.

LEMMA 15. *If $\mathcal{F}(g) \in \partial\Omega$, there exists a straight line passing through the following set of points*

$$\Big\{\big(v^{(1)}, g(v^{(1)})\big);\ v^{(1)} \in A\Big\} \cup \left\{\left(v^{(2)}, \frac{g\big(v^{(2)}+0\big) + g\big(v^{(2)}\big)}{2}\right);\ v^{(2)} \in B\right\}$$
$$\cup \left\{\left(\frac{v^{(3)}+v^{(4)}}{2}, g\left(\frac{v^{(3)}+v^{(4)}}{2}\right)\right);\ (v^{(3)}, v^{(4)}) \in C\right\}$$
$$\cup \Big\{\big(v^{(5)}, g(v^{(5)})\big);\ v^{(5)} \in D\Big\}.$$

P r o o f. Consider an arbitrary finite set of elements from $A \dots D$. We shall derive, first of all, that $\mathcal{F}(g) = X$ can be decomposed as $X = \sum_{i=0}^{N} a_i + B_i X^{(i)}$

[12] Elements belonging to the defined sets will be denoted by fixed letters.

(setting as usual $X^{(i)} = \mathcal{F}(g_i)$, where g_i is the linear expansion of $g\big|_{(v_i, v_{i+1})}$ given by (9)), where $X^{(i)} \to X_0^{(i)}$ and

$$X_0^{(i)} \in \left\{(0,0,0), \left(1, \tfrac{1}{2}, 1\right), \left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}\right), \left(1-v, \tfrac{1}{2}(1-v^2), 1-v\right), \left(u, \tfrac{1}{2}u, u^2\right)\right\}.$$
$$(u, v \in (0,1)),$$

for such i which correspond to the choice of elements from $A \dots D$. Further, with the help of neighbourhoods of these limiting vectors (Lemma 7), the corresponding neighbourhood of X can be immediately (Lemmas 14, 14') found. We shall complete the proof using the assumption $X \in \partial\Omega$.

To do this, let us discuss the following four cases.

a) Let g be a distribution function that has a non-zero finite left derivative at $v^{(1)} \in (0,1]$. Suppose that the independent variable point (v_i, u_i) tends to fixed $(v_{i+1}, u_{i+1}) = \left(v^{(1)}, g(v^{(1)})\right)$. Using these assumptions, the following limit can be established

$$X^{(i)} \to \left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

Here we shall only show the case $X_3^{(i)} \to \frac{1}{3}$; the cases $X_1^{(i)} \to \frac{1}{2}$ and $X_2^{(i)} \to \frac{1}{3}$ are completely similar.

Indeed, we can write

$$g(x) = g(v_{i+1}) + g'(v_{i+1})(x - v_{i+1}) + \omega(x, v_{i+1})(x - v_{i+1})$$

for all $x \in [v, v_{i+1}]$, where $\omega(x, v_{i+1}) \to 0$ as $x \to v_{i+1}$. Substituting that into the formula (10), after some arrangements, we find

$$\begin{aligned}
X_3^{(i)} = \frac{1}{\left(\dfrac{g(v_{i+1}) - g(v_i)}{v_{i+1} - v_i}\right)^2 (v_{i+1} - v_i)^3} & \Big\{ \big(g(v_{i+1}) - g(v_i)\big)^2 (v_{i+1} - v_i) \\
& + g'(v_{i+1})^2 \frac{1}{3}(v_{i+1} - v_i)^3 + \omega_1(v_i, v_{i+1}) \frac{1}{3}(v_{i+1} - v_i)^3 \\
& + 2g'(v_{i+1})\omega_2(v_i, v_{i+1}) \frac{1}{3}(v_{i+1} - v_i)^3 \\
& - 2\big(g(v_{i+1}) - g(v_i)\big) g'(v_{i+1}) \frac{1}{2}(v_{i+1} - v_i)^2 \\
& + 2\big(g(v_{i+1}) - g(v_i)\big)\omega_3(v_i, v_{i+1}) \frac{1}{2}(v_{i+1} - v_i)^2 \Big\}.
\end{aligned}$$

where

$$\omega_1(v_i, v_{i+1})\frac{1}{3}(v_{i+1}-v_i)^3 = \int\limits_{v_i}^{v_{i+1}} \omega(x, v_{i+1})^2(x-v_{i+1})^2 \, \mathrm{d}x \,,$$

$$\omega_2(v_i, v_{i+1})\frac{1}{3}(v_{i+1}-v_i)^3 = \int\limits_{v_i}^{v_{i+1}} \omega(x, v_{i+1})(x-v_{i+1})^2 \, \mathrm{d}x \,,$$

$$\omega_3(v_i, v_{i+1})\frac{1}{3}(v_{i+1}-v_i)^2 = \int\limits_{v_i}^{v_{i+1}} \omega(x, v_{i+1})(x-v_{i+1}) \, \mathrm{d}x \,,$$

and, since $\omega_1, \omega_2, \omega_3 \to 0$ as $v_i \to v_{i+1}$, we finally obtain

$$X_3^{(i)} \to \frac{1}{g'(v_{i+1})^2}\left\{g'(v_{i+1})^2 + \frac{1}{3}g'(v_{i+1})^2 - g'(v_{i+1})^2\right\} = \frac{1}{3} \,.$$

b) Let us consider the case that g has a jump in $v^{(2)} \in (0,1)$. We choose variable points (v_j, u_j), (v_{j+1}, u_{j+1}) such that $v_j < v^{(2)} < v_{j+1}$, $u_j = g(v_j)$, $u_{j+1} = g(v_{j+1})$, $v_j, v_{j+1} \to v^{(2)}$, and

$$\frac{v^{(2)} - v_j}{v_{j+1} - v_j} = v \,,$$

where $v \in (0,1)$ is an arbitrary constant. Then, if $(u_j, v_j, u_{j+1}, v_{j+1})$ runs through these variables, we have

$$X^{(j)} \to \left(1 - v, \tfrac{1}{2}(1-v^2), 1 - v\right).$$

c) Begin with the case when $\left(v^{(3)}, v^{(4)}\right)$ is any interval of a constant value of g, where $0 < g < 1$. In the same way as we choose the variables $(u_j, v_j, u_{j+1}, v_{j+1})$, now select $(u_k, v_k, u_{k+1}, v_{k+1})$ such that $v_k \le v^{(3)} < v^{(4)} \le v_{k+1}$, $u_k, u_{k+1} \to g$, and

$$\frac{g - u_k}{u_{k+1} - u_k} = u \,.$$

Then

$$X^{(k)} \to \left(u, \frac{1}{2}u, u^2\right).$$

For a later application of Lemma 14 in this proof, we note that for neighbouring $v^{(2)}$ and $(v^{(3)}, v^{(4)})$ the corresponding variables coincide. In this case we take $(v_{j+1}, u_{j+1}) = (v_k, u_k) \to (v^{(2)}, g)$, preferably. Then the difference $X - X_0$ in the proof of Lemma 14 can be expressed as

$$X - X_0 = B_i\left(X^{(i)} - X_0^{(i)}\right) + B_j\left(X^{(j)} - X_0^{(j)}\right) + B_k\left(X^{(k)} - X_0^{(k)}\right)$$
$$+ \left(\int_{v^{(2)}}^{v_k} (g - u_k)\,\mathrm{d}x,\ \int_{v^{(2)}}^{v_k} x(g - u_k)\,\mathrm{d}x,\ \int_{v^{(2)}}^{v_k} (g - u_k)^2\,\mathrm{d}x \right),$$

and the final vector on the right-hand side is $O(t^2)$. Thus again the limit in (16') $(X - X_0)/t \to 0$ as $t \to 0$.

d) Let us suppose that the distribution function g has the Dini derivatives $D^+g = D^-g = +\infty$ and $D_+g = D_-g = 0$ at a continuity point $v^{(5)} \in (0,1)$. Having in mind the geometrical interpretation of the Dini derivatives, by selecting two suitable sequences of variable vectors $(u_s, v_s, u_{s+1}, v_{s+1})$ one can guarantee the existence of the limits

$$X^{(s)} \to (0,0,0) \quad \text{and} \quad X^{(s)} \to \left(1, \tfrac{1}{2}, 1\right).$$

These cases are obtained in the limit when all of the sides of the rectangles shown in Fig. 6 suitably tend to zero.

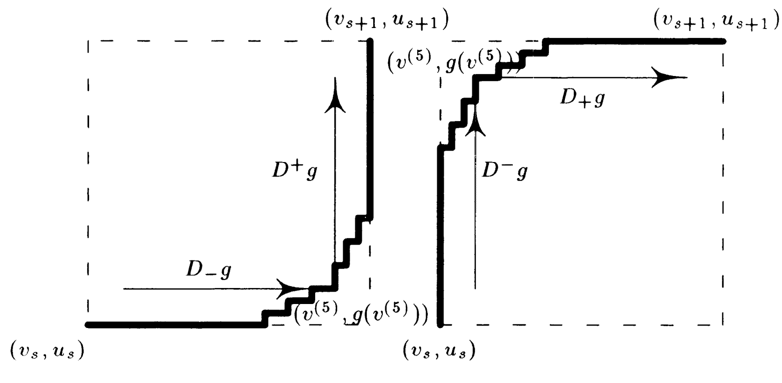


Figure 6.

For our further aims, to apply Lemmas 14 and 14', we need to establish the limits in (iv). To do this, we have already found in Lemma 7 the neighbourhoods of all limiting vectors from a)–d). When $(u_i, v_i, u_{i+1}, v_{i+1}), \dots$

$\dots,(u_s,v_s,u_{s+1},v_{s+1})$ run through the variables in a) – d), we verify that

$$\frac{B_i(3,1,2)}{(u_{i+1}-u_i)(v_{i+1}-v_i)} \to \left(3,\ 3v^{(1)},\ 6g(v^{(1)})\right),$$
$$\frac{B_j(1,v,1)}{(u_{j+1}-u_j)(v_{j+1}-v_j)} \to \left(1,\ v^{(2)},\ g(v^{(2)}+0)+g(v^{(2)})\right),$$
$$\frac{B_k(1,\frac{1}{2},2u)}{(u_{k+1}-u_k)(v_{k+1}-v_k)} \to \left(1,\ \frac{v^{(3)}+v^{(4)}}{2},\ 2g\left(\frac{v^{(3)}+v^{(4)}}{2}\right)\right),$$
$$\frac{B_s(1,1,1)}{(u_{s+1}-u_s)(v_{s+1}-v_s)} \to \left(1,\ v^{(5)},\ 2g(v^{(5)})\right),$$
$$\frac{B_s(-1,0,-2)}{(u_{s+1}-u_s)(v_{s+1}-v_s)} \to \left(-1,\ -v^{(5)},\ -2g(v^{(5)})\right).$$

Finally, the corresponding vector product of limit vectors has the co-ordinate X_3 different from zero for all interesting cases. Thus the additional parallelity assumption in (v) of Lemma 14' is valid.

As a consequence of these lemmas, using $\mathcal{F}(g)\in\partial\Omega$, we easily obtain that all the limiting vectors are co-planar and we have therefore shown Lemma 15. In conclusion, it should only be noted that in cases a) – d) we can choose an arbitrary finite number of elements from $A\dots D$, respectively, and at these there must exist independently specified variables $(u_n,v_n,u_{n+1},v_{n+1})$.[13] □

For the sake of more clarity, we shall give a geometrical illustration of this result in Fig. 7.

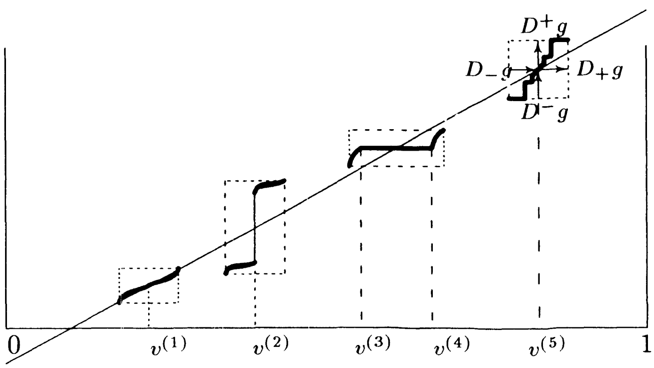


Figure 7.

[13] This assumption may be unrealizable if $v^{(2)}$ and $(v^{(3)},v^{(4)})$ are neighbouring, and therefore we use (20') or the note in c).

Lemma 15 implies the following restriction upon the nature of those distribution functions which can be $\mathcal{F}$-mapped on the surface $\partial\Omega$.

LEMMA 16. *In order that the distribution function* $g\colon [0,1] \to [0,1]$ *should satisfy* $\mathcal{F}(g) \in \partial\Omega$, *it is necessary that either*

(i) g *is continuous in* $(0,1)$ *and the part of its graph included in the open square* $(0,1)^2$ *takes the form of a line-segment in* $(0,1)^2$ (see a list of these graphs in Fig. 2). *Express it as* $y = ax + b$. *If* $a \neq 0$, *then* $\mathcal{F}(g)$ *possesses a half-spherical neighbourhood having the normal vector* $\left(b, a, -\frac{1}{2}\right)$;

or

(ii) g *is a step-function such that all midpoints of its jumps and intervals of constancy from the open square* $(0,1)^2$ *lie on a common straight line. Write it as* $y = ax + b$ *and suppose that at least one step*[14] *of the graph of* g *lies in the open square* $(0,1)^2$. *Then a half-spherical neighbourhood of* $\mathcal{F}(g)$ *can be found, with the normal vector* $\left(-b, -a, \frac{1}{2}\right)$.

Proof. Let $\mathcal{F}(g) \in \partial\Omega$. We adopt the notations of Lemma 15 and assume first that g has a jump in $v_0^{(2)} \in (0,1)$. Claim: It is not possible to choose a sequence of points of type $v^{(1)}$ so that $v^{(1)} \to v_0^{(2)}$. Suppose, on the contrary, that

$$\left(v^{(1)}, g(v^{(1)})\right) \to \left(v_0^{(2)}, g(v_0^{(2)})\right),$$

or

$$\left(v^{(1)}, g(v^{(1)})\right) \to \left(v_0^{(2)}, g(v_0^{(2)} + 0)\right).$$

Since, by Lemma 15, all of these $(v^{(1)}, g(v^{(1)}))$ and $\left(v_0^{(2)}, \dfrac{g(v_0^{(2)}+0) + g(v_0^{(2)})}{2}\right)$ must be lying on a fixed straight-line, the only possibility is that

$$\left(v^{(1)}, g(v^{(1)})\right) \to \left(v_0^{(2)}, \frac{g(v_0^{(2)}+0) + g(v_0^{(2)})}{2}\right).$$

This is impossible, and the claim is proved.

[14] A step of g consists of a vertical segment associated to a jump of g and the neighbouring interval of constancy of g.

A similar analysis can be done for sequences of $v^{(2)}$, $\left(v^{(3)}, v^{(4)}\right)$ and $v^{(5)}$ tending to $v_0^{(2)}$. Thus, for a given $v_0^{(2)}$, there exists a suitable ε so that g has, on $\left(v_0^{(2)} - \varepsilon, v_0^{(2)}\right) \cup \left(v_0^{(2)}, v_0^{(2)} + \varepsilon\right)$, the following properties: g is continuous; g has derivative zero at all points of differentiability; g cannot have a point of type $v^{(5)}$ and an interval $\left(v^{(3)}, v^{(4)}\right)$. Since (see G a r g [2]) *for any continuous strictly increasing function f which has a zero-derivative almost everywhere* [15] *there exists a residual set of points with $D^+f = D^-f = +\infty$, $D_+f = D_-f = 0$*, then g is a constant function on $\left(v_0^{(2)} - \varepsilon, v_0^{(2)}\right)$ and $\left(v_0^{(2)}, v_0^{(2)} + \varepsilon\right)$.

Along the same lines, it can be shown that if g has an interval $\left(v^{(3)}, v^{(4)}\right)$ of constant value of g, $0 < g < 1$, the boundary points $v^{(3)}$ and $v^{(4)}$ are the jump-points of g. In an alternative proof one uses the transformation Λ from Lemma 1.

Collecting all these results, we obtain that whenever the function g has at most one point of type $v^{(2)}$ or an interval of type $\left(v^{(3)}, v^{(4)}\right)$, then g is a step-function.

To complete the possible cases, let us assume that g is continuous in $(0, 1)$, but has no element of type $\left(v^{(3)}, v^{(4)}\right)$. Then the unit interval can be split into three sub-intervals, say $[0, c]$, (c, d), $[d, 1]$, so that g takes the value 0 in $[0, c]$, 1 in $[d, 1]$, and is strictly increasing in (c, d). If in addition to this, the set of points $v^{(1)}$ is dense in (c, d), then the set of points $\left(v^{(1)}, g(v^{(1)})\right)$ is also dense in the graph of $g\big|_{(c,\,d)}$, and according to Lemma 15, the graph is a line-segment. If g cannot have $v^{(1)}$-point in a sub-interval $(e, f) \subset (c, d)$, then the restriction $g\big|_{(e,\,f)}$ of g is singular. Making use of G a r g 's (above mentioned) theorem, we may argue that the set of points $v^{(5)}$ is dense in (e, f). But then, since g is continuous, the set of points $\left(v^{(5)}, g(v^{(5)})\right)$ is also dense in the graph of $g\big|_{(e,\,f)}$. Using here Lemma 15, one derives that this graph forms a line-segment. This is impossible because of the existence of $v^{(5)}$.

Let us proceed to find an expression for the half-spherical neighbourhoods of points $\mathcal{F}(g)$ of g specified by (i) and (ii). We start with a decomposition $\mathcal{F}(g) = \sum_{i=0}^{N} a_i + B_i X^{(i)}$ constructed as in the proof of Lemma 5.

In case (i) we may assume that (v_i, u_i) and (v_{i+1}, u_{i+1}) are chosen from the straight-line $y = ax + b$ with suitable i. Then $X^{(i)} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$, and we begin with the construction of that half-spherical neighbourhood

$$\left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}\right) + \left[\pm(3, 1, 2), \pm(3, 2, 4), (0, -1, 0)\right]. \tag{18}$$

[15] such function is said to be *singular*.

There is a point $\left(\frac{1}{2}, \frac{5}{16}, \frac{1}{3}\right)$ lying under $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$. Since it may be put in the form

$$\left(\tfrac{1}{2}, \tfrac{5}{16}, \tfrac{1}{3}\right) = X\left(\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{3}{4}\right),$$

by virtue of the expression (14) for $X(u_1, v_1, u_2, v_2)$ and since

$$\frac{\partial X}{\partial v_2} = \left(-\tfrac{2}{3}, -\tfrac{1}{2}, -\tfrac{8}{9}\right), \qquad \frac{\partial X}{\partial u_2} = \left(\tfrac{3}{8}, \tfrac{3}{16}, \tfrac{1}{14}\right),$$

we find that $\left(\frac{1}{2}, \frac{5}{16}, \frac{1}{3}\right)$ has a planar neighbourhood in Ω. Summing up the result and using convexity, provided by Lemma 2, under the lines parallel to the X_2-axis, we shall then complete our proof of (18).

Now the decomposition of $\mathcal{F}(g)$ and Lemma 8 imply

$$\mathcal{F}(g) + \left[\pm B_i(3,1,2),\ \pm B_i(3,2,4),\ B_i(0,-1,0)\right] \subset \Omega. \tag{19}$$

One easily sees that if (c,d,e) is a normal vector to (18) (compare the definition over Lemma 11), then $(c,d,e)B_i^{-1}$ is a normal vector to (19). Since (18) has $\left(0,1,-\frac{1}{2}\right)$ as its normal vector, and

$$\begin{aligned}\left(0,1,-\tfrac{1}{2}\right)B_i^{-1} &= \frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}\left(u_i - v_i\frac{u_{i+1}-u_i}{v_{i+1}-v_i},\ \frac{u_{i+1}-u_i}{v_{i+1}-v_i},\ -\frac{1}{2}\right)\\ &= \frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}\left(b, a, -\tfrac{1}{2}\right),\end{aligned}$$

then the half-spherical neighbourhood (19) of the image $\mathcal{F}(g)$ has a normal vector $\left(b, a, -\frac{1}{2}\right)$ as asserted.

Using the same procedure from the above construction, we can now find a half-spherical neighbourhood of $\mathcal{F}(g)$, where g is specified by (ii). Assume that there are neighbouring $v^{(2)}$ and $\left(v^{(3)}, v^{(4)}\right)$. We distinguish two cases, depending on whether $v^{(2)} = v^{(3)}$ or $v^{(2)} = v^{(4)}$. Observe that the transformation Λ transfers immediately the case $v^{(2)} = v^{(4)}$ to $v^{(2)} = v^{(3)}$, so that it remains to consider the case $v^{(2)} = v^{(3)}$. In this case $\mathcal{F}(g)$ can be decomposed into the sum $\mathcal{F}(g) = \sum_{i=0}^{N} a_i + B_i X^{(i)}$, where some $X^{(i)}$ may be put in the form $X^{(i)} = \mathcal{F}(\tilde{g})$ for a one-step function $\tilde{g}$. We shall use the representation for $\tilde{g}$ given in (1), where $\tilde{g} = g(u_1, v_1, u_2, v_2)$, $0 < u_1 = u_2 < 1$, $0 < v_1 < 1$, $v_2 = 1$, and we again use the expression (14) that gives $X^{(i)} = X(u_1, v_1, u_2, v_2)$. Now let $y = ax + b$ be the line joining the points $\left(v^{(2)}, \dfrac{g(v^{(2)}+0) + g(v^{(2)})}{2}\right)$.

$\left(\frac{v^{(3)}+v^{(4)}}{2}, g\left(\frac{v^{(3)}+v^{(4)}}{2}\right)\right)$, and $y = \tilde{a}x + \tilde{b}$ joining $\left(v_1, \frac{u_1}{2}\right)$ and $\left(\frac{v_1+1}{2}, u_1\right)$. Graphically, see Fig. 8.

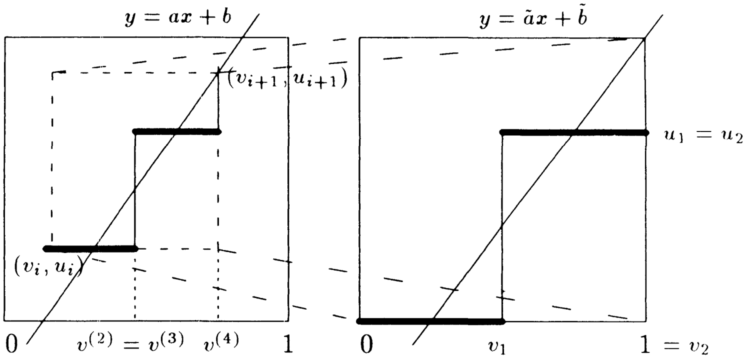


Figure 8.

Now we can show that the partial derivatives of X with respect to the parameters v_1, v_2, u_1, and u_2 are

$$\frac{\partial X}{\partial v_1} = -u_1(1, v_1, u_1), \qquad \frac{\partial X}{\partial v_2} = (-1+u_1)(1, 1, 1+u_1),$$
$$\frac{\partial X}{\partial u_1} = (1-v_1)\left(\tfrac{1}{2}, \tfrac{1}{6}(1+2v_1), u_1\right), \qquad \frac{\partial X}{\partial u_2} = (1-v_1)\left(\tfrac{1}{2}, (2+v_1), u_1\right).$$

Moreover, the increments $\mathrm{d}v_1$, $\mathrm{d}v_2$, $\mathrm{d}u_1$, and $\mathrm{d}u_2$ must satisfy $\mathrm{d}v_2 \le 0$ and $\mathrm{d}u_1 \le \mathrm{d}u_2$, and, in addition, the differential of X becomes

$$\mathrm{d}X = \frac{\partial X}{\partial v_1}\mathrm{d}v_1 + \frac{\partial X}{\partial v_2}\mathrm{d}v_2 + \left(\frac{\partial X}{\partial u_1} + \frac{\partial X}{\partial u_2}\right)\left(\frac{\mathrm{d}u_2 + \mathrm{d}u_1}{2}\right)$$
$$+ \left(\frac{\partial X}{\partial u_2} - \frac{\partial X}{\partial u_1}\right)\left(\frac{\mathrm{d}u_2 - \mathrm{d}u_1}{2}\right).$$

So we arrive at

$$\mathcal{F}(\tilde{g}) + \left[\pm(1, v_1, u_1), \pm\left(1, \frac{1+v_1}{2}, 2u_1\right), \left(0, \frac{1-v_1}{6}, 0\right), (1, 1, 1+u_1)\right] \subset \Omega. \tag{20}$$

Now we make use of the inner products

$$(-\tilde{b}, -\tilde{a}, \tfrac{1}{2})(1, v_1, u_1) = 0, \qquad (-\tilde{b}, -\tilde{a}, \tfrac{1}{2})\left(1, \frac{1+v_1}{2}, 2u_1\right) = 0,$$
$$(-\tilde{b}, -\tilde{a}, \tfrac{1}{2})\left(0, \frac{1-v_1}{6}, 0\right) < 0, \qquad (-\tilde{b}, -\tilde{a}, \tfrac{1}{2})(1, 1, 1+u_1) = \frac{1}{2} - u_1,$$

and conclude that (20) will be either a half-spherical neighbourhood of $\mathcal{F}(\tilde{g})$ with normal vector $\left(-\tilde{b},-\tilde{a},\frac{1}{2}\right)$ if $\frac{1}{2}\le u_1$, or a spherical neighbourhood for $u_1<\frac{1}{2}$.

But again the decomposition of $\mathcal{F}(g)$, Lemma 8, and

$$
\begin{aligned}
&\left(-\tilde{b},-\tilde{a},\tfrac{1}{2}\right)B_i^{-1}\\
&=\left(-\tilde{b}(u_{i+1}-u_i)+\tilde{a}v_i\frac{u_{i+1}-u_i}{v_{i+1}-v_i}-u_i,\ -\tilde{a}\frac{u_{i+1}-u_i}{v_{i+1}-v_i},\ \frac{1}{2}\right)\frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}\\
&=\left(-b,-a,\tfrac{1}{2}\right)\frac{1}{(u_{i+1}-u_i)^2(v_{i+1}-v_i)}
\end{aligned}
$$

yield that

$$
\mathcal{F}(g)+\left[\pm B_i(1,v_1,u_1),\ \pm B_i\left(1,\frac{1+v_1}{2},2u_1\right),\ B_i\left(0,\frac{1-v_1}{6},0\right)\right]\subset\Omega \qquad (20')
$$

is a half-spherical neighbourhood of $\mathcal{F}(g)$ with the normal vector $\left(-b,-a,\frac{1}{2}\right)$, and the proof is complete. □

Having done all this, we may now conclude the proof of Theorem 2.

§6. Completion of the proof.

Let g run through the set of all one-jump and constant distribution functions with graphs in Fig. 5. Then its $\mathcal{F}$-images, X, form two curve-edges of $\partial\Omega$. That the moment problem $X=\mathcal{F}(g)$ is uniquely solvable in g, we have seen in the proof of Lemma 4. Observe that, for all other g, we have showed a construction of half-spherical or spherical neighbourhood of $\mathcal{F}(g)$ in Ω. This is given by the proof of Lemmas 15 and 16.Therefore, henceforth we shall require that g should not be a one-jump or a constant distribution function.

For brevity, we introduce an *upper boundary surface* $\overline{\partial}\Omega$ and a *lower boundary surface* $\underline{\partial}\Omega$ of Ω with respect to the plane $X_1\times X_3$:

$$
\begin{aligned}
\overline{\partial}\Omega&=\left\{(X_1,\sup X_2,X_3);\ X\in\Omega\right\},\\
\underline{\partial}\Omega&=\left\{(X_1,\inf X_2,X_3);\ X\in\Omega\right\}.
\end{aligned}
$$

Here sup and inf are taken over all $X\in\Omega$ for which X_1 and X_3 are fixed.

The next two lemmas are needed to prove that the upper boundary surface of Ω coincides with $\mathcal{F}$-images of g from Lemma 16, part (i), i.e. of types from Fig. 2, and for a construction of the lower boundary surface we need a reduction of g from part (ii) of Lemma 16 to one-step functions in Fig. 3. The finiteness of the number of solutions $X=\mathcal{F}(g)$ in distribution functions g is also established, for $X\in\partial\Omega$. It should be noted that our argument is very strongly based on Lemma 11.

LEMMA 17. *We have*

$$\overline{\partial}\Omega = \bigcup_{1\le i\le 4} \Pi_i \,.$$

P r o o f . Suppose that g satisfies condition (i) and $\tilde{g}$ condition (ii) from Lemma 16. For the points $X = \mathcal{F}(g)$ and $\tilde{X} = \mathcal{F}(\tilde{g})$ we have, according to this lemma, two half-spherical neighbourhoods in Ω, with normal vectors $\left(b, a, -\frac{1}{2}\right)$ and $\left(-\tilde{b}, -\tilde{a}, \frac{1}{2}\right)$, respectively. Now we state the result: *If* $X, \tilde{X} \in \partial\Omega$ *and* $X_1 = \tilde{X}_1$, $X_3 = \tilde{X}_3$, *then* $X_2 > \tilde{X}_2$. Indeed, let us assume, to the contrary, that $X_2 \le \tilde{X}_2$. Then the points X and $\tilde{X}$ (or, in the case $X_2 = \tilde{X}_2$, its small shift) satisfy all the three conditions in Lemma 11, and thus they have spherical neighbourhoods in Ω; this is a contradiction. We thus obtain a separation of $\mathcal{F}$-images of g which are described in (i) and (ii) of Lemma 16 either to the upper and lower surfaces of $\partial\Omega$, respectively, or to the interior of Ω.

Now we use that the functions g which satisfy (i) of Lemma 16 can have only the graphs of types $g^{(1)} \dots g^{(4)}$ from Fig. 2. Then, using notations (2), let $\Pi_1 \dots \Pi_4$ be their $\mathcal{F}$-images in Ω. By virtue of the expression $g(u_1, v_1, u_2, v_2)$ in (1) for $g^{(1)} \dots g^{(4)}$, we obtain a parametric representation of $\Pi_1 \dots \Pi_4$. Eliminating parameters from these equations, we obtain the canonical equation of $\Pi_1 \dots \Pi_4$ as in (4). One sees immediately that any intersection $\Pi_i \cap \Pi_j$ of two different surfaces from $\Pi_1 \dots \Pi_4$ coincide with the intersection of their boundaries. If a point X runs through this common curve, one can also compute the identity $g^{(i)} = g^{(j)}$ at the equation $X = \mathcal{F}\big(g^{(i)}\big) = \mathcal{F}\big(g^{(j)}\big)$, $(1 \le i, j \le 4)$. In an alternative proof, one uses the normal vectors $\left(b_i, a_i, -\frac{1}{2}\right)$ and $\left(b_j, a_j, -\frac{1}{2}\right)$ to half-spherical neighbourhoods of $\mathcal{F}\big(g^{(i)}\big)$ and $\mathcal{F}\big(g^{(j)}\big)$, respectively, constructed as in the proof of (i) of Lemma 16. Then the direction vector considered in Lemma 10 can be found as the vector product

$$\left(b_i, a_i, -\tfrac{1}{2}\right) \times \left(b_j, a_j, -\tfrac{1}{2}\right).$$

Applying $\mathcal{F}(g_i) = \mathcal{F}(g_j) \in \partial\Omega$ and Lemma 10, we find that the first co-ordinate of the product must be zero. The only possibility is that $a_i = a_j$, and consequently $b_i = b_j$ and $g^{(i)} = g^{(j)}$. □

In fact, *the mapping* $\mathcal{F}$ *specifies a one-to-one correspondence between the functions of Fig. 2 and the points* X *of the upper boundary surface of* Ω.

LEMMA 18. *We have*

$$\underline{\partial}\Omega = \bigcup_{5\le i\le 7} \Pi_i \,.$$

P r o o f . Let us consider one-step functions listed in Fig. 3. Their expression $g(u_1, v_1, u_2, v_2)$ in (4) is determined by two collections of conditions $v_1 = 0$, $0 < u_1 = u_2 < 1$, $0 < v_2 < 1$ or $0 < v_1 < 1$, $0 < u_1 = u_2 < 1$, $v_2 = 1$. Using the $\mathcal{F}$-mapping, we can extend the surfaces Π_5, Π_6 (without boundary) to the following enlarged sets

$$\Pi_8 = \{\mathcal{F}(g);\ v_1 = 0,\ 0 < u_1 = u_2 < 1,\ 0 < v_2 < 1\},$$
$$\Pi_9 = \{\mathcal{F}(g);\ 0 < v_1 < 1,\ 0 < u_1 = u_2 < 1,\ v_2 = 1\},$$

respectively. Taking into account expression (14), we can rewrite parametrically

$$\Pi_8 = \left\{ \begin{pmatrix} 1-v_2+v_2u_2 \\ \frac{1}{2}-\frac{1}{2}v_2^2(1-u_2) \\ 1-v_2+v_2u_2^2 \end{pmatrix};\ 0 < u_2 < 1,\ 0 < v_2 < 1 \right\},$$
$$\Pi_9 = \left\{ \begin{pmatrix} (1-v_1)u_1 \\ \frac{1}{2}-\frac{1}{2}(1-u_1)-\frac{1}{2}v_1^2u_1 \\ (1-v_1)u_1^2 \end{pmatrix};\ 0 < u_1 < 1,\ 0 < v_1 < 1 \right\}.$$

Since the parameters u_2 and v_2 in Π_8 can be eliminated as

$$u_2 = \frac{X_1 - X_3}{1 - X_1}, \qquad v_2 = \frac{(1 - X_1)^2}{1 + X_3 - 2X_1}, \tag{21}$$

the expression for Π_8 takes the form

$$\Pi_8 = \left\{ (X_1, X_2, X_3);\ X_2 = \frac{1}{2} - \frac{1}{2}\frac{(1 - X_1)^3}{1 + X_3 - 2X_1}, \right.$$
$$\left. X_1^2 < X_3 < X_1,\ 0 < X_1 < 1 \right\}.$$

Similarly, in the case Π_9,

$$u_1 = \frac{X_3}{X_1}, \qquad 1 - v_1 = \frac{X_1^2}{X_3}, \tag{22}$$

and we can write

$$\Pi_9 = \left\{ (X_1, X_2, X_3);\ X_2 = X_1 - \frac{1}{2}\frac{X_1^3}{X_3}, \right.$$
$$\left. X_1^2 < X_3 < X_1,\ 0 < X_1 < 1 \right\}.$$

Together with Lemma 3, we obtain that the projections of Π_8 and Π_9 on the $X_1 \times X_3$-plane are the same as the projection of the domain Ω.

Moreover, it is easy to verify that Π_8 and Π_9 intersect in a curve

$$\Pi_7 = \left\{ \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{16X_3}, X_3 \right);\ \frac{1}{4} < X_3 < \frac{1}{2} \right\},$$

where $\frac{1}{2} = X_1$, and we simply deduce that every $X \in \Pi_8$ with $X_1 < \frac{1}{2}$ must lie under Π_9 and every $X \in \Pi_9$ with $X_1 > \frac{1}{2}$ must lie under Π_8, in both cases with respect to the $X_1 \times X_3$-plane.

Recall that every point from $\Pi_8 \cup \Pi_9$ is the $\mathcal{F}$-image of a one-step function and we may apply (ii) of Lemma 16, so that then each one has a half-spherical neighbourhood with a normal vector $\left(-b, -a, \frac{1}{2}\right)$, where the second co-ordinate is negative. But then we make again use of Lemma 11 and conclude that all the points from Π_8 lying above Π_9 and all the points from Π_9 lying above Π_8 must lie in the interior of Ω. We simply write this relation as $\Pi_8^0 \cup \Pi_9^0 \subset \operatorname{int} \Omega$, where

$$\Pi_8^0 = \left\{ \mathcal{F}(g);\ v_1 = 0,\ 0 < u_1 = u_2 < 1,\ 0 < v_2 < 1,\ v_2(1 - u_2) < \frac{1}{2} \right\},$$

$$\Pi_9^0 = \left\{ \mathcal{F}(g);\ 0 < v_1 < 1,\ 0 < u_1 = u_2 < 1,\ v_2 = 1,\ (1 - v_1)u_1 < \frac{1}{2} \right\}.$$

Just note that an $\mathcal{F}$-image of g belonging to (ii) of Lemma 16 and having at least two jumps or two intervals of constancy entirely lying within $[0,1]^2$ can be expressed as $\mathcal{F}(g) = \sum_{i=0}^{N} a_i + B_i X^{(i)}$, where $X^{(i)} \in \Pi_9^0$ for some i. We can visualize this situation as in Fig. 9.

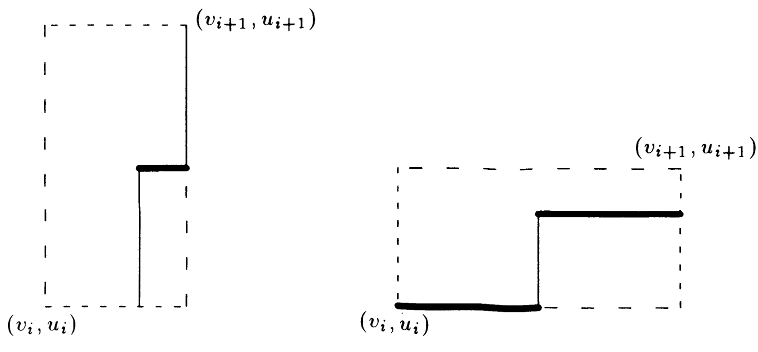


Figure 9.

Now we apply Lemma 12 to get $\mathcal{F}(g) \in \operatorname{int} \Omega$. □

We sum up the above-mentioned results in the following: *Every point of the lower boundary surface of Ω is the $\mathcal{F}$-image of a one-step function $g(u_1, v_1, u_2, v_2)$, where either $v_1 = 0$, $0 < u_1 = u_2 < 1$, $0 < v_2 < 1$ and $\frac{1}{2} \le v_2(1-u_2)$, or $0 < v_1 < 1$, $0 < u_1 = u_2 < 1$, $v_2 = 1$ and $\frac{1}{2} \le (1-v_1)u_1$.*

To prove the theorem, it remains to establish the finiteness of the number of solutions of $X = \mathcal{F}(g)$, where X is a point of the lower boundary surface of Ω. In the case $X_1 \ne \frac{1}{2}$ such a solution g is the only one because the parameters u_1, v_1, u_2, v_2 in $g(u_1, v_1, u_2, v_2)$ are uniquely specified by X_1, X_3 and (21) if $X_1 < \frac{1}{2}$, and by (22) if $X_1 > \frac{1}{2}$. The exceptional case $X_1 = \frac{1}{2}$ corresponds to exactly two solutions of (21) and (22) which form $g^{(7)}$ and $g^{(7^*)}$ in our theorem. It should be noted that the argument in Lemma 10 does not work in the case where $X \in \Pi_7$. To see this, using the representations $X = \mathcal{F}(g^{(7)})$ and $X = \mathcal{F}(g^{(7^*)})$, and applying part (i) of Lemma 16, we can find two half-spherical neighbourhoods of X in Ω with normal vectors

$$\left(3X_3 - 1,\ -8X_3^2,\ \frac{1}{2}\right), \qquad \left(8X_3^2 - 3X_3,\ -8X_3^2,\ \frac{1}{2}\right).$$

Then their vector product has a zero co-ordinate in the first place, and so condition (ii) in Lemma 10 is not satisfied. Thus Theorem 2 is proved. □

5. Concluding remarks

The neighbourhood-constructing technique described above can be used for solutions of a moment problem in an arbitrary dimension. But in our 3-dimensional case, I. Korec has informed the author about a simplified construction of the upper boundary surface of Ω, which can be established without the technical apparatus of neighbourhoods. We shall demonstrate now briefly the method.

To determine the distribution function $g\colon [0,1] \to [0,1]$ at which a conditional maximum can be attained by the integral $\int_0^1 x g(x)\,\mathrm{d}x$ under the condition $\int_0^1 g(x)\,\mathrm{d}x = X_1$ and $\int_0^1 g^2(x)\,\mathrm{d}x = X_3$, where X_1 and X_3 are constants, one should form the auxiliary system of linear equations

$$\begin{aligned} \mathrm{d}g(x_1)\,\mathrm{d}x_1 + \mathrm{d}g(x_2)\,\mathrm{d}x_2 + \mathrm{d}g(x_3)\,\mathrm{d}x_3 &= 0\,, \\ \mathrm{d}g(x_1)x_1\,\mathrm{d}x_1 + \mathrm{d}g(x_2)x_2\,\mathrm{d}x_2 + \mathrm{d}g(x_3)x_3\,\mathrm{d}x_3 &= \varepsilon\,, \\ \mathrm{d}g^2(x_1)\,\mathrm{d}x_1 + \mathrm{d}g^2(x_2)\,\mathrm{d}x_2 + \mathrm{d}g^2(x_3)\,\mathrm{d}x_3 &= 0\,. \end{aligned} \tag{23}$$

Here $0 < x_1 < x_2 < x_3 < 1$ are arbitrary three points of continuity of g; $\mathrm{d}g(x_1)$, $\mathrm{d}g(x_2)$, $\mathrm{d}g(x_3)$ are unknown small increments of g; and $\varepsilon > 0$ is sufficiently small. Write

$$D = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 2g(x_1) & 2g(x_2) & 2g(x_3) \end{pmatrix}.$$

a) If $0 < g(x_1) \le g(x_2) \le g(x_3) < 1$, then $\det D = 0$. Indeed, suppose as contrary, that $\det D \neq 0$. Then (23) has a solution $\mathrm{d}g(x_1)$, $\mathrm{d}g(x_2)$, $\mathrm{d}g(x_3)$. Substituting the variations $g(x_1)+\mathrm{d}g(x_1)$, $g(x_2)+\mathrm{d}g(x_2)$, $g(x_3)+\mathrm{d}g(x_3)$ into the graph of g and rearranging them in ascending order, we obtain a distribution function $\tilde{g}$ with $\int_0^1 \tilde{g}(x)\,\mathrm{d}x = X_1$, $\int_0^1 \tilde{g}^2(x)\,\mathrm{d}x = X_3$ and $\int_0^1 x\tilde{g}(x)\,\mathrm{d}x > \int_0^1 xg(x)\,\mathrm{d}x$; this is a contradiction.

b) If $0 = g(x_1) < g(x_2) \le g(x_3) < 1$, the only possibility is that $0 = g(x_1) < g(x_2) < g(x_3) < 1$, because $0 = g(x_1) < g(x_2) = g(x_3) < 1$ gives the solution $\mathrm{d}g(x_1) = 0$; this is a contradiction as in a).

c) Now, assume that $0 = g(x_1) < g(x_2) < g(x_3) < 1$. Let $y = ax + b$ be the line joining the points $\big(x_2, g(x_2)\big)$ and $\big(x_3, g(x_3)\big)$. Then $ax_1 + b \le 0$, because in the opposite case we have $\mathrm{d}g(x_1) > 0$, and again a contradiction as in a).

Summing up the results a) – c), we easily obtain that g cannot have a jump in $(0,1)$, and along the same lines it can be shown that g cannot have an interval of constant value of g, $0 < g < 1$. Thus g *maximizing* $\int_0^1 xg(x)\,\mathrm{d}x$ *and preserving* $\int_0^1 g(x)\,\mathrm{d}x$ *and* $\int_0^1 g^2(x)\,\mathrm{d}x$ *is described in Fig.* 2.

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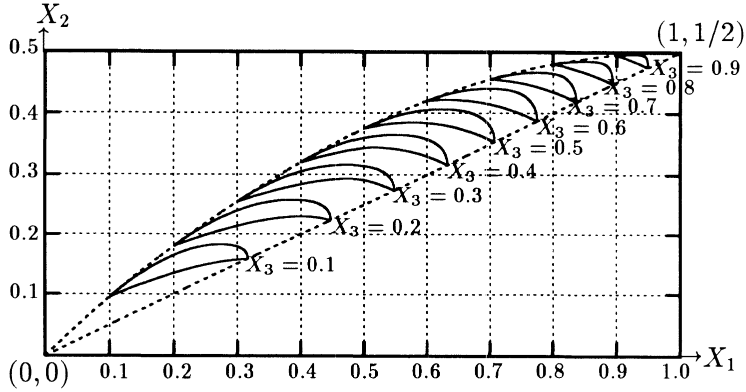

Figure 10. The cuts of Ω by planes perpendicular to X_3-axis.

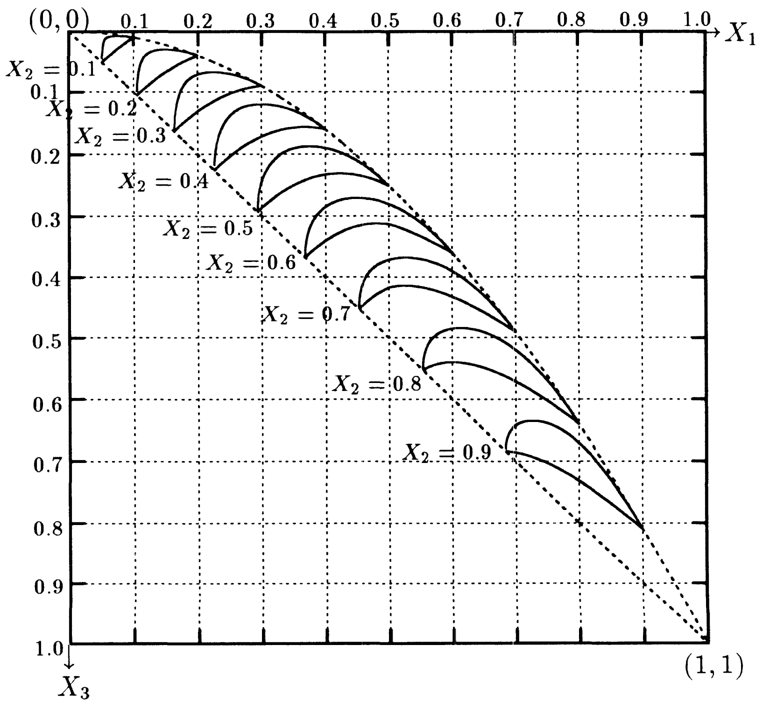

Figure 11. The cuts of Ω by planes perpendicular to X_2-axis.

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Mathematical Institute
Slavak Academy of Sciences
Štefánikova ul. 49
SK-814 73 Bratislava
Slovakia