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Dedicated to Academician Štefan Schwarz
on the occasion of his 80th birthday

IDENTITIES INVOLVING COVERING SYSTEMS I

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(Communicated by Milan Paštéka)

ABSTRACT. It is shown in this paper (following an idea of J. Beebee) how the notion of the covering system of congruences can be used to generalize some identities involving Bernoulli numbers and polynomials.

1. Introduction

In [5], V. Namias used Stirling's series

$$\Gamma(n) = n^{n-1}\,\mathrm{e}^{-n}\,\sqrt{2\pi n}\,\exp\left(\sum_{m=2}^{\infty}\frac{B_m}{m(m-1)n^{m-1}}\right) \tag{1}$$

and the Gauß-Legendre duplication and triplication formula for the gamma function Γ to derive the following two recurrence relations for the Bernoulli numbers B_m:

$$B_m = \frac{1}{2(1-2^m)}\sum_{s=0}^{m-1} 2^s\binom{m}{s}B_s\,,$$

$$B_m = \frac{1}{3(1-3^m)}\sum_{s=0}^{m-1} 3^s(1+2^{m-s})\binom{m}{s}B_s\,.$$

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V. Namias further conjectured that an infinity of related recurrences can be proved for Bernoulli numbers, all giving the same Stirling's series (1).

The first recurrence coincides essentially with identity (S_{III}) proved by J. Stern in [11].

In [2], E. Y. Deeba and D. M. Rodriguez proved Namias' conjecture with the recurrence relations

$$B_m = \frac{1}{n(1-n^m)} \sum_{s=0}^{m-1} n^s \binom{m}{s} B_s \sum_{t=1}^{n-1} t^{m-s} . \tag{2}$$

Their proof is based on an elementary procedure (without any recall to Gauß-Legendre multiplication formula), using the elementary finite geometric series summation

$$\frac{1-\mathrm{e}^{nx}}{1-\mathrm{e}^{x}} = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{j^m x^m}{m!} \tag{3}$$

and the formal power series generating function for Bernoulli numbers.

In [1], J. Beebee observed on one hand that this reasoning can be further simplified, using the Raabe multiplication formula for the Bernoulli polynomials (see (15)). On the other hand, he observed that the partition of the set of non-negative integers into n arithmetic progressions with respect to the same modulus n, which lies behind (2), can be replaced by an arbitrary disjoint covering system of arithmetic progressions. These enabled him to extend further the result of
E. Y. Deeba and D. M. Rodriguez. Moreover, distinct disjoint covering systems formally lead to different recurrence relations. J. Beebee showed that this result can even be reversed in the sense that his relations characterize the underlying system of arithmetical sequences as a disjoint covering system. In this paper, we shall present a further extension of these formulas to systems more general than disjoint covers.

Consider a system of congruence classes

$$a_t \pmod{n_t}, \quad 0 \le a_t < n_t , \qquad t = 1, 2, \dots, k, \quad k > 1 . \tag{4}$$

Let a real valued function $\mu(t) = \mu_t$ be defined on the system (4). Then the function $\mathfrak{m}$ introduced in [7],

$$\mathfrak{m}(n) = \sum_{t=1}^{k} \mu_t \chi_t(n) , \qquad n \in \mathbb{Z},$$

where χ_t is the indicator of the class $a_t \pmod{n_t}$, and $\mathbb{Z}$ the set of all integers, is the so-called *covering function* of the system (4), and the system (4) is called $(\mu, \mathfrak{m})$-*covering*. The function $\mathfrak{m}$ is periodic and its (least nonnegative) period, say n_0, divides the l.c.m. $[n_1, n_2, \dots, n_k]$. In what follows n_0 will always denote the (least nonnegative) period of the covering function $\mathfrak{m}$ of the corresponding $(\mu, \mathfrak{m})$-covering system (4).

The notion of the $(\mu, \mathfrak{m})$-covering system includes some previously investigated notions. For instance, if $\mathfrak{m}(n) = 1$ for every $n \in \mathbb{Z}$, then the $(\mu, 1)$-covering systems are just the ε-covering systems from [12]. Further, if the function μ is constant and equal to 1, and if $\mathfrak{m}(n) \geq 1$ for every $n \in \mathbb{Z}$, then we obtain the so called *covering systems*. The notion of covering system was introduced by P. Erdős in the thirties as a tool in the disproof of a question of Romanoff from additive number theory (for more details see [3]).

One of the simplest nontrivial covering system is

$$0 \pmod 2, \quad 0 \pmod 3, \quad 1 \pmod 6, \quad 5 \pmod 6. \tag{5}$$

Its covering function

$$\mathfrak{m}(n) = \begin{cases} 1 & \text{if } n = 1, 2, 3, 4, 5, \\ 2 & \text{if } n = 0 \end{cases}$$

has period $n_0 = 6$.

If the covering function of a covering system (4) is constant, say $\mathfrak{m}(n) = m$ for every $n \in \mathbb{Z}$, $\mu(t) = 1$,then the system is called an m *times covering* ([8]). Prototypes of m times covering systems are *disjoint covering systems*, which correspond to the case $m = 1$. The system

$$0 \pmod n, \quad 1 \pmod n, \quad \dots, \quad n-1 \pmod n \tag{6}$$

is the most trivial example of a disjoint covering system. One of the surprising results involving disjoint covering systems says that there is no disjoint covering system with all the moduli distinct. One wide class of disjoint covering system, the so called *natural covering systems* ([6]), can be obtained by successive splitting of the set of all the integers into arithmetic progression, e.g.

$$\begin{array}{llll} 1 \pmod 2, & 2 \pmod 4, & \dots, & 2^{f-1} \pmod{2^f}, \\ 0 \pmod{n2^f}, & 2^f \pmod{n2^f}, & \dots, & (n-1)2^f \pmod{n2^f} \end{array} \tag{7}$$

for arbitrary positive integers f, n.

Obviously, m disjoint covering systems taken together form an m times covering system. Our question whether there is an m times covering system which is not a union of m disjoint covering systems was answered by S. L. G. Choi, who essentially constructed the following twice covering system ([8]) which is not a union of two disjoint covering systems:

$$\begin{array}{lllll} 1 \pmod{2}, & 0 \pmod{3}, & 1 \pmod{3}, & 2 \pmod{6}, & 0 \pmod{10}, \\ 4 \pmod{10}, & 6 \pmod{10}, & 8 \pmod{10}, & 2 \pmod{15}, & 5 \pmod{30}, \\ 11 \pmod{30}, & 12 \pmod{30}, & 22 \pmod{30}, & 23 \pmod{30}, & 29 \pmod{30}. \end{array} \tag{8}$$

The reader may consult [9] for more information about covering systems and related notions.

2. Identities

We shall employ the following generating series for the Bernoulli polynomials $B_r(x)$

$$\frac{z\,\mathrm{e}^{xz}}{\mathrm{e}^z - 1} = \sum_{r=0}^{\infty} B_r(x) \frac{z^r}{r!}, \qquad |z| < 2\pi, \tag{9}$$

and for the Bernoulli numbers

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{r=0}^{\infty} B_r \frac{z^r}{r!}, \qquad z < 2\pi.$$

These expansions immediately imply that $B_r = B_r(0)$ for every $r \in \mathbb{Z}^*$, where $\mathbb{Z}^*$ is the set of nonnegative integers, and the identity

$$\sum_{r=0}^{\infty} B_r(x) \frac{z^r}{r!} = \mathrm{e}^{xz} \sum_{r=0}^{\infty} B_r \frac{z^r}{r!}$$

implies the explicit formula

$$B_r(x) = \sum_{t=0}^{r} \binom{r}{t} x^{r-t} B_t. \tag{10}$$

In 1973, A. S. Fraenkel [4] proved that (4) is a disjoint covering systems if and only if

$$B_r = \sum_{t=0}^{k} n_t^{r-1} B_r\Big(\frac{a_t}{n_t}\Big) \tag{11}$$

for every $r \in \mathbb{Z}^*$. In [7], this relation was extended to general systems of congruences, namely:

LEMMA 1. *The following statements are equivalent:*

A. *The system* (4) *is* $(\mu, \mathfrak{m})$*-covering.*
B. *For every* $r \in \mathbb{Z}^*$ *we have*

$$n_0^{r-1} \sum_{n=0}^{n_0-1} \mathfrak{m}(n) B_r\Big(\frac{n}{n_0}\Big) = \sum_{t=1}^{k} \mu_t n_t^{r-1} B_r\Big(\frac{a_t}{n_t}\Big). \tag{12}$$

For m times covering systems the left hand side has a more simple form ([8]):

LEMMA 2. *The system* (4) *is* m *times covering* ($m \in \mathbb{Z}$, $m \geq 1$) *if and only if*

$$mB_r = \sum_{t=1}^{k} n_t^{r-1} B_r\Big(\frac{a_t}{n_t}\Big) \tag{13}$$

for every $r \in \mathbb{Z}^*$.

Since $B_0(x)$ is identically equal to 1, we immediately have ([7]):

LEMMA 3. *If a system* (4) *is a* $(\mu, \mathfrak{m})$*-covering, then*

$$\sum_{i=1}^{k} \frac{\mu_i}{n_i} = \frac{1}{n_0} \sum_{n=0}^{n_0-1} \mathfrak{m}(n). \tag{14}$$

J. Beebee observed that Fraenkel's identity (11), when applied to the disjoint covering system (6), reduces to a special case ($x = 0$) of the Raabe multiplication formula [10; pp. 23–28]:

$$B_r(x) = n^{r-1} \sum_{t=0}^{n-1} B_r\Big(\frac{x+t}{n}\Big). \tag{15}$$

Raabe's formula plays an important role in the theory of the Bernoulli polynomials.

Using a classical rearrangement of absolutely convergent series the following generalization of Raabe's multiplication formula can be proved:

THEOREM 1. *Let x be any real number. Then a system* (4) *is* $(\mu, \mathfrak{m})$*-covering if and only if*

$$n_0^{r-1} \sum_{n=0}^{n_0-1} \mathfrak{m}(n) B_r\Big(\frac{x+n}{n_0}\Big) = \sum_{t=1}^{k} \mu_t n_t^{r-1} B_r\Big(\frac{x+a_t}{n_t}\Big) \tag{16}$$

holds for every $r \in \mathbb{Z}^*$.

Proof. As in [7], the fact that (4) is $(\mu, \mathfrak{m})$-covering is equivalent with the identity

$$\mathfrak{m}(0) + \mathfrak{m}(1)y + \mathfrak{m}(2)y^2 + \cdots = \sum_{t=1}^{k} \mu_t y^{a_t}(1 + y^{n_t} + y^{2n_t} + \ldots), \qquad |y| < 1,$$

and this is equivalent with

$$\sum_{n=0}^{n_0-1} \mathfrak{m}(n) \frac{y^n}{1-y^{n_0}} = \sum_{t=1}^{k} \mu_t \frac{y^{a_t}}{1-y^{n_t}} .$$

Then the substitution $y = \mathrm{e}^z$ and corresponding algebraic manipulation give

$$\frac{1}{n_0} \sum_{n=0}^{n_0-1} \mathfrak{m}(n) \frac{n_0 z\, \mathrm{e}^{n_0 z \frac{x+n}{n_0}}}{\mathrm{e}^{n_0 z} - 1} = \sum_{t=0}^{k} \frac{\mu_t}{n_t} \frac{n_t z\, \mathrm{e}^{n_t z \frac{x+a_t}{n_t}}}{\mathrm{e}^{n_t z} - 1} . \tag{17}$$

Substituting the generating series (9) for the Bernoulli polynomials shows that the relation (17) is equivalent to

$$\sum_{r=0}^{\infty} n_0^{r-1} \Big(\sum_{n=0}^{n_0-1} \mathfrak{m}(n) B_r\Big(\frac{x+n}{n_0}\Big)\Big) \frac{z^r}{r!} = \sum_{r=0}^{\infty} \Big(\sum_{t=1}^{k} \mu_t n_t^{r-1} B_r\Big(\frac{x+a_t}{n_t}\Big)\Big) \frac{z^r}{r!} .$$

which yields (16), and the proof is finished.

The identity (17) can be used to prove another form of (16). Namely, if $\{x\}$ denotes the fractional part of x, then

$$\{x\} = \{x + a\} = \Big\{\frac{x+a}{n}\Big\} n$$

for every $a, n \in \mathbb{Z}^*$, $n > 0$, $a < n$. Thus (16) can be written in the form

$$n_0^{r-1} \sum_{n=0}^{n_0-1} \mathfrak{m}(n) B_r\Big(\Big\{\frac{x+n}{n_0}\Big\}\Big) = \sum_{t=1}^{k} \mu_t n_t^{r-1} B_r\Big(\Big\{\frac{x+a_t}{n_t}\Big\}\Big) . \tag{18}$$

There is an interesting formula due to Hermite involving the integer part function[1]:

$$[nx] = [x] + \left[x + \frac{1}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right],$$

or, equivalently,

$$[x] = \left[\frac{x}{n}\right] + \left[\frac{x+1}{n}\right] + \cdots + \left[\frac{x+n-1}{n}\right]$$

for every real number x and positive integer n.

Hermite's identities are in fact a consequence of the Raabe multiplication formula for the first Bernoulli polynomial $B_1(x) = x - \frac{1}{2}$ as the following generalization shows:

THEOREM 2. *If a system* (4) *is* $(\mu, \mathfrak{m})$*-covering, then for every real number* x *we have*

$$\mathfrak{m}(0)\left[\frac{x}{n_0}\right] + \mathfrak{m}(1)\left[\frac{x+1}{n_0}\right] + \cdots + \mathfrak{m}(n_0 - 1)\left[\frac{x+n_0-1}{n_0}\right]$$
$$= \mu_1\left[\frac{x+a_1}{n_1}\right] + \mu_2\left[\frac{x+a_2}{n_2}\right] + \cdots + \mu_k\left[\frac{x+a_k}{n_k}\right].$$

For the proof simply subtract (18) from (16) with $r = 1$.

COROLLARY 1. *If* (4) *is an* m *times covering system, then for every real number* x *we have*

$$m[x] = \left[\frac{x+a_1}{n_1}\right] + \left[\frac{x+a_2}{n_2}\right] + \cdots + \left[\frac{x+a_k}{n_k}\right].$$

Beebee [1] proved the following generalization of (2):

System (4) *is a disjoint covering system if and only if*

$$\sum_{t=1}^{k} \frac{1}{n_t} = 1, \tag{19}$$

$$B_r = \frac{1}{1 - \sum\limits_{j=1}^{k} n_j^{r-1}} \sum_{s=0}^{r-1} \binom{r}{s} B_s \sum_{t=1}^{k} n_t^{r-1} \left(\frac{a_t}{n_t}\right)^{r-s} \tag{20}$$

for every positive integer r.

Note that in every covering system (4) the classes can be rearranged in such a way that $a_1 = 0$, and thus in (20) the summation in the last sum can run from $t = 2$.

Now we can prove the following generalization of this result:

[1] I would like to thank Professor A. Schinzel for calling my attention to it

THEOREM 3. *System* (4) *is a* $(\mu, \mathfrak{m})$*-covering if and only if*

$$\begin{aligned}&\left(n_0^{r-1}\sum_{n=0}^{n_0-1}\mathfrak{m}(n)-\sum_{t=1}^{k}\mu_t n_t^{r-1}\right)B_r\\&=\sum_{s=0}^{r-1}\binom{r}{s}B_s\cdot\left(\sum_{t=1}^{k}\mu_t a_t^{r-s}n_t^{s-1}-n_0^{s-1}\sum_{n=1}^{n_0-1}\mathfrak{m}(n)n^{r-s}\right)\end{aligned}\tag{21}$$

for every $r \in \mathbb{Z}^*$.

The proof is based on Lemma 1 and formula (10).

Note that for $r = 0$ the right-hand side of (21) is empty, which (again) implies (14). This avoids formal splitting into two seemingly nonconnected parts (19) and (20).

For covering systems this yields:

COROLLARY 1. *A system* (4) *is a covering with covering function* $\mathfrak{m}$ *if and only if*

$$\begin{aligned}&\left(n_0^{r-1}\sum_{n=0}^{n_0-1}\mathfrak{m}(n)-\sum_{t=1}^{k}n_t^{r-1}\right)B_r\\&=\sum_{s=0}^{r-1}\binom{r}{s}B_s\cdot\left(\sum_{t=1}^{k}a_t^{r-s}n_t^{s-1}-n_0^{s-1}\sum_{n=1}^{n_0-1}\mathfrak{m}(n)n^{r-s}\right)\end{aligned}$$

for every $r \in \mathbb{Z}^*$.

For m times covering systems this yields a formula closer to that proved by J. Beebee:

COROLLARY 2. *A system* (4) *is* m *times covering if and only if*

$$\left(m-\sum_{t=1}^{k}n_t^{r-1}\right)B_r=\sum_{s=0}^{r-1}\binom{r}{s}B_s\sum_{t=1}^{k}a_t^{r-s}n_t^{s-1}$$

for every $r \in \mathbb{Z}^*$.

In the introduction, we gave some examples of covering systems. Substitution of such concrete examples in the proved recurrences does not always lead to nice formulas because of the irregular assembly structure of general covering systems. So, e.g. covering system (5) gives:

COROLLARY 3. *For every positive integer r we have*

$$B_r = \frac{1}{3 \cdot 2^r + 2 \cdot 3^r - 5 \cdot 6^r} \sum_{s=0}^{r-1} \binom{r}{s} B_s \cdot \left(6^s(2^{r-s} + 3^{r-s} + 4^{r-s})\right).$$

Systems of type (7) together with Corollary 2 yield the following recurrence relations:

COROLLARY 4. *For positive integers r, n, and $f \in \mathbb{Z}^*$, we have*

$$B_r = \frac{1}{2^{f+1}(2^r - 1) + 2^{fr}\left(2n^r - 2^r(n^r + 1)\right)} \sum_{s=0}^{r-1} \binom{r}{s} B_s \cdot$$

$$\cdot \left(2^{fr+s} - 2^{s+f} + n^{s-1}\left(2^{r(f+1)} - 2^{fr+1}\right) \sum_{t=1}^{n-1} t^{r-s}\right).$$

For $f = 0$, $r = m$, this identity reduces to (2) because in this case (7) reduces to (6). For $f = 1$ we obtain:

COROLLARY 5. *For positive integers r and n we have*

$$B_r = \frac{2^r}{2 - 2^r(n^r + 1)} \sum_{s=0}^{r-1} \binom{r}{s} B_s \cdot \left(2^{s-r} + n^{s-1} \sum_{t=1}^{n-1} t^{r-s}\right).$$

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