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TWO STAGE LINEAR MODEL WITH CONSTRAINTS

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ABSTRACT. A two stage linear model with constraints on parameters is under consideration. These constraints occurs in the second stage, however parameters of the both stages are involved in them. The problem is to construct a linear estimator in the second stage under a condition that the estimator of the first stage parameter must not be changed.

Introduction

In the multistage linear model [1], [2], [4], [6], [7] a linear system of constraints on parameters of the ith and the preceding stages may occur. The parameters of the preceding stages are not known and in the ith stage estimators of them are at our disposal only. A linear estimator in the ith stage respecting the mentioned system of constraints belongs to a subclass of the class of all linear unbiased estimators. In this subclass there does not exist the jointly efficient estimator (cf. [3, p. 58 and 156]); it is reasonable to seek an estimator optimal with respect to some risk function ([3, p. 267]).

The aim of the paper is to contribute to a solution of the estimation problem in the two stage linear model, where the mentioned constraints occur in the second stage.

1. A motivation example

Let $\hat{\theta}_1$, $\hat{\theta}_2$, Y_1, Y_2 be stochastically independent random variables with mean values θ_1, θ_2, β_1, β_2 and with the variances τ_1^2, τ_2^2, σ_1^2, σ_2^2. This will

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be denoted as

$$\begin{pmatrix}\hat\theta_1\\ \hat\theta_2\\ Y_1\\ Y_2\end{pmatrix} \sim \left(\begin{pmatrix}\theta_1\\ \theta_2\\ \beta_1\\ \beta_2\end{pmatrix}, \begin{pmatrix}\tau_1^2, & 0, & 0, & 0\\ 0, & \tau_2^2, & 0, & 0\\ 0, & 0, & \sigma_1^2, & 0\\ 0, & 0, & 0, & \sigma_2^2\end{pmatrix}\right).$$

The parametric space of this model is $\underline{\Theta} = \{(\theta_1,\theta_2,\beta_1,\beta_2)' : (1,-1,1,1)\cdot \cdot(\theta_1,\theta_2,\beta_1,\beta_2)' = 0\}$. Let θ_1, θ_2 be parameters of the first stage and β_1, β_2 parameters of the second stage; $\beta_1+\beta_2 = \theta_2-\theta_1$. E.g., an experiment the aim of which is to determine a height difference β_1 between points A and P and a height difference β_2 between points P and B if the height of the point A is θ_1 and the height of the point B is θ_2. The estimators $\hat\theta_1$ and $\hat\theta_2$ of the heights θ_1 and θ_2 are given from the first stage, the measurement Y_1, Y_2 of the parameters β_1, β_2 is performed in the second stage. The estimates of the first stage must not be changed in the second stage, i.e., the estimators $\tilde\beta_1 = \alpha_{11}Y_1+\alpha_{12}Y_2+\alpha_{13}\hat\theta_1+\alpha_{14}\hat\theta_2$ and $\tilde\beta_2 = \alpha_{21}Y_1+\alpha_{22}Y_2+\alpha_{23}\hat\theta_1+\alpha_{24}\hat\theta_2$ have to fulfil the condition $\tilde\beta_1+\tilde\beta_2 = \hat\theta_2-\hat\theta_1$. As the estimators are to be unbiased they must fulfil the following conditions

(i) $\forall\{(\theta_1,\theta_2,\beta_1,\beta_2)' : \beta_1+\beta_2+\theta_1-\theta_2 = 0\}$ $E(\tilde\beta_i \mid \theta_1,\theta_2,\beta_1,\beta_2) = \beta_i$, $i=1,2$,

(ii) $\tilde\beta_1+\tilde\beta_2 = \hat\theta_2-\hat\theta_1$.

The condition (i) is fulfilled for such a system of coefficients α_{ij}, $i=1,2$; $j=1,\dots,4$, that

$$\mathcal{M}\left(\begin{pmatrix}\alpha_{11}-1, & \alpha_{12}, & \alpha_{13}, & \alpha_{14}\\ \alpha_{21}, & \alpha_{22}-1, & \alpha_{23}, & \alpha_{24}\end{pmatrix}'\right) \subset \mathcal{M}\left((1, \ 1, \ 1, \ -1)'\right),$$

where $\mathcal{M}$ denotes the columns space of the matrix in the parenthesis ().

The unbiased estimators of the parameters β_1 and β_2 are

$$\tilde\beta_1 = Y_1 + k_1(Y_1+Y_2+\hat\theta_1-\hat\theta_2), \qquad k_1\in\mathcal{R}^1,$$
$$\tilde\beta_2 = Y_2 + k_2(Y_1+Y_2+\hat\theta_1-\hat\theta_2), \qquad k_2\in\mathcal{R}^1,$$

where a symbol $\mathcal{R}^p$ denotes the p-dimensional Euclidean space. The class of such estimators is denoted $\mathcal{U}_\beta$.

If the unbiased linear estimators fulfil the condition (ii), then $k_1+k_2=-1$.

In the class of linear unbiased estimators (the condition (i) is fulfilled, the condition (ii) need not be fulfilled) the jointly efficient estimator of the parameters β_1, β_2 is

$$\begin{pmatrix} \hat{\beta}_1^* \\ \hat{\beta}_2^* \end{pmatrix} = \begin{pmatrix} Y_1 - [\sigma_1^2/(\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2)](Y_1 + Y_2 + \hat{\theta}_1 - \hat{\theta}_2) \\ Y_2 - [\sigma_2^2/(\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2)](Y_1 + Y_2 + \hat{\theta}_1 - \hat{\theta}_2) \end{pmatrix}.$$

The estimator obviously does not fulfil the condition (ii) because of

$$(-\sigma_1^2 - \sigma_2^2)/(\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2) \neq -1.$$

An estimator fulfilling both of the conditions (i) and (ii) is denoted by a tilde:

$$\tilde{\boldsymbol{\beta}} = \mathbf{Y} + \begin{pmatrix} k \\ -1-k \end{pmatrix} (Y_1 + Y_2 + \hat{\theta}_1 - \hat{\theta}_2).$$

The class of such estimators is denoted $\tilde{\mathcal{U}}_{\boldsymbol{\beta}}$.

The problem is how to choose the number k in order to attain an estimator $\tilde{\boldsymbol{\beta}}$ optimal with respect to a given criterion of optimality. In the following an estimator $\tilde{\boldsymbol{\beta}}$ will be considered as the optimal one if it minimizes the risk function

$$R(\tilde{\boldsymbol{\beta}}, \boldsymbol{\beta}) = \mathrm{Tr}\left[\mathbf{H}\,\mathrm{Var}_{\boldsymbol{\beta}}(\tilde{\boldsymbol{\beta}})\right], \qquad \tilde{\boldsymbol{\beta}} \in \tilde{\mathcal{U}}_{\boldsymbol{\beta}}.$$

The matrix $\mathbf{H}$ is chosen by a statistician and it is usually positively definite. For a given matrix $\mathbf{H}$ the optimal estimator $\tilde{\boldsymbol{\beta}}$ is characterized by a number

$$k = \frac{-\left[H_{11}\sigma_1^2 - H_{12}(2\sigma_1^2 + \tau_1^2 + \tau_2^2) + H_{22}(\sigma_1^2 + \tau_1^2 + \tau_2^2)\right]}{\left[(H_{11} - 2H_{12} + H_{22})(\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2)\right]},$$

where

$$\mathbf{H} = \begin{pmatrix} H_{11}, & H_{12} \\ H_{21}, & H_{22} \end{pmatrix}, \qquad H_{11} + H_{22} - 2H_{12} \neq 0.$$

If $\mathbf{H} = \mathbf{e}_1\mathbf{e}_1'$ ($\mathbf{e}_1' = (1,0)$), then $\tilde{\beta}_1 = \hat{\beta}_1^*$, if $\mathbf{H} = \mathbf{e}_2\mathbf{e}_2'$ ($\mathbf{e}_2 = (0,1)'$), then $\tilde{\beta}_2 = \hat{\beta}_2^*$, etc.

This example demonstrates an unpleasant consequence of the condition (ii) which prevents us to find the jointly efficient unbiased linear estimator in the class $\tilde{\mathcal{U}}_{\boldsymbol{\beta}}$.

2. Two stage linear model with constraints

In the following (cf. [1]) let

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim \left(\begin{pmatrix} \mathbf{X}_1, & \mathbf{O} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\beta} \end{pmatrix}; \begin{pmatrix} \boldsymbol{\Sigma}_{11}, & \mathbf{O} \\ \mathbf{O}, & \boldsymbol{\Sigma}_{22} \end{pmatrix} \right), \qquad (2.1)$$
$$\underline{\Theta} = \left\{ (\boldsymbol{\theta}', \boldsymbol{\beta}')' : \ \mathbf{B}\boldsymbol{\beta} + \mathbf{C}\boldsymbol{\theta} + \boldsymbol{a} = \mathbf{0} \right\};$$

$\mathbf{X}_1$, $\mathbf{D}$, $\mathbf{X}_2$ are known $n_1 \times k_1$, $n_2 \times k_1$, $n_2 \times k_2$ matrices, $\boldsymbol{\theta}$, $\boldsymbol{\beta}$ are k_1- and k_2-dimensional unknown vectors and $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{22}$ are known matrices, $\mathbf{B}$, $\mathbf{C}$ are known $q \times k_2$, $q \times k_1$ matrices and $\boldsymbol{a}$ is a given vector. It must hold: $\mathcal{M}(\mathbf{D}') \subset \mathcal{M}(\mathbf{X}'_1)$, $\mathcal{M}(\mathbf{C}) \subset \mathcal{M}(\mathbf{B})$ and $\boldsymbol{a} \in \mathcal{M}(\mathbf{C})$. The vector $\boldsymbol{\theta}$ is the parameter of the first stage, the vector $\boldsymbol{\beta}$ is the parameter of the second stage. From the first stage the unbiased estimator $\hat{\boldsymbol{\theta}} = (\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{Y}_1$ and its covariance matrix $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}} = (\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}$ are at our disposal only.

The aim is to determine an estimator of the parameter $\boldsymbol{\beta}$ on the basis of the random vector $\mathbf{Y}_2 - \mathbf{D}\hat{\boldsymbol{\theta}}$, where $\mathbf{Y}_2$ is the observation vector of the second stage and on the basis of the estimator $\hat{\boldsymbol{\theta}}$.

Definition 2.1. *The model* (2.1) *is regular if the rank of the matrix* $\mathbf{X}_1$ *is* $R(\mathbf{X}_1) = k_1$, $R(\mathbf{X}_2) = k_2$, $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{22}$ *are positively definite matrices,* $R(\mathbf{B}) = q$, $R(\mathbf{C}\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}\mathbf{C}') = q$, *where* $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}} = \mathrm{Var}(\hat{\boldsymbol{\theta}}) = (\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}$.

Lemma 2.2. *The class* $\mathcal{U}_{\boldsymbol{\beta}}$ *of unbiased estimators of the parameter* $\boldsymbol{\beta}$ *in the regular model* (2.1) *is*

$$\mathcal{U}_{\boldsymbol{\beta}} = \left\{ \left[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{E}\mathbf{B}\mathbf{X}_2^-\right](\mathbf{Y}_2 - \mathbf{D}\hat{\boldsymbol{\theta}}) + \mathbf{E}\mathbf{C}\hat{\boldsymbol{\theta}} + \mathbf{E}\boldsymbol{a} : \right.$$
$$\left. \mathbf{Z} \ \textit{an arbitrary } k_2 \times n_2 \ \textit{matrix}, \ \mathbf{E} \ \textit{an arbitrary } k_2 \times q \ \textit{matrix} \right\},$$

where $\mathbf{X}_2^-$ *is an arbitrary but fixed matrix from the class* $\mathcal{X}_2^-$ *of the g-inverses* $\mathbf{X}_2^-$ *of the matrix* $\mathbf{X}_2$ (*a matrix* $\mathbf{X}_2^-$ *is a solution of the equation* $\mathbf{X}_2\mathbf{X}_2^-\mathbf{X}_2 = \mathbf{X}_2$; *in detail cf.* [5]).

P r o o f. A linear estimator $\hat{\boldsymbol{\beta}} = \mathbf{T}_1(\mathbf{Y}_2 - \mathbf{D}\hat{\boldsymbol{\theta}}) + \mathbf{T}_2\hat{\boldsymbol{\theta}} + \boldsymbol{d}$ is unbiased estimator of $\boldsymbol{\beta}$ if and only if $\mathbf{T}_1\mathbf{X}_2\boldsymbol{\beta} + \mathbf{T}_2\boldsymbol{\theta} + \boldsymbol{d} = \boldsymbol{\beta}$ for each vector $\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\theta} \end{pmatrix} \in \left\{ (\boldsymbol{u}', \boldsymbol{v}')' : \mathbf{B}\boldsymbol{u} + \mathbf{C}\boldsymbol{v} + \boldsymbol{a} = \mathbf{0} \right\} = \underline{\Theta}$.

Thus

$$\forall \left\{ (\boldsymbol{\beta}', \boldsymbol{\theta}')' \in \underline{\Theta} \right\} \ \left[(\mathbf{T}_1\mathbf{X}_2 - \mathbf{I}), \mathbf{T}_2\right] \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\theta} \end{pmatrix} + \boldsymbol{d} = \mathbf{0}$$
$$\iff \exists \left\{ \mathbf{E} : \ k_2 \times q \ \text{matrix} \right\} \ \mathbf{X}_2'\mathbf{T}_1' - \mathbf{I} = \mathbf{B}'\mathbf{E}' \wedge \mathbf{T}_2' = \mathbf{C}'\mathbf{E}' \wedge \boldsymbol{d} = \mathbf{E}\boldsymbol{a}.$$

The equation $\mathbf{T}_1\mathbf{X}_2 - \mathbf{I} = \mathbf{EB}$ has a solution $\mathbf{T}_1$ for any $k_2 \times q$ matrix $\mathbf{E}$, because of $(\mathbf{I} + \mathbf{EB})\mathbf{X}_2^-\mathbf{X}_2 = \mathbf{I} + \mathbf{EB}$ (cf. [5, Lemma 2.2.4]; the implications $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X'}_2) \implies \mathbf{BX}_2^-\mathbf{X}_2 = \mathbf{B}$ and $R(\mathbf{X}_2) = k_2 \implies \mathbf{X}_2^-\mathbf{X}_2 = \mathbf{I}$ are utilized). Thus the class of all solutions for a given $\mathbf{E}$ is

$$\{\mathbf{T}_1 : \ \mathbf{T}_1 = \mathbf{X}_2^- + \mathbf{EBX}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) : \ \mathbf{Z} \text{ an arbitrary } k_2 \times n_2 \text{ matrix}\},$$

where $\mathbf{X}_2^-$ is any fixed element of $\mathcal{X}_2^-$. It is obvious how to finish the proof.

Lemma 2.3. *The class of all linear unbiased estimators $\tilde{\beta}$ of the parameter β based on the vectors $\mathbf{Y}_2 - \mathbf{D}\hat{\theta}_2$ and $\hat{\theta}_2$ and fulfilling the condition $\mathbf{B}\tilde{\beta} + \mathbf{C}\hat{\theta} + \mathbf{a} = 0$ is*

$$\begin{aligned}\tilde{\mathcal{U}}_\beta = \{&(\mathbf{I} - \mathbf{B}^-\mathbf{B})[\mathbf{X}_2^- + \mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{BX}_2^-](\mathbf{Y}_2 - \mathbf{D}\hat{\theta}) \\ &+ [-\mathbf{B}^- + (\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{W}_2]\mathbf{C}\hat{\theta} + (\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{W}_2\mathbf{a} - \mathbf{B}^-\mathbf{a} : \\ &\mathbf{W}_1 \ \textit{an arbitrary } k_2 \times n_2 \ \textit{matrix}, \ \mathbf{W}_2 \ \textit{an arbitrary } k_2 \times q \ \textit{matrix}\},\end{aligned}$$

where $\mathbf{X}_2^-$ and $\mathbf{B}^-$ are arbitrary but fixed matrices from $\mathcal{X}_2^-$ and $\mathcal{B}^-$, respectively.

P r o o f. An estimator $\tilde{\beta}$ (Lemma 2.2) fulfils the condition

$$\mathbf{B}\tilde{\beta} + C\hat{\theta} + \mathbf{a} = 0$$

if and only if

$$\begin{aligned}&\mathbf{B}[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\mathbf{X}_2\beta + (\mathbf{C} + \mathbf{BEC})\theta + \mathbf{BE}\mathbf{a} \\ &\quad + \mathbf{B}[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\varepsilon + (\mathbf{C} + \mathbf{BEC})\varepsilon_{\hat{\theta}} = 0\end{aligned}$$

for each $(\beta', \theta')' \in \underline{\Theta}$ and for each (or almost each) realization of the random vectors $\varepsilon = \mathbf{Y}_2 - \mathbf{D}\hat{\theta} - \mathbf{X}_2\beta$ and $\varepsilon_{\hat{\theta}} = \hat{\theta} - \theta$. The expression

$$\mathbf{B}[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\mathbf{X}_2\beta + (\mathbf{C} + \mathbf{BEC})\theta + \mathbf{BE}\mathbf{a}$$

equals zero vector for each $(\beta', \theta')' \in \underline{\Theta}$ since $\tilde{\beta} \in \mathcal{U}_\beta$. As $P\{\varepsilon \in \mathcal{M}[\Sigma = \Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}']\} = 1$ and $P\{\varepsilon_{\hat{\theta}} \in \mathcal{M}[\Sigma_{\hat{\theta}} = (\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1')^{-1}]\} = 1$ and the matrices Σ and $\Sigma_{\hat{\theta}}$ are regular with respect to our assumption, i.e., $\mathcal{M}(\Sigma) = \mathcal{R}^{n_2}$, $\mathcal{M}(\Sigma_{\hat{\theta}}) = \mathcal{R}^{k_1}$, we obtain the following equations for unknown matrices $\mathbf{Z}$ and $\mathbf{E}$:

$$\begin{aligned}\mathbf{BX}_2^- + \mathbf{BZ}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{BEBX}_2^- &= \mathbf{O}, \\ \mathbf{C} + \mathbf{BEC} &= \mathbf{O}.\end{aligned}$$

This can be written as follows:

$$\mathbf{B}(\mathbf{Z},\mathbf{E})\begin{pmatrix}\mathbf{O}, & \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-\\ \mathbf{C}, & \mathbf{B}\mathbf{X}_2^-\end{pmatrix} = (-\mathbf{C}, -\mathbf{B}\mathbf{X}_2^-)\,.$$

A particular solution is $\mathbf{Z}=\mathbf{X}_2^-$ and $\mathbf{E}=-\mathbf{B}^-$ thus the system is consistent (the implications $\mathcal{M}(\mathbf{B}')\subset\mathcal{M}(\mathbf{X}_2')\implies \mathbf{B}\mathbf{X}_2^-\mathbf{X}_2=\mathbf{B}$ and $\mathcal{M}(\mathbf{C})\subset\mathcal{M}(\mathbf{B})\implies \mathbf{B}\mathbf{B}^-\mathbf{C}=\mathbf{C}$ and [5, Lemma 2.2.4] are utilized). Then with respect to [5, Theorem 2.3.2] the class of all solutions of this system is

$$\Big\{(\mathbf{Z},\mathbf{E}) = \mathbf{B}^-(-\mathbf{C},-\mathbf{B}\mathbf{X}_2^-)\begin{pmatrix}\mathbf{O}, & \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-\\ \mathbf{C}, & \mathbf{B}\mathbf{X}_2^-\end{pmatrix}^- + (\mathbf{W}_1,\mathbf{W}_2)$$

$$- \mathbf{B}^-\mathbf{B}(\mathbf{W}_1,\mathbf{W}_2)\begin{pmatrix}\mathbf{O}, & \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-\\ \mathbf{C}, & \mathbf{B}\mathbf{X}_2^-\end{pmatrix}\begin{pmatrix}\mathbf{O}, & \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-\\ \mathbf{C}, & \mathbf{B}\mathbf{X}_2\end{pmatrix}^- :$$

$$\mathbf{W}_1 \text{ any } k_2\times n_2 \text{ matrix}, \ \mathbf{W}_2 \text{ any } k_2\times q \text{ matrix}\Big\}.$$

The matrices $\mathbf{X}_2^-$ and $\mathbf{B}^-$ are arbitrary but fixed g-inverses of the matrices $\mathbf{X}_2$ and $\mathbf{B}$, respectively. If the g-inverse

$$\begin{pmatrix}\mathbf{O}, & \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-\\ \mathbf{C}, & \mathbf{B}\mathbf{X}_2^-\end{pmatrix}^- = \begin{pmatrix}\mathbf{O}, & \mathbf{O}\\ \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-, & \mathbf{X}_2\mathbf{B}^-\end{pmatrix}$$

is used (a reader can easily verify that it is a g-inverse), we obtain general solution in the form

$$\mathbf{Z} = \mathbf{W}_1 - \mathbf{B}^-\mathbf{B}\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)\,,$$

$$\mathbf{E} = \mathbf{W}_2 - \mathbf{B}^-\mathbf{B}\mathbf{B}^- - \mathbf{B}^-\mathbf{B}\mathbf{W}_2\mathbf{B}\mathbf{B}^-\,.$$

After substituting $\mathbf{Z}$ and $\mathbf{E}$ into Lemma 2.2 we obtain the sought estimator either in the form

$$\tilde{\boldsymbol\beta} = \big\{\mathbf{X}_2^- + (\mathbf{I}-\mathbf{B}^-\mathbf{B})\big[\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{B}\mathbf{X}_2^-\big] - \mathbf{B}^-\mathbf{B}\mathbf{X}_2^-\big\}(\boldsymbol{Y}_2 - \mathbf{D}\hat{\boldsymbol\theta})$$

$$+ \big[-\mathbf{B}^- + (\mathbf{I}-\mathbf{B}^-\mathbf{B})\mathbf{W}_2]\mathbf{C}\hat{\boldsymbol\theta} + (\mathbf{I}-\mathbf{B}^-\mathbf{B})\mathbf{W}_2\boldsymbol{a} - \mathbf{B}^-\boldsymbol{a}\,,$$

or in the form

$$\tilde{\boldsymbol\beta} = (\mathbf{I}-\mathbf{B}_2^-\mathbf{B})[\mathbf{X}_2^- + \mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{B}\mathbf{X}_2^-\big]\boldsymbol{Y}$$

$$+ (\mathbf{I}-\mathbf{B}^-\mathbf{B})\big[-\mathbf{X}_2^-\mathbf{D} - \mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)\mathbf{D} - \mathbf{W}_2\mathbf{B}\mathbf{X}_2^-\mathbf{D} + \mathbf{W}_2\mathbf{C}\big]\hat{\boldsymbol\theta} - \mathbf{B}^-\mathbf{C}\hat{\boldsymbol\theta}$$

$$+ (\mathbf{I}-\mathbf{B}^-\mathbf{B})\mathbf{W}_2\boldsymbol{a} - \mathbf{B}^-\boldsymbol{a}\,.$$

In the following the notation $\mathbf{A}^-_{m(\mathbf{N})}$ means a matrix with the properties $\mathbf{A}\mathbf{A}^-_{m(\mathbf{N})}\mathbf{A}=\mathbf{A}$, and $\mathbf{N}\mathbf{A}^-_{m(\mathbf{N})}\mathbf{A} = \mathbf{A}'(\mathbf{A}^-_{m(\mathbf{N})})'\mathbf{N}$; here $\mathbf{A}$ is an arbitrary $m\times n$ matrix and $\mathbf{N}$ is a p.s.d. $n\times n$ matrix (cf. [5]).

THEOREM 2.4. *In the class $\mathcal{U}_{\boldsymbol\beta}$ from Lemma 2.2 there exists the jointly efficient estimator $\hat{\boldsymbol\beta}^*$ of the vector $\boldsymbol\beta$,*

$$\hat{\boldsymbol\beta}^* = \left((\mathbf{X}_2', \quad \mathbf{B}')^-_{m(\mathbf{S})}\right)' \begin{pmatrix} \mathbf{Y}_2 - \mathbf{D}\hat{\boldsymbol\theta} \\ -\mathbf{C}\hat{\boldsymbol\theta} - \boldsymbol{a} \end{pmatrix},$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11}, & \mathbf{S}_{12} \\ \mathbf{S}_{21}, & \mathbf{S}_{22} \end{pmatrix},$$

$$\mathbf{S}_{11} = \boldsymbol\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\boldsymbol\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}', \qquad \mathbf{S}_{12} = \mathbf{D}(\mathbf{X}_1'\boldsymbol\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}',$$
$$\mathbf{S}_{21} = \mathbf{C}(\mathbf{X}_1'\boldsymbol\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}', \qquad \mathbf{S}_{22} = \mathbf{C}(\mathbf{X}_1'\boldsymbol\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'.$$

Proof. Let us consider any linear function $f(\boldsymbol\beta) = \boldsymbol{p}'\boldsymbol\beta$, $\boldsymbol\beta \in \left\{(\mathbf{I}, \mathbf{O})\begin{pmatrix}\boldsymbol\beta \\ \boldsymbol\theta\end{pmatrix} : \mathbf{B}\boldsymbol\beta + \mathbf{C}\boldsymbol\theta + \boldsymbol{a} = \mathbf{0}\right\}$, and its unbiased linear estimator $\boldsymbol{p}'\big([\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\mathbf{Y}_2 + \{-[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\mathbf{D} + \mathbf{EC}\}\hat{\boldsymbol\theta}\big) + \boldsymbol{p}'\mathbf{E}\boldsymbol{a}$ from Lemma 2.2. Its variance is

$$\begin{aligned}
&\varphi(\mathbf{E}, \mathbf{Z}) \\
&= \boldsymbol{p}'[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\boldsymbol\Sigma_{22}\{(\mathbf{X}_2^-)' + [\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2']\mathbf{Z}' + (\mathbf{X}_2^-)'\mathbf{B}'\mathbf{E}'\}\boldsymbol{p} \\
&\quad + \boldsymbol{p}'\{[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\mathbf{D} - \mathbf{EC}\}\cdot \\
&\quad \cdot \boldsymbol\Sigma_{\hat{\boldsymbol\theta}}(\mathbf{D}'\{(\mathbf{X}_2^-)' + [\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2']\mathbf{Z}' + (\mathbf{X}_2^-)'\mathbf{B}'\mathbf{E}'\} - \mathbf{C}'\mathbf{E}')\boldsymbol{p}.
\end{aligned}$$

Its extremal value is attained for matrices $\mathbf{E}$, $\mathbf{Z}$ fulfilling the equations

$$\partial\varphi(\mathbf{E}, \mathbf{Z})/\partial\mathbf{E} = \mathbf{O},$$
$$\partial\varphi(\mathbf{E}, \mathbf{Z})/\partial\mathbf{Z} = \mathbf{O}.$$

These equations can be written in the form:

$$\begin{aligned}
\boldsymbol{pp}'\{[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-][\mathbf{S}_{11}(\mathbf{X}_2^-)'\mathbf{B}' - \mathbf{S}_{12}] - \mathbf{ES}_{21}(\mathbf{X}_2^-)'\mathbf{B}' + \mathbf{ES}_{22}\} &= \mathbf{O}, \\
\boldsymbol{pp}'\{[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-]\mathbf{S}_{11}[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'] - \mathbf{ES}_{21}[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2']\} &= \mathbf{O}.
\end{aligned} \tag{2.2}$$

Now it is necessary to find the considered g-inverse of the matrix $(\mathbf{X}_2', \mathbf{B}')$, to express it in the form

$$\left((\mathbf{X}_2', \mathbf{B}')^-_{m(\mathbf{S})}\right)' = (\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-, -\mathbf{E})$$

and to prove that the proper choice of the matrices $\mathbf{X}_2^-$, $\mathbf{E}$, $\mathbf{Z}$ from the right-hand side fulfil the equations (2.2) for any $\boldsymbol{p}$.

Let us take into account the relations

$$(\mathbf{X}_2', \mathbf{B}')^-_{m(\mathbf{S})} = \mathbf{S}^{-1}\begin{pmatrix}\mathbf{X}_2\\ \mathbf{B}\end{pmatrix}\left((\mathbf{X}_2', \mathbf{B}')\mathbf{S}^{-1}\begin{pmatrix}\mathbf{X}_2\\ \mathbf{B}\end{pmatrix}\right)^{-1},$$

$$\mathbf{S}^{-1} = \begin{pmatrix}\mathbf{S}_{11}, & \mathbf{S}_{12}\\ \mathbf{S}_{21}, & \mathbf{S}_{22}\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}\mathbf{S}_{11}^{-1} + \mathbf{S}_{11}^{-1}\mathbf{S}_{12}(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}\mathbf{S}_{21}\mathbf{S}_{11}^{-1}, & -\mathbf{S}_{11}^{-1}\mathbf{S}_{12}(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}\\ -(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}\mathbf{S}_{21}\mathbf{S}_{11}^{-1}, & (\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}\end{pmatrix}$$

$$= \begin{pmatrix}(\mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21})^{-1}, & -(\mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21})^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\\ -\mathbf{S}_{22}^{-1}\mathbf{S}_{21}(\mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21})^{-1}, & \mathbf{S}_{22}^{-1} + \mathbf{S}_{22}^{-1}\mathbf{S}_{21}(\mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21})^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\end{pmatrix}, \tag{2.3}$$

then we obtain

$$\begin{aligned}&\left((\mathbf{X}_2', \mathbf{B}')^-_{m(\mathbf{S})}\right)'\\ &= \Big(\big[\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2 + (\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}(\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)\big]^{-1}\cdot\\ &\quad\cdot\big[\mathbf{X}_2'\mathbf{S}_{11}^{-1} - (\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\big],\\ &\quad\big[\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2 + (\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}(\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)\big]^{-1}\cdot\\ &\quad\cdot(\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}\Big)\\ &= \big(\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{E}\mathbf{B}\mathbf{X}_2^-, -\mathbf{E}\big)\end{aligned}$$

where

$$\begin{aligned}\mathbf{X}_2^- &= (\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1},\\ \mathbf{E} &= -\big[\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2 + (\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})'(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}(\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)\big]^{-1}\cdot\\ &\quad\cdot(\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1},\\ \mathbf{Z} &= \mathbf{E}\mathbf{S}_{21}\mathbf{S}_{11}^{-1}.\end{aligned}$$

It is easy to verify that these matrices $\mathbf{X}_2^-$, $\mathbf{E}$, $\mathbf{Z}$ fulfil the equations (2.2). The following relationships must be taken into account:

$$\begin{aligned}
&\left[\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2+(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)\right]^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1}\\
&=(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1}-(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2')\cdot\\
&\quad\cdot\left[\mathbf{S}_{22}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12}+(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'\right]^{-1}\cdot\\
&\quad\cdot(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{EBX}_2^- -\mathbf{ZX}_2\mathbf{X}_2^- =\mathbf{E}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)\mathbf{X}_2^-\\
&=-\left[\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2+(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)\right]^{-1}\cdot\\
&\quad\cdot(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'(\mathbf{S}_{22}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12})^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1}\\
&=-(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'\left[\mathbf{S}_{22}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12}+(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)\cdot\right.\\
&\quad\left.\cdot(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)'\right]^{-1}(\mathbf{B}-\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1}
\end{aligned}$$

(cf. (2.3)). As the equations (2.2) are solved for any vector $\boldsymbol{p}$, the choice of the matrices $\mathbf{X}_2^-$, $\mathbf{E}$ and $\mathbf{Z}$ leads to the joint efficient estimator.

Corollary 2.5. *If the two stage models (frequently occurring)*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}\sim\left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{O}\\ \mathbf{O}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\theta\\ \beta\end{pmatrix},\begin{pmatrix}\Sigma_{11}, & \mathbf{O}\\ \mathbf{O}, & \Sigma_{22}\end{pmatrix}\right]\qquad(\textit{i.e. } \mathbf{D}=\mathbf{O}),$$

$\underline{\Theta}=\{(\theta',\beta')':\ \mathbf{B}\beta+\mathbf{C}\theta+\boldsymbol{a}=\mathbf{0}\}$, *is under consideration, then*

$$\beta_1^*=\left[(\mathbf{X}_2',\mathbf{B}')^-_{m\begin{pmatrix}\Sigma_{22}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{C}\Sigma_{\hat\theta}\mathbf{C}'\end{pmatrix}}\right]'\begin{pmatrix}\mathbf{Y}_2\\ -\mathbf{C}\hat\theta-\boldsymbol{a}\end{pmatrix},$$

where

$$\left[(\mathbf{X}_2',\mathbf{B}')^-_{m\begin{pmatrix}\Sigma_{22}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{C}\Sigma_{\hat\theta}\mathbf{C}'\end{pmatrix}}\right]'=(\mathbf{X}_2^-+\mathbf{Z}(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)+\mathbf{EBX}_2^-,-\mathbf{E}),$$

and

$$\begin{aligned}
\mathbf{X}_2^- &=(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{22}^{-1},\\
\mathbf{E}&=-(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\left[\mathbf{C}\Sigma_{\hat\Theta}\mathbf{C}'+\mathbf{B}(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\right]^{-1},\\
\mathbf{Z}&=\mathbf{O}.
\end{aligned}$$

The proof is analogous to the preceding one.

Remark 2.6. The estimator $\hat{\beta}^*$ from Theorem 2.4 does not fulfil the condition $\mathbf{B}\hat{\beta}^* + \mathbf{C}\hat{\theta} + \mathbf{a} = \mathbf{0}$; after some simple but tedious calculation the following result can be obtained:

$$\begin{aligned}\mathbf{B}\hat{\beta}^* + \mathbf{C}\hat{\theta} + \mathbf{a} &= \mathbf{B}\left[(\mathbf{X}_2', \mathbf{B}')^-_{m(\mathbf{S})}\right]' \begin{pmatrix} \mathbf{Y}_2 - \mathbf{D}\hat{\theta} \\ -\mathbf{C}\hat{\theta} - \mathbf{a}\end{pmatrix} + \mathbf{C}\hat{\theta} + \mathbf{a} \\ &= [\mathbf{I} - \mathbf{M}_1]\mathbf{M}_2^{-1}(\mathbf{B}\bar{\beta} + \mathbf{C}\hat{\theta} + \mathbf{a}) - \mathbf{M}_1\mathbf{M}_2^{-1}\mathbf{S}_{21}\mathbf{S}_{11}^{-1} \cdot \\ &\quad \cdot [\mathbf{I} - \mathbf{X}_2(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1}](\mathbf{Y}_2 - \mathbf{D}\hat{\theta}),\end{aligned}$$

where

$$\mathbf{M}_1 = \mathbf{B}(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)',$$

$$\mathbf{M}_2 = \mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12} + (\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)(\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{X}_2)',$$

$$\bar{\beta} = (\mathbf{X}_2'\mathbf{S}_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{S}_{11}^{-1}(\mathbf{Y}_2 - \mathbf{D}\hat{\theta}) \qquad \text{(an unbiased but not efficient estimator of } \beta\text{)}.$$

Remark 2.7. If the two stage model from Corollary 2.5 is considered, then

$$\begin{aligned}&\mathbf{B}\hat{\beta}^* + \mathbf{C}\hat{\theta} + \mathbf{a} \\ &= \mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{Y}_2 + \mathbf{C}\hat{\theta} + \mathbf{a} \\ &\quad - \mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' + \mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1} \cdot \\ &\quad \cdot \{\mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{Y}_2 + \mathbf{C}\hat{\theta} + \mathbf{a}\} \\ &= \mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'[\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' + \mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1} \cdot \\ &\quad \cdot \{\mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{Y}_2 + \mathbf{C}\hat{\theta} + \mathbf{a}\}.\end{aligned}$$

If $\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' = \mathbf{O}$, i.e., the vector $\mathbf{C}\theta + \mathbf{a}$ is known, then $\hat{\beta}^*$ is the jointly efficient estimator and at the same time it fulfils the condition $\mathbf{B}\hat{\beta}^* + \mathbf{C}\theta + \mathbf{a} = \mathbf{0}$. In this case we obtain

$$\begin{aligned}\hat{\beta}^* &= \left[(\mathbf{X}_2', \mathbf{B}')^-_{m\begin{pmatrix}\Sigma_{22}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O}\end{pmatrix}}\right]' \begin{pmatrix}\mathbf{Y} \\ -\mathbf{C}\theta - \mathbf{a}\end{pmatrix} \\ &= \{\mathbf{I} - (\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\}\bar{\beta} \\ &\quad - (\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\Sigma_{11}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}(\mathbf{C}\theta + \mathbf{a})\end{aligned}$$

(cf. [3, p. 152]).

THEOREM 2.8. *Let* $\mathbf{H}$ *be a given* $k_2 \times k_2$ *p.s.d. matrix and let an estimator* $\tilde{\beta}$ *from* $\tilde{\mathcal{U}}_{\beta}$ *be optimal* ($\mathbf{H}$*-optimal*) *if it minimizes the function* $\phi(\tilde{\beta}) = \operatorname{Tr}\left[\mathbf{H}\operatorname{Var}(\tilde{\beta})\right]$, $\tilde{\beta} \in \tilde{\mathcal{U}}_{\beta}$. *Then the matrices* $\mathbf{X}^-$, $\mathbf{B}^-$, $\mathbf{W}_1$ *and* $\mathbf{W}_2$ *from Lemma* 2.3 *are solutions of the following equations*

$$\mathbf{U}_1(\mathbf{W}_1, \mathbf{W}_2)\begin{pmatrix} \mathbf{V}_1, & \mathbf{T}_1 \\ \mathbf{V}_2, & \mathbf{T}_2 \end{pmatrix} = (\mathbf{P}_1, \mathbf{P}_2) \tag{2.5}$$

where

$$\mathbf{U}_1 = \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}],$$

$$\mathbf{V}_1 = (\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)\left[\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\right]\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right],$$

$$\begin{aligned} \mathbf{V}_2 = {} & \mathbf{B}\mathbf{X}_2^-\left[\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\right]\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right] \\ & - \mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right], \end{aligned}$$

$$\begin{aligned} \mathbf{P}_1 = {} & -\left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-\left[\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\right]\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right] \\ & - \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}\mathbf{B}^-\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right], \end{aligned}$$

$$\mathbf{T}_1 = \left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right]\left\{\left[\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\right](\mathbf{X}_2^-)'\mathbf{B}' - \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'\right\},$$

$$\begin{aligned} \mathbf{T}_2 = {} & \mathbf{B}\mathbf{X}_2^-\left[\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\right](\mathbf{X}_2^-)'\mathbf{B}' + \mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' \\ & - \mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'(\mathbf{X}_2^-)'\mathbf{B}' - \mathbf{B}\mathbf{X}_2^-\mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}', \end{aligned}$$

$$\begin{aligned} \mathbf{P}_2 = {} & -\left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-\left[\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'\right](\mathbf{X}_2^-)'\mathbf{B}' \\ & + \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}\mathbf{B}^-\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' \\ & - \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}\mathbf{B}^-\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'(\mathbf{X}_2^-)'\mathbf{B}' \\ & + \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-\mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'. \end{aligned}$$

Proof. Let

$$\begin{aligned} \tilde{\beta} = {} & \left[[\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^- + [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{W}_2\mathbf{B}\mathbf{X}_2^-\right]\boldsymbol{Y}_2 \\ & + \left\{-\mathbf{B}^-\mathbf{C} - [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-\mathbf{D} - [\mathbf{I} - \mathbf{B}^-\mathbf{B}][\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)\mathbf{D} \right. \\ & \left. - [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{W}_2(\mathbf{B}\mathbf{X}_2\mathbf{D} - \mathbf{C})\right\}\hat{\theta} + [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{W}_2\boldsymbol{a} - \mathbf{B}^-\boldsymbol{a} \\ = {} & (\mathbf{A}_1 + \mathbf{A}_2\mathbf{W}_1\mathbf{A}_3 + \mathbf{A}_2\mathbf{W}_2\mathbf{A}_4)\boldsymbol{Y}_2 + (\mathbf{A}_5 - \mathbf{A}_2\mathbf{W}_1\mathbf{A}_6 - \mathbf{A}_2\mathbf{W}_2\mathbf{A}_7)\hat{\theta} + \boldsymbol{c}, \end{aligned}$$

where

$$\mathbf{A}_1 = [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-, \quad \mathbf{A}_2 = [\mathbf{I} - \mathbf{B}^-\mathbf{B}], \quad \mathbf{A}_3 = \left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right],$$

$$\mathbf{A}_4 = \mathbf{B}\mathbf{X}_2^-, \quad \mathbf{A}_5 = \mathbf{B}^-\mathbf{C} - [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-\mathbf{D}, \quad \mathbf{A}_6 = (\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)\mathbf{D},$$

$$\mathbf{A}_7 = \mathbf{B}\mathbf{X}_2^-\mathbf{D} - \mathbf{C}, \quad \boldsymbol{c} = [\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{W}_2\boldsymbol{a} - \mathbf{B}^-\boldsymbol{a}.$$

In the following let $\operatorname{vec}(\mathbf{A}) = (\boldsymbol{a}_1', \dots, \boldsymbol{a}_n')'$, where $\mathbf{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ is an arbitrary $m \times n$ matrix; the symbol $\otimes$ will denote the tensor multiplication of matrices.

Thus

$$\begin{aligned}
&\phi(\mathbf{W}_1, \mathbf{W}_2) = \operatorname{Tr}\left[\mathbf{H}\operatorname{Var}(\tilde{\boldsymbol{\beta}})\right]\\
&= \operatorname{Tr}\left(\mathbf{H}\left(\mathbf{A}_1 + \mathbf{A}_2(\mathbf{W}_1, \mathbf{W}_2)\begin{pmatrix}\mathbf{A}_3\\ \mathbf{A}_4\end{pmatrix}\right)\Sigma_{22}\left(\mathbf{A}_1' + (\mathbf{A}_3', \mathbf{A}_4')\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\mathbf{A}_2'\right)\right)\\
&\quad + \operatorname{Tr}\left(\mathbf{H}\left(\mathbf{A}_5 - \mathbf{A}_2(\mathbf{W}_1, \mathbf{W}_2)\begin{pmatrix}\mathbf{A}_6\\ \mathbf{A}_7\end{pmatrix}\right)\Sigma_{\hat{\boldsymbol{\theta}}}\left(\mathbf{A}_5' - (\mathbf{A}_6', \mathbf{A}_7')\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\mathbf{A}_2'\right)\right)\\
&= \left(\operatorname{vec}\left(\Sigma_{22}\left(\mathbf{A}_1' + (\mathbf{A}_3', \mathbf{A}_4')\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\mathbf{A}_2'\right)\mathbf{H}\right)\right)'\operatorname{vec}\left(\mathbf{A}_1' + (\mathbf{A}_3', \mathbf{A}_4')\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\mathbf{A}_2'\right)\\
&\quad + \left(\operatorname{vec}\left(\Sigma_{\hat{\boldsymbol{\theta}}}\left(\mathbf{A}_5' - (\mathbf{A}_6', \mathbf{A}_7')\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\mathbf{A}_2'\right)\mathbf{H}\right)\right)'\operatorname{vec}\left(\mathbf{A}_5' - (\mathbf{A}_6', \mathbf{A}_7')\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\mathbf{A}_2'\right)\\
&= \left(\left[\operatorname{vec}(\mathbf{A}_1')\right]'(\mathbf{H} \otimes \Sigma_{22}) + \left(\operatorname{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\right)'\left((\mathbf{A}_2'\mathbf{H}) \otimes \left(\begin{pmatrix}\mathbf{A}_3\\ \mathbf{A}_4\end{pmatrix}\Sigma_{22}\right)\right)\right)\cdot\\
&\quad \cdot\left(\operatorname{vec}(\mathbf{A}_1') + \left[\mathbf{A}_2' \otimes (\mathbf{A}_3', \mathbf{A}_4')\right]\operatorname{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\right)\\
&\quad + \left(\left[\operatorname{vec}(\mathbf{A}_5')\right]'(\mathbf{H} \otimes \Sigma_{\hat{\boldsymbol{\theta}}}) - \left(\operatorname{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\right)'\left((\mathbf{A}_2'\mathbf{H}) \otimes \left(\begin{pmatrix}\mathbf{A}_6\\ \mathbf{A}_7\end{pmatrix}\Sigma_{\hat{\boldsymbol{\theta}}}\right)\right)\right)\cdot\\
&\quad \cdot\left(\operatorname{vec}(\mathbf{A}_5') - \left[\mathbf{A}_2 \otimes (\mathbf{A}_6', \mathbf{A}_7')\right]\operatorname{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\partial\phi(\mathbf{W}_1, \mathbf{W}_2)/\partial\operatorname{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\\
&= 2\left((\mathbf{A}_2'\mathbf{H}\mathbf{A}_2) \otimes \left(\begin{pmatrix}\mathbf{A}_3\\ \mathbf{A}_4\end{pmatrix}\Sigma_{22}(\mathbf{A}_3', \mathbf{A}_4') + \begin{pmatrix}\mathbf{A}_6\\ \mathbf{A}_7\end{pmatrix}\Sigma_{\hat{\boldsymbol{\theta}}}(\mathbf{A}_6', \mathbf{A}_7')\right)\right)\operatorname{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}\\
&\quad + 2\left((\mathbf{A}_2'\mathbf{H}) \otimes \left(\begin{pmatrix}\mathbf{A}_3\\ \mathbf{A}_4\end{pmatrix}\Sigma_{22}\right)\right)\operatorname{vec}(\mathbf{A}'_1)\\
&\quad - 2\left((\mathbf{A}_2'\mathbf{H}) \otimes \left(\begin{pmatrix}\mathbf{A}_6\\ \mathbf{A}_7\end{pmatrix}\Sigma_{\hat{\boldsymbol{\theta}}}\right)\right)\operatorname{vec}(\mathbf{A}_5').
\end{aligned}$$

The equation

$$\left(\frac{1}{2}\partial\phi(\mathbf{W}_1,\mathbf{W}_2)/\partial\,\mathrm{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}=\right)$$

$$\left((\mathbf{A}_2'\mathbf{H}\mathbf{A}_2)\otimes\left(\begin{pmatrix}\mathbf{A}_3\\ \mathbf{A}_4\end{pmatrix}\Sigma_{22}(\mathbf{A}_3',\mathbf{A}_4')+\begin{pmatrix}\mathbf{A}_6\\ \mathbf{A}_7\end{pmatrix}\Sigma_{\hat{\boldsymbol\theta}}(\mathbf{A}_6,\mathbf{A}_7)\right)\right)\mathrm{vec}\begin{pmatrix}\mathbf{W}'_1\\ \mathbf{W}'_2\end{pmatrix}$$
$$+\left((\mathbf{A}_2'\mathbf{H})\otimes\left(\begin{pmatrix}\mathbf{A}_3\\ \mathbf{A}_4\end{pmatrix}\Sigma_{22}+\begin{pmatrix}\mathbf{A}_6\\ \mathbf{A}_7\end{pmatrix}\Sigma_{\hat{\boldsymbol\theta}}\right)\right)\begin{pmatrix}\mathrm{vec}(\mathbf{A}_1')\\ -\mathrm{vec}(\mathbf{A}_5')\end{pmatrix}=\mathbf{O}$$

has obviously solution for $\mathrm{vec}\begin{pmatrix}\mathbf{W}_1'\\ \mathbf{W}_2'\end{pmatrix}$ and in this solution the function $\phi(\cdot)$ attains its minimum; the last equation can be rewritten in the form

$$\mathbf{U}_1(\mathbf{W}_1,\mathbf{W}_2)\begin{pmatrix}\mathbf{V}_1, & \mathbf{T}_1\\ \mathbf{V}_2, & \mathbf{T}_2\end{pmatrix}=(\mathbf{P}_1,\mathbf{P}_2)$$

given in the assertion of the theorem.

R e m a r k 2.9. Let $\mathcal{W}_{k_2,n_2}$ and $\mathcal{W}_{k_2,q}$, respectively, be spaces of matrices of the size $k_2\times n_2$ and $k_2\times q$, respectively. Let $\langle\mathbf{W}_1,\mathbf{V}_1\rangle=\mathrm{Tr}(\mathbf{W}_1'\mathbf{V}_1)$, $\mathbf{W}_1,\mathbf{V}_1\in\mathcal{W}_{k_2,n_2}$ and let $\langle\mathbf{W}_2,\mathbf{V}_2\rangle=\mathrm{Tr}(\mathbf{W}_2'\mathbf{V}_2)$, $\mathbf{W}_2,\mathbf{V}_2\in\mathcal{W}_{k_2,q}$. Let $\mathcal{S}_{k_2}$ be a space of $k_2\times k_2$ symmetric matrices with the inner product $\langle\mathbf{S}_1,\mathbf{S}_2\rangle=\mathrm{Tr}(\mathbf{S}_1\mathbf{S}_2)$, $\mathbf{S}_1,\mathbf{S}_2\in\mathcal{S}_{k_2}$. The image $\mathcal{C}_\phi$ of the cartesian product $\mathcal{W}_{k_2,n_2}\times\mathcal{W}_{k_2,q}$, when the function (cf. Lemma 2.3)

$$\begin{aligned}&\phi(\mathbf{W}_1,\mathbf{W}_2)\\ &=\mathrm{Var}\Big((\mathbf{I}-\mathbf{B}^-\mathbf{B})\big[\mathbf{X}_2^-+\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)+\mathbf{W}_2\mathbf{B}\mathbf{X}_2^-\big]\boldsymbol{Y}_2\\ &\quad+\big\{-\mathbf{B}^-\mathbf{C}+(\mathbf{I}-\mathbf{B}^-\mathbf{B})\big[-\mathbf{X}_2^-\mathbf{D}-\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)\mathbf{D}-\mathbf{W}_2\mathbf{B}\mathbf{X}_2^-\mathbf{D}+\mathbf{W}_2\mathbf{C}\big]\big\}\hat{\boldsymbol\theta}\\ &\quad+(\mathbf{I}-\mathbf{B}^-\mathbf{B})\mathbf{W}_2\boldsymbol{a}-\mathbf{B}^-\boldsymbol{a}\Big)\end{aligned}$$

is used, is a set included in the convex cone $\mathcal{C}$ of the positive semidefinite matrices in $\mathcal{S}_{k_2}$. Let $k_0=\min\{k:\ k\in\mathcal{R},\ \mathcal{V}_k\cap\mathcal{C}_\phi\neq\emptyset\}$, where

$$\mathcal{V}_k=k\mathbf{K}+\mathcal{K}_{\mathbf{H}}=\{k\mathbf{H}+\mathbf{K}:\ \mathbf{K}\in\mathcal{K}_{\mathbf{H}}\},$$
$$\mathcal{K}_{\mathbf{H}}=\{\mathbf{S}:\ \mathbf{S}\in\mathcal{S}_{k_2},\ \mathrm{Tr}(\mathbf{S}\mathbf{H})=0\}.$$

Then there exist matrices $\mathbf{W}_1^0$, $\mathbf{W}_2^0$ with property $\phi(\mathbf{W}_1^0,\mathbf{W}_2^0)\in\mathcal{V}_{k_0}\cap\mathcal{C}_\phi$ and these matrices are a solution of the equation (2.5).

By Theorem 2.8 the problem of a numerical determination of the $\mathbf{H}$-optimal estimator $\tilde{\boldsymbol{\beta}}$ is solved. Nevertheless a structure of such an estimator is not transparent. It would be useful sometimes to know an explicit expression of matrices $\mathbf{X}_2^-$, $\mathbf{B}^-$, $\mathbf{W}_1$, $\mathbf{W}_2$ for a given $\mathbf{H}$. Another problem can be formulated in such a way that the matrices $\mathbf{X}_2^-$, $\mathbf{B}^-$, $\mathbf{W}_1$, $\mathbf{W}_2$ are chosen and a matrix $\mathbf{H}$ is to be found with respect to which the estimator given by the matrices $\mathbf{X}_2^-$, $\mathbf{B}^-$, $\mathbf{W}_1$, $\mathbf{W}_2$ is $\mathbf{H}$-optimal. A partial answer to these questions is given by the following corollary.

COROLLARY 2.10.

a) *Let* $\mathbf{H} = \mathbf{BTB}'$, *where* $\mathbf{T}$ *is any symmetric* $k_2 \times k_2$ *matrix. Then any estimator* $\tilde{\boldsymbol{\beta}} \in \mathcal{U}_{\boldsymbol{\beta}}$ *is* $\mathbf{H}$*-optimal.*

b) *Let* $\mathbf{D} = \mathbf{O}$ *(a model from Corollary* 2.5*) and let*

$$\mathbf{H} = \mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2 \ \left(= \left\{\mathrm{Var}\left[(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{Y}_2\right]\right\}^{-1}\right).$$

Then

$$\begin{aligned}\tilde{\boldsymbol{\beta}} &= \left\{\mathbf{I} - (\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\left[\mathbf{B}(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\right]^{-1}\mathbf{B}\right\}(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{Y}_2 \\ &\quad - (\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\left[\mathbf{B}(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\right]^{-1}(\mathbf{C}\hat{\boldsymbol{\theta}} + \boldsymbol{a}) \\ &= \left(\mathbf{I} - \mathbf{B}^-_{m(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)}\mathbf{B}\right)\left[(\mathbf{X}_2')^-_{m(\Sigma_{22})}\right]'\mathbf{Y}_2 - \mathbf{B}^-_{m(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)}(\mathbf{C}\hat{\boldsymbol{\theta}} + \boldsymbol{a}) \in \tilde{\mathcal{U}}_{\boldsymbol{\beta}}\end{aligned}$$

is $\mathbf{H}$*-optimal. This assertion is proved by verifying that the choice*

$$\mathbf{X}_2^- = \left[(\mathbf{X}'_2)^-_{m(\Sigma_{22})}\right]', \qquad \mathbf{W}_1 = \mathbf{O},$$
$$\mathbf{B}^- = \mathbf{B}^-_{m(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)}, \qquad \mathbf{W}_2 = -\mathbf{B}^-_{m(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)}$$

satisfy (2.5).

c) *Let* $\mathbf{D} = \mathbf{O}$ *and* $\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2 = \sigma^2\mathbf{I}$. *Then the estimator* $\tilde{\boldsymbol{\beta}}$ *from* b) *minimizes a value of the quantity* $\mathrm{Tr}\left(\mathrm{Var}(\tilde{\boldsymbol{\beta}})\right)$, $\tilde{\boldsymbol{\beta}} \in \tilde{\mathcal{U}}_{\boldsymbol{\beta}}$.

Remark 2.11. In the case b) from Corollary 2.10 let the spectral decomposition of the matrix $(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2)^{-1}$ be $\mathbf{Q}_1\mathbf{D}_1\mathbf{Q}_1'$, $\mathbf{Q}_1\mathbf{Q}_1' = \mathbf{I}$, $\mathbf{D}_1 = \mathrm{Diag}(d_{1,11}, \dots, d_{1,k_2,k_2})$ and the spectral decomposition of the matrix $\mathbf{V} = \mathrm{Var}(\tilde{\boldsymbol{\beta}})$ be $\mathbf{V} = \mathbf{Q}_2\mathbf{D}_2\mathbf{Q}_2'$, $\mathbf{Q}_2\mathbf{Q}_2' = \mathbf{I}$, $\mathbf{D}_2 = \mathrm{Diag}(d_{2,11}, \dots, d_{2,k_2,k_2})$. Then $\mathrm{Tr}\left[\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2\mathbf{V}\right] = \mathrm{Tr}\left[\mathbf{D}_1^{-1}\mathbf{Q}_1'\mathbf{Q}_2\mathbf{D}_2(\mathbf{Q}_1'\mathbf{Q}_2)'\right]$. The matrix $\mathbf{Q}_1'\mathbf{Q}_2 = \mathbf{F}$ is orthogonal, $\mathbf{F} = (\boldsymbol{f}_1, \boldsymbol{f}_2, \dots, \boldsymbol{f}_k)$ and

$$\mathrm{Tr}(\mathbf{X}_2'\Sigma_{22}^{-1}\mathbf{X}_2\mathbf{V}) = \sum_{i=1}^{k_2}\sum_{j=1}^{k_2} f_{i,j}^2 d_{2,ii}/d_{1,jj},$$

where $f_{i,j} = \{\mathbf{f}_i\}_{j.}$.

R e m a r k 2.12. Since $\mathrm{Var}(\tilde{\beta})$ is a singular matrix for each $\tilde{\beta} \in \tilde{\mathcal{U}}_{\beta}$, the image $\phi(\mathcal{W}_{k_2,j_2} \times \mathcal{W}_{k_2,q})$ from Remark 2.9 is included into the boundary of the convex cone $\mathcal{C}$ (the boundary consists of positive semidefinite matrices which are not regular). If for a given $\tilde{\beta} \in \phi(\mathcal{W}_{k_2,n_2} \times \mathcal{W}_{k_2,q})$ a supporting hyperplane $\mathcal{H}$ of $\mathcal{C}$ at the point $\tilde{\beta}$ is constructed, then $\tilde{\beta}$ is $\mathbf{H}$-optimal with respect to any $\mathbf{H} \perp (\mathcal{H} - \tilde{\beta})$. Thus (2.5) can be solved with respect to $\mathbf{H}$ for any given $\mathbf{X}_2^-$, $\mathbf{B}^-$, $\mathbf{W}_1$, $\mathbf{W}_2$; in this case (2.4) can be rewritten as

$$[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}(\mathbf{K}_1, \mathbf{K}_2) = (\mathbf{O}, \mathbf{O}),$$

where

$$\begin{aligned}\mathbf{K}_1 = &(\mathbf{I} - \mathbf{B}^-\mathbf{B})[\mathbf{X}_2^- + \mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{B}\mathbf{X}_2^-][\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'] \cdot \\ &\cdot [\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'] + [\mathbf{B}^- - (\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{W}_2]\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'],\end{aligned}$$

$$\begin{aligned}\mathbf{K}_2 = &(\mathbf{I} - \mathbf{B}^-\mathbf{B})[\mathbf{X}_2^- + \mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{B}\mathbf{X}_2^-][\Sigma_{22} + \mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'] \cdot \\ &\cdot (\mathbf{X}_2^-)'\mathbf{B}' - [\mathbf{B}^- - (\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{W}_2]\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' \\ &+ [\mathbf{B}^- - (\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{W}_2]\mathbf{C}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'(\mathbf{X}_2^-)'\mathbf{B}' \\ &- (\mathbf{I} - \mathbf{B}^-\mathbf{B})[\mathbf{X}_2^- + \mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{B}\mathbf{X}_2^-]\mathbf{D}(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'.\end{aligned}$$

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