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# ON DISTRIBUTION OF SEQUENCES OF INTEGERS

MILAN PAŠTÉKA *) — ŠTEFAN PORUBSKÝ **) [1]

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ABSTRACT. In the paper we introduce on sequences of integers a notion analogical to that of distribution function. Due to some topological peculiarities of the set of integers we shall study more general notions of distribution measure and distribution density.

## 1. Introduction

In 1916 Herman Weyl in his famous paper [12] introduced the notion of uniformly distributed sequences of real numbers modulo 1. This notion was subsequently generalized in various ways. One of them stems from I. Niven [6] who in 1961 introduced the notion of uniform distribution of integers. Another generalization can be done via the notion of asymptotic distribution function mod 1 which was initiated by Schoenberg [11].

The aim of this paper is to join this two approaches. However the structure of positive integers gives us small space for the study of "distribution functions". Therefore we shall use a more general notion of distribution measure and we shall investigate the sequences of integers from the point of view of the uniform distribution in compact spaces of so called polyadic numbers. This is a generalization of Meijer's method [5], [4] in the space of $g$-adic numbers. Nevertheless, instead of "distribution function" we shall use "$\Gamma$-distribution".

In the first chapter we prove an existence theorem (Theorem 1), which is an analog of the existence theorem known for the distribution function. Then we shall focus on the properties of $\Gamma$-distributed sequences and the distribution measures. The principal result is given in chapter 3 which establishes (Theorem 3) that an integer sequence is $\Gamma$-distributed if and only if it is uniformly distributed in the space of polyadic numbers with respect to distribution measures; (Corollary 3 of Th. 6) any integer polynomial sequence is distributed. In chapter 4 we collected technical results.

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In the sequel we will denote by $\mathbb{N}$ the set of all nonnegative integers and by $\mathbb{R}$ the set of all real numbers.

## 2. $\Gamma$-distribution

The following sequences $\Gamma$ will be of fundamental importance for us. Let

$$\Gamma = \big\{h(j,m):\ 0 \le j < m,\ \ m \in \mathbb{N} \setminus \{0\}\big\} \tag{1}$$

be a system of nonnegative real numbers. Given a sequence $w = \big\{w(n)\big\}$ of positive integers, a sequence $\big\{w(n)\big\}$ will be called $\Gamma$-*distributed* if and only if

$$\lim_{N\to\infty} \frac{\#\big\{n \le N\,;\ \ w(n) \equiv j \pmod{m}\big\}}{N} = h(j,m) \tag{2}$$

for every $j, m \in \mathbb{N}$ and $0 \le j < m$.

The first natural question: *Which conditions on the system $\Gamma$ do guarantee the existence of a $\Gamma$-distributed sequence $\big\{w(n)\big\}$?*

Let us denote

$$j + \langle m \rangle = \{j + km\,;\ \ k = 0,1,2,\dots\}$$

for $j, m \in \mathbb{N}$ and $0 \le j < m$. If we write $j + \langle m \rangle$, we shall always tacitly suppose that $m \ne 0$ and instead of $0 + \langle d \rangle$ we shall simply write $\langle d \rangle$.

Further, for any set $S \subset \mathbb{N}$ and $N \in \mathbb{N}$ let

$$A(N,w,S) = \#\big\{n \le N\,;\ \ w(n) \in S\big\}.$$

Clearly, for two disjoint sets $S_1, S_2 \subset \mathbb{N}$ we have

$$A(N,w,S_1 \cup S_2) = A(N,w,S_1) + A(N,w,S_2)\,, \tag{3}$$

and for every $N$, $j$, $m$ we have

$$\#\big\{n \le N\,;\ \ w(n) \equiv j \pmod{m}\big\} = A\big(N,w,j + \langle m \rangle\big)\,.$$

Since every set $j+\langle m \rangle$ can be represented in the form of a disjoint decomposition

$$j + \langle m \rangle = \bigcup_{r=0}^{k-1} j + rm + \langle km \rangle \tag{4}$$

for arbitrary $k \in \mathbb{N}$ and $A(N,w,\mathbb{N}) = N$, the existence of a $\Gamma$-distributed sequence implies that

$$h(0,1) = 1 \tag{5}$$

and

$$h(j,m) = \sum_{r=0}^{k-1} h(j+rm, km) \tag{6}$$

for every $k, m \in \mathbb{N}$, $0 \le j < m$. In other words: if $m \mid m_1$, then

$$h(j,m) = \sum_{r=0}^{\frac{m_1}{m}-1} h(j+rm, m_1)\,. \tag{7}$$

Thus (5) and (6) are necessary conditions for the existence of a $\Gamma$-distributed sequence.

A system $\Gamma$ satisfying conditions (5) and (6) will be called a *distribution.* The fact that the distribution properties are also sufficient for the existence of a $\Gamma$-distributed sequences will follow from Theorem 1.

However before stating this theorem, we have to recall some preliminaries from the theory of the uniform distribution in compact spaces. Various parts presented here can be found in monograph [3, Chapter 3].

Let $X$ be a compact separable Hausdorff space with a probability measure $P$ which is a regular normed Borel measure in $X$. A sequence $\{x_n\}$ of elements in $X$ will be called *$P$-uniformly distributed in $X$* (or shortly $P$-u.d. in $X$) if

$$\lim_{N\to\infty} \frac{1}{n} \sum_{n=1}^{N} f(x_n) = \int f \,\mathrm{d}P$$

for every continuous functions $f\colon X \to \mathbb{R}$ (c.f. [3, p. 171]).

The next three results form a springboard for us:

**THEOREM A.** ([3, p. 175]) *A sequence $\{x_n\}$ is $P$-u.d. in $X$ if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\le N\\ x_n\in M}} 1 = P(M)$$

*for every Borel set $M$ with $P\big(\overline{M} \setminus \operatorname{Int}(M)\big) = 0$.*

**THEOREM B.** ([3, p. 183]) *Let $S$ be the set of all $P$-u.d. sequences in $X$ viewed as a subset of $X^\infty$. Let $P_\infty$ be the product measure on $X^\infty$. Then $P_\infty(S) = 1$.*

**COROLLARY.** *There exists at least one sequence of elements in $X$ which is $P$-u.d.*

In 1962 E. V. Novoselov [7] introduced the space $\Omega$ of polyadic numbers in the following way: For every $n \in \mathbb{N}$, let $\varphi_n$ be the indicator of positive integers not divisible by $n$, i.e.

$$\varphi_n(m) = \begin{cases} 1 & \text{if } n \nmid m, \\ 0 & \text{if } n \mid m \end{cases}$$

for $n = 1, 2, \dots$. Then the function

$$d(x,y) = \sum_{n=1}^{\infty} \frac{\varphi_n(x-y)}{2^n}$$

is a metric on $\mathbb{N}$. Unfortunately, the set $\mathbb{N}$ endowed with this metric is not a complete metric space, it is only relatively compact. The completion $\Omega$ of $(\mathbb{N}, d)$ is called the space of *polyadic* numbers. The set $\Omega$ is obviously a compact metric space with respect to the extension of the metric $d$ on $\Omega$. Moreover, the operations of addition and multiplication can be in a natural way extended from $\mathbb{N}$ to continuous operations on $\Omega$. Both for these operations and their extensions we shall use the standard symbols $+$, $\cdot$.

Let us remark that probably the polyadic numbers are the original result of Prüfer's work [10]. For a general survey of polyadic numbers, we refer to [9, Chapter 3.5].

The next result shows that also the process of division with remainder can be extended in a natural way to the whole $\Omega$.

**THEOREM C.** ([7]) *For every $\alpha \in \Omega$ and $m \in \mathbb{N} \setminus \{0\}$ there exist uniquely determined elements $\beta \in \Omega$ and $j \in \mathbb{N}$ such that $0 \le j < m$ and*

$$\alpha = m \cdot \beta + j .$$

The extension of the metric $d$ to the whole $\Omega$ is also given by the same rule

$$d(\alpha,\beta)=\sum_{n=1}^{\infty}\frac{\varphi_n(\alpha-\beta)}{2^n} \tag{8}$$

for $\alpha,\beta\in\Omega$, where again for $\gamma\in\Omega$

$$\varphi_n(\gamma)=\begin{cases}1 & \text{if } n\nmid\gamma,\\ 0 & \text{if } n\mid\gamma.\end{cases}$$

Let for $\gamma\in\Omega$ symbol $(\gamma)$ denote the principal ideal generated by $\gamma$, and $\alpha+(\gamma)$ the residue class containing the element $\alpha$. Theorem C implies that for every $m\in\mathbb{N}$ the set $\Omega$ can be represented in the form of the disjoint decomposition

$$\Omega=\bigcup_{j=0}^{m-1} j+(m)\,. \tag{9}$$

The set $j+(m)$ is closed (see [7]), but (9) implies that it is simultaneously also an open set. Moreover it follows from (8) that the system

$$\{j+(m)\,;\ 0\le j<m\,,\ \ m\in\mathbb{N}\}$$

forms a base of open sets in $\Omega$.

Put

$$\mathcal{D}=\left\{\bigcup_{i=1}^{k} l_i+(m_i)\,;\ 0\le l_i<m_i\,,\ \ m_i,k\in\mathbb{N}\right\}\cup\{\emptyset\}\,.$$

It is easy to see that $\mathcal{D}$ is an algebra of sets. Our next aim is to give a construction for measures on $\mathcal{D}$ which depend upon our system (1) of nonnegative real numbers. To do this we shall need the following two lemmas:

**Lemma 1.** *Let* $\Gamma$ *satisfy* (6). *If*

$$l+(m)=\bigcup_{i=1}^{k} l_i+(m_i) \tag{10}$$

*is a disjoint decomposition, then*

$$h(l,m)=\sum_{i=1}^{k} h(l_i,m_i)\,.$$

P r o o f. From (10) we have

$$\frac{1}{m} = \frac{1}{m_1} + \cdots + \frac{1}{m_k}.$$

If $M = [m, m_1, \ldots, m_k]$, then for every $i = 1, 2, \ldots, k$ we have

$$l_i + (m_i) = \bigcup_{p=0}^{\frac{M}{m_i}-1} l_i + pm_i + (M)$$

and according to (7)

$$h(l_i, m_i) = \sum_{p=0}^{\frac{M}{m_i}-1} h(l_i + pm_i, M). \tag{11}$$

Therefore the set $l + (m)$ can be represented as a disjoint decomposition of the form

$$l + (m) = \bigcup_{i=1}^{k} \bigcup_{p=0}^{\frac{M}{m_i}-1} l_i + pm_i + (M).$$

Since $0 \le l_i + pm_i < M$ for all $i, p$, $0 \le p < \frac{M}{m_i}$,

$$A = \left\{ l_i + pm_i ;\ 0 \le p < \frac{M}{m_i},\ i = 1, 2, \ldots k \right\}$$
$$\subset \left\{ l, l + m, \ldots, l + \left(\frac{M}{m} - 1\right) m \right\} = B.$$

To show that the sets $A$, $B$ coincide, it is enough to show that both have the same number of elements. The set $B$ has obviously $\frac{M}{m}$ elements. On the other hand, since the sets $l_i + pm_i + (M)$ for $i = 0, \ldots, \frac{M}{m_i} - 1$ and $i = 1, 2, \ldots, k$ are mutually disjoint, the numbers $l_i + pm_i$ for $p = 0, \ldots, \frac{M}{m_i} - 1$, $i = 1, 2, \ldots, k$ are distinct. Therefore the set $A$ has

$$\frac{M}{m_1} + \cdots + \frac{M}{m_k} = M\left(\frac{1}{m_1} + \cdots + \frac{1}{m_k}\right) = \frac{M}{m}$$

elements, as the set $B$ has, and consequently $A = B$. Now (11) and (7) imply that

$$\sum_{i=1}^{k} h(l_i, m_i) = \sum_{i=1}^{k} \sum_{p=0}^{\frac{M}{m_i}-1} h(l_i + pm_i, M) = \sum_{l' \in A} h(l', M) = h(l, m),$$

and the proof is complete.

**Lemma 2.** *Let $l_1+(m_1),\dots,l_k+(m_k)$ and $l'_1+(m'_1),\dots,l'_r+(m'_r)$ be two systems of pairwise disjoint sets and let $h$ satisfy* (6). *If*

$$\bigcup_{i=1}^{k} l_i+(m_i)=\bigcup_{j=1}^{r} l'_j+(m'_j)\,,$$

*then*

$$\sum_{i=1}^{k} h(l_i,m_i)=\sum_{j=1}^{r} h(l'_j,m'_j)\,.$$

Proof. We will proceed by induction on $k$. The case $k=1$ is covered by Lemma 1. Let the induction hypothesis be true for $k-1$ and take

$$\bigcup_{i=1}^{k} l_i+(m_i)=\bigcup_{j=1}^{r} l'_j+(m'_j)\,.$$

Splitting every class $l'_j+(m'_j)$ into $m_k$ disjoint classes modulo $m_k m'_j$ we obtain a new system on the right-hand side, e.g.

$$\bigcup_{i=1}^{k} l_i+(m_i)=\bigcup_{j=1}^{rm_k} l''_j+(m''_j)\,, \tag{12}$$

where all $m''_j$ are multiples of $m_k$. Thus every class on the right-hand side of (12) is either a subset of $l_k+(m_k)$ or it is disjoint with it. Rearrange the sets on the right-hand side of (12) in such a way that $r_1$ is the maximal integer sharing the property

$$1\le j\le r_1 \implies l_k+(m_k)\cap l''_j+(m''_j)\neq\emptyset\,.$$

Then the disjointness property and $k>1$ imply that

$$\bigcup_{i=1}^{k-1} l_i+(m_i)=\bigcup_{j=r_1+1}^{rm_k} l''_j+(m''_j)$$

together with

$$l_k+(m_k)=\bigcup_{j=1}^{r_1} l''_j+(m''_j)\,.$$

Now the induction hypothesis and Lemma 1 imply

$$\sum_{i=1}^{k-1} h(l_i, m_i) = \sum_{j=r_1+1}^{rm_k} h(l''_j, m''_j)$$

and

$$h(l_k, m_k) = \sum_{j=1}^{r_1} h(l''_j, m''_j).$$

Now it is enough to add these two equalities and to use (7) to finish the proof.

Lemma 2 provides a tool for introducing a measure on $\mathcal{D}$.

Every set from $\mathcal{D}$ can be represented in a form of a disjoint decomposition, e.g. (4). If $H \in \mathcal{D}$ has such a disjoint decomposition

$$H = \bigcup_{i=1}^{k} \ell_i + (m_i),$$

we put

$$\Delta_\Gamma(H) = \sum_{i=1}^{k} h(l_i, m_i).$$

Lemma 2 and compactness of $\Omega$ imply that $\Delta_\Gamma(H)$ is a measure on $\mathcal{D}$. If

$$P^\star_\Gamma(S) = \inf\left\{ \sum_{i=1}^{\infty} \Delta_\Gamma(H_i)\,;\ S \subset \bigcup_{i=1}^{\infty} H_i\,,\ H_i \in \mathcal{D} \right\}$$

for $S \subset \Omega$, then $P^\star_\Gamma$ is an outer measure on $2^\Omega$. Moreover, the subsystem

$$\mathcal{D}_\Gamma = \bigl\{ S \subset \Omega;\ P^\star_\Gamma(S) + P^\star_\Gamma(\Omega \setminus S) = 1 \bigr\}$$

is a $\sigma$-algebra and the contraction

$$P_\Gamma = P^\star_\Gamma |_{\mathcal{D}_\Gamma}$$

is a measure on the $\sigma$-algebra $\mathcal{D}_\Gamma$ (see [2]). For $j, m \in \mathbb{N}$, $0 \le j < m$ we plainly have

$$P^\star_\Gamma\bigl(j + (m)\bigr) = h(j, m). \tag{13}$$

Therefore $j + (m) \in \mathcal{D}_\Gamma$. Thus the fact that $\bigl\{ j + (m);\ j, m \in \mathbb{N} \bigr\}$ is the base of open sets on $\Omega$ implies that $P_\Gamma$ is a Borel regular probability measure on $\Omega$.

Now we are in the position to state the announced Theorem 1:

**THEOREM 1.** *For every distribution* $\Gamma$ *there exists a sequence* $\{w(n)\}$ *of positive integers which is* $\Gamma$*-distributed.*

Proof. By Corollary 1 of Theorem B there exists a sequence $\{\alpha_n\}$, $\alpha_n \in \Omega$, $n = 1, 2, \dots$, which is $P_\Gamma$-uniformly distributed in $\Omega$. The set $\mathbb{N}$ is dense in $\Omega$, thus there exists a sequence of positive integers $\{x_n\}$ such that $d(x_n, \alpha_n) \to 0$ for $n \to \infty$. If $g$ is a continuous function on $\Omega$ it is also uniformly continuous. Thus

$$\lim_{n\to\infty} |g(x_n) - g(\alpha_n)| = 0 .$$

Therefore

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(x_n) = \int g \, \mathrm{d}P_\Gamma .$$

This equation also shows that $\{x_n\}$ is a uniformly distributed sequence in $\Omega$.

Theorem C now implies for $x \in \mathbb{N}$:

$$x \equiv j \pmod m \iff x \in j + (m) .$$

If $w(n) = x_n$ for $n = 1, 2, \dots$, then

$$A\big(N, w, j + \langle m\rangle\big) = \sum_{\substack{n \le N \\ x_n \in j+(m)}} 1 . \tag{14}$$

Since $\overline{j+(m)} = \operatorname{Int}\big(j+(m)\big)$, $\overline{j+(m)} \setminus \operatorname{Int}\big(j+(m)\big) = \emptyset$. Thus from (13), (14) and Theorem A we have

$$\lim_{N\to\infty} \frac{A\big(N, w, j + \langle m\rangle\big)}{N} = h(j, m) ,$$

and the proof is complete.

The measure $P_\Gamma$ will be called a *distribution measure* of the sequence $\{w(n)\}$.

We now give two examples of distributions:

Example 1. Let $\sum\limits_{n=1}^{\infty} a_n = 1$ be an infinite series with nonnegative elements. It is a routine task to show that the values

$$h(j, m) = \sum_{n \equiv j \pmod m} a_n$$

satisfy conditions (5) and (6) and therefore the system $\Gamma$ consisting from these values is a distribution.

Example 2. Let $B_n(x)$, $n = 1, 2, \dots$ be the $n$-th Bernoulli polynomial defined by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi .$$

It is known that in the class of all polynomials $B(x)$ the Bernoulli polynomial $B_n(x)$ together with the value of $B(0)$ are completely characterized by the difference relations

$$B(x+1) - B(x) = nx^{n-1}$$

and the interpolation equation

$$B(mx) = m^{m-1} \sum_{k=0}^{m-1} B\Big(x + \frac{k}{m}\Big) .$$

If we define

$$h(j, m) = m^{s-1} B_s\Big(\frac{j}{m}\Big)$$

for every fixed $s = 0, 1, 2, \dots$ , then

$$\sum_{r=0}^{m-1} h(j + rm, km) = h(j, m) .$$

In particular for $s = 0$ we have

$$h(j, m) = m^{-1} B_0\Big(\frac{j}{m}\Big) = \frac{1}{m}$$

for $B_0(x) = 1$.

This together implies that every triangular matrix (1) which entries satisfy relations (6) and are given through polynomials has apart from the value $B_s(0)$ the form

$$\Gamma = \Big\{ m^{s-1} B_s\Big(\frac{j}{m}\Big) ; \ 0 \le j < m , \ \ m = 1, 2, \dots \Big\}$$

for every given $s = 0, 1, 2, \dots$ .

Note that this example is within the scope of the previous one because of the connection with the partial zeta function. For if, $0 \le x < m$ and $x$ is coprime with $m$, we define

$$\zeta(x,t) = \sum_{n \equiv x \pmod m} n^{-t}.$$

Then this Dirichlet series converges only for $\mathrm{Re}(t) > 1$, but it can be analytically continued to the whole complex plane and Hurwitz has shown that

$$\zeta(x, 1-s) = -\frac{1}{s} m^{s-1} B_s\Big(\frac{x}{m}\Big).$$

As an application of Theorem 1 to Buck's measure density $\mu^\star$ (see [1]) we have:

**THEOREM 2.**

a) *If $h(j,m) > 0$, for all $j, m \in \mathbb{N}$, $0 \le j < m$, then*

$$\mu^\star\big(\{w(n);\ n = 1,2,\dots\}\big) = 1.$$

b) *If $S \subset \mathbb{N}$ and $\mu^\star(S) = 1$, then $S$ can be rearranged into a $\Gamma$-distributed sequence.*

## 3. Transformations of $\Gamma$-distributed sequences

In this part we shall study properties of transformations of sequences which are related to preservation of $\Gamma$-distribution of sequences of positive integers. Therefore let $\omega(n)$ be a sequence of positive integers. A mapping $F\colon \mathbb{N} \to \mathbb{N}$ will be called $\Gamma$*-preserving* if for each $\Gamma$-distributed sequence $\{\omega(n)\}$ also the sequence $\{F(\omega(n))\}$ is $\Gamma$-distributed. Since one sequence can be $\Gamma$-distributed for a distribution $\Gamma$ and not $\Gamma_1$-distributed for $\Gamma_1 \ne \Gamma$, we introduce the following notion:

A sequence $\{\omega(n)\}$ will be called *distributed* if it is $\Gamma$-distributed for some distribution $\Gamma$. A mapping $F$ will be called *distribution preserving* (or shortly d. preserving) if for every distributed sequence $\{\omega(n)\}$ also the sequence $F(\omega(n))$ is distributed.

Further we shall use the notion of a polyadically continuos mapping. A mapping $F$ will be called *polyadic continuous* (shortly p.c.) if and only if

$$\forall M\ \exists N\ \forall u_1, u_2;\ u_1 \equiv u_2 \pmod{N!} \implies F(u_1) \equiv F(u_2) \pmod{M!}.$$

It is easy to see that such p.c. mapping can be extended to a uniformly continuous mapping

$$F : \Omega \to \Omega .$$

It follows from (8) that a function $g : \Omega \to R$ is continuous if and only if

$$\forall \varepsilon > 0 \ \exists N \ \forall \alpha_1, \alpha_2 \in \Omega ; \quad \alpha_1 \equiv \alpha_2 \pmod{N!} \implies |g(\alpha_1) - g(\alpha_2)| < \varepsilon .$$

Thus the system of such periodic real valued functions whose periods are positive integers is dense in the space $C(\Omega)$ of all continuous real valued functions on $\Omega$. Analogically the system of all indicators of remainder classes $j + (m)$ is linearly dense in $C(\Omega)$. Since every periodic real valued function, with positive integeral period is continuous, Theorem A implies:

**THEOREM 3.** *A sequence $\{w(n)\}$ is $\Gamma$-distributed if and only if it is $P_\Gamma$-uniformly distributed in $\Omega$.*

We shall use this fact in proving some properties of transformations of sequences.

**THEOREM 4.** *A p.c. mapping $F$ is $\Gamma$-preserving if and only if for every real valued continuous function $g$ on $\Omega$*

$$\int g \, \mathrm{d}P_\Gamma = \int g \circ F \, \mathrm{d}P_\Gamma .$$

Proof. Let $\{w(n)\}$ be a $\Gamma$-distributed sequence. The functions $g$ and $g \circ F$ are continuous and thus

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g\big(w(n)\big) = \int g \, \mathrm{d}P_\Gamma$$

and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g\big(F\big(w(n)\big)\big) = \int g \circ F \, \mathrm{d}P_\Gamma .$$

This yields the assertion.

**THEOREM 5.** *A p.c. mapping $F$ is $\Gamma$-preserving if and only if there exist at most one $\Gamma$-distributed sequence $\{w(n)\}$ such that $\{F(w(n))\}$ is $\Gamma$-distributed.*

P r o o f . The necessary condition is trivial. For the sufficient part let $\{w(n)\}$ be such a sequence. Then

$$\int g \,\mathrm{d}P_\Gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g\big(w(n)\big) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g\big(F\big(w(n)\big)\big) = \int g \circ F \,\mathrm{d}P_\Gamma$$

and the proof is complete.

A special case of $\Gamma$-distribution is the uniform distribution in $\mathbb{Z}$ ([6]), in which case $h(j,m) = \dfrac{1}{m}$ for all $h(j,m) \in \Gamma$.

The sequence $\{n\}$ is uniformly distributed in $\mathbb{Z}$, therefore we have:

**COROLLARY 1.** *A p.c. mapping $F$ is u.d. preserving if and only if the sequence $\{F(n)\}$ is uniformly distributed in $\mathbb{Z}$.*

**COROLLARY 2.** *Let $F\colon \mathbb{N} \to \mathbb{N}$ be a p.c. mapping. If there exists $x$ such that the sequence of iterations $\{x, F(x), F\big(F(x)\big), \dots\}$ is $\Gamma$-distributed then $F$ is $\Gamma$-preserving.*

In the sequel we shall need the following known result:

**THEOREM D.** [2, p. 384] *Let $X$ be a compact topological space. If $\varphi$ is a nonnegative linear functional on $C(X)$ (the space of all continuous real valued functions), then there exists a unique regular Borel measure $P$ such that for every $g \in C(X)$ we have*

$$\varphi(g) = \int g \,\mathrm{d}P .$$

**THEOREM 6.** *Each p.c. mapping $F\colon \mathbb{N} \to \mathbb{N}$ is d. preserving.*

P r o o f . Let $\{w(n)\}$ be a $\Gamma$-distribution sequence. If $g\colon \Omega \to \mathbb{R}$ is a continuous function, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g\big(F\big(w(n)\big)\big) = \int g \circ F \,\mathrm{d}P_\Gamma .$$

If we put

$$\varphi(g) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g\big(F\big(w(n)\big)\big) ,$$

then Theorem D implies the existence of a regular Borel measure $P$ such that

$$\varphi(g) = \int g \, \mathrm{d}P .$$

The system $\Gamma_1 = \{P(j + (m)); \ 0 \le j < m, \ m \in \mathbb{N}\}$ obviously forms a distribution and the regularity of $P$ implies that $P = P_{\Gamma_1}$. Theorem 3 implies that $\{F(w(n))\}$ is a $\Gamma_1$-distributed sequence and the proof is complete.

Theorem 6 implies the following corollaries:

**COROLLARY 1.** *If $F\colon \mathbb{N} \to \mathbb{N}$ is a p.c. mapping, then $\{F(n)\}$ is a distributed sequence.*

**COROLLARY 2.** *If $F\colon \mathbb{N} \to \mathbb{N}$ is a such mapping that for $m, n \in \mathbb{N}$ we have*

$$(m - n) \mid (F(m) - F(n)) ,$$

*then $\{F(n)\}$ is a distributed sequence.*

**COROLLARY 3.** *If $F(x)$ is a polynomial with nonnegative integeral coefficients, then $\{F(n)\}$ is a distributed sequence.*

## 4. Continuos functions on $\Omega$

In this chapter we will study continuous functions on $\Omega$ with relationship to the $\Gamma$-distribution, or to the distribution measure. We shall use the following notions:

Let $m \in \mathbb{N}$, $S \subset \Omega$. A set $S_1 \subset S$ will be called a set of *representatives* of $S$ modulo $m$ if

(i) $\forall \alpha \in S \ \exists \alpha_1 \in S_1 ; \ \alpha_1 \equiv \alpha \pmod{m}$ ,
(ii) $\forall \alpha_1, \alpha_2 \in S_1 ; \ \alpha_1 \equiv \alpha_2 \pmod{m} \implies \alpha_1 = \alpha_2$ .

Let $\{B_n\}$ be a sequence of positive integers. We say that $\{B_n\}$ is *complete* if

(iii) $\forall d \in \mathbb{N} \ \exists n_0 ; \ \forall n \ge n_0 ; \ d \mid B_n$ ,
(iv) $B_{n-1} \mid B_n$ , $n = 2, 3, \dots$ .

The equation (8) implies that the function

$$g\colon \Omega \to \mathbb{R}$$

is continuous if and only if

$$\forall \varepsilon > 0 \ \exists n ; \ \forall \alpha_1, \alpha_2 \in \Omega ; \ \alpha_1 \equiv \alpha_2 \pmod{B_n} \implies |g(\alpha_1) - g(\alpha_2)| < \varepsilon .$$

Denote by $\alpha \bmod m$, $\alpha \in \Omega$, $m \in \mathbb{N}$ the remainder of $\alpha$ after division by $m$ (see Theorem C).

**THEOREM 7.** *Let $g\colon \Omega \to R$ be a continuous function and $S \subset \Omega$ be a closed set. If $\{B_n\}$ is a complete sequence and $S_n$ is a set of representatives of $S$ modulo $B_n$, then*

$$\int_S g \,\mathrm{d}P_\Gamma = \lim_{n\to\infty} \sum_{\alpha\in S_n} g(\alpha) h(\alpha \bmod B_n,\, B_n)\,.$$

P r o o f. The set $S$ is closed. Thus

$$S = \bigcap_{n=1}^{\infty} S + (B_n)\,.$$

Since the measure is quasicontinuous from above,

$$\int_S g \,\mathrm{d}P_\Gamma = \lim_{n\to\infty} \int_{S+(B_n)} g \,\mathrm{d}P_\Gamma\,.$$

$S$ is also a compact set, therefore $g$ is a uniformly continuous on $S$. Consequently for every $\varepsilon > 0$ there exists $n_0$ such that for $n \ge n_0$

$$\alpha \equiv \beta \pmod{B_n} \implies |g(\alpha) - g(\beta)| < \varepsilon \tag{15}$$

for $\alpha, \beta \in S$.

Clearly

$$\int_{S+(B_n)} g \,\mathrm{d}P_\Gamma = \sum_{\alpha\in S_n} \int_{\alpha+(B_n)} g \,\mathrm{d}P_\Gamma. \tag{16}$$

If

$$g(\alpha_n') = \max\{g(\beta)\,;\ \beta \in \alpha + (B_n)\}\,,$$
$$g(\alpha_n'') = \min\{g(\beta)\,;\ \beta \in \alpha + (B_n)\}\,,$$

then (15) implies

$$|g(\alpha_n') - g(\alpha_n'')| < \varepsilon\,.$$

Since

$$g(\alpha_n'') \le g(\beta) \le g(\alpha_n')\,,$$

$$g(\alpha_n'') h(\alpha \bmod B_n, B_n) \le \int_{\alpha+(B_n)} g \,\mathrm{d}P_\Gamma \le g(\alpha_n') h(\alpha \bmod B_n, B_n)\,.$$

In particular, for $\beta = \alpha$ we also have

$$g(\alpha_n'')h(\alpha \bmod B_n, B_n) \leq g(\alpha)h(\alpha \bmod B_n, B_n) \leq g(\alpha_n')h(\alpha \bmod B_n, B_n).$$

Thus, for $n \geq n_0$, we obtain

$$\Big| \int\limits_{\alpha+(B_n)} g \, \mathrm{d}P_\Gamma - g(\alpha)h(\alpha \bmod B_n, B_n) \Big| \leq \varepsilon \cdot h(\alpha \bmod B_n, B_n).$$

This, together with the relation (16) leads to the

$$\Big| \int\limits_{S+(B_n)} \cdot g \, \mathrm{d}P_\Gamma - \sum_{\alpha \in S_n} g(\alpha)h(\alpha \bmod B_n, B_n) \Big| < \varepsilon$$

for $n \geq n_0$. The proof is complete.

If $S \subset \Omega$ is a closed set, then its characteristic function $\chi_S$ is continuous and Theorem 7 implies

$$P_\Gamma(S) = \lim_{n \to \infty} \sum_{\alpha \in S_n} h(\alpha \bmod B_n, B_n).$$

**COROLLARY 1.** *Let $g \colon \Omega \to \mathbb{R}$ be a continuous function and $\{B_n\}$ be a complete sequence. Then*

$$\int g \, \mathrm{d}P_\Gamma = \lim_{n \to \infty} \sum_{j=0}^{B_n - 1} g(j)h(j, B_n).$$

## 5. Distribution density

For $S \subset \mathbb{N}$ we define

$$\mu_\Gamma^*(S) = P_\Gamma(\overline{S}).$$

The following assertion shows that $\mu_\Gamma^*$ has the properties of the so called strong submeasure.

**THEOREM 8.** *For every $S_1, S_2 \subset \mathbb{N}$ we have*

(v) $S_1 \subset S_2 \implies \mu_\Gamma^*(S_1) \leq \mu_\Gamma^*(S_2)$,
(vi) $\mu_\Gamma^*(S_1 \cup S_2) + \mu_\Gamma^*(S_1 \cap S_2) \leq \mu_\Gamma^*(S_1) + \mu_\Gamma^*(S_2)$,
(vii) $\mu_\Gamma^*(\mathbb{N}) = 1$.

Proof. Properties (v) and (vii) are trivial. Therefore we prove only (vi). Clearly

$$\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}.$$

Thus

$$\mu^*_\Gamma(S_1 \cup S_2) + P_\Gamma(\overline{S_1} \cap \overline{S_2}) = \mu^*_\Gamma(S_1) + \mu^*_\Gamma(S_2).$$

But

$$\overline{S_1 \cap S_2} \subset \overline{S_1} \cap \overline{S_2},$$

and so

$$\mu^*_\Gamma(S_1 \cap S_2) \le P_\Gamma(\overline{S_1} \cap \overline{S_2})$$

and (vi) follows.

The set $S \subset \mathbb{N}$ will be called $\Gamma$*-measurable* if

$$\mu^*_\Gamma(S) + \mu^*_\Gamma(\mathbb{N} \setminus S) = 1.$$

Denote by $\mathcal{A}_\Gamma$ the system of all $\Gamma$-measurable sets. Theorem 8 implies that $\mathcal{A}_\Gamma$ is an algebra of sets and that the restriction

$$\mu_\Gamma = \mu^*_\Gamma|\mathcal{A}_\Gamma$$

is a finitely additive measure on $\mathcal{A}_\Gamma$. The submeasure $\mu^*_\Gamma$ will be called the distribution density.

If $S \subset \mathbb{N}$ and $m \in \mathbb{N}$, then put

$$H(S,m) = \sum_{j \in S \mod m} h(j,m).$$

Since $S \mod m = \overline{S} \mod m$, Theorem 7 gives:

**THEOREM 9.** *If $\{B_n\}$ is a complete sequence of positive integers, then*

$$\mu^*_\Gamma(S) = \lim_{n \to \infty} H(S, B_n)$$

*for every $S \subset \mathbb{N}$.*

Let $\mathcal{A}$ be the system of all sets of the form $a_1 + \langle m_1 \rangle \cup \dots \cup a_k + \langle m_k \rangle$. Then $\mathcal{A} \subset \mathcal{A}_\Gamma$ and the compactness of $\Omega$ implies:

**THEOREM 10.** *For every $S \subset \mathbb{N}$ we have*

$$\mu^*_\Gamma(S) = \inf\{\mu_\Gamma(H);\ S \subset H,\ H \in \mathcal{A}\}$$

Theorem 10 gives us the following possibility for characterization of the algebra $\mathcal{A}_\Gamma$:

**THEOREM 11.** *Let $S \subset \mathbb{N}$. Then $S \in \mathcal{A}_\Gamma$ if and only if for every $\varepsilon > 0$ there exist $S_1, S_2 \in \mathcal{A}$ such that*

$$S_1 \subset S \subset S_2$$

*and*

$$\mu_\Gamma(S_2) - \mu_\Gamma(S_1) < \varepsilon .$$

This characterization yields

**THEOREM 12.** *For every $S \in \mathcal{A}_\Gamma$ we have*

$$\lim_{N\to\infty} \frac{A(N,w,S)}{N} = \mu_\Gamma(S)$$

*for every $\Gamma$-distributed sequence $\{w(n)\}$.*

Recall the well-known Darboux property:

The set function $\nu$ has the Darboux property on the set system $\mathcal{P}$ if for every sets $S_1, S_2 \in \mathcal{P}$, $S_1 \subset S_2$ we have

$$\{\nu(S);\ S_1 \subset S \subset S_2,\ S \in \mathcal{P}\} = [\nu(S_1), \nu(S_2)] .$$

A partial answer gives us the following Theorem proved in [8]:

**THEOREM E.** *Let $\mathcal{P} \subset P(\mathbb{N})$ be an algebra of sets and $\nu$ be a finitely additive measure on $\mathcal{P}$. If $\nu$ satisfies the following two conditions:*

(a) *If $A \subset \mathbb{N}$ and*

$$\inf\{\nu(C);\ C \in \mathcal{P},\ A \subset C\} = v = \sup\{\nu(B);\ B \in \mathcal{P},\ B \subset A\},$$

*then $A \in \mathcal{P}$ and $\nu(A) = v$.*

(b) *For each $M \in \mathcal{P}$ and $\varepsilon > 0$ there exists mutually disjoint sets $D_j \in \mathcal{P}$, $j = 1, 2, \dots, s$ such that $M = \bigcup_{j=1}^{s} D_j$ and $\nu(D_j) < \varepsilon$, $j = 1, \dots, s$,*

*then $\nu$ has the Darboux property on $\mathcal{P}$.*

Since the distribution density $\mu_\Gamma$ satisfies the condition (a) on $\mathcal{A}_\Gamma$, we have:

**THEOREM 13.** *If for each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $h(j,m) < \varepsilon$, $0 \le j \le m-1$, then $\mu_\Gamma$ has the Darboux property on $\mathcal{A}_\Gamma$.*

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*) *Mathematical Institute*
*Slovak Academy of Sciences*
*Štefánikova 49*
*814 73 Bratislava*
*Slovakia*

**) *Department of Mathematics*
*Prague Institute of Chemical Technology*
*Technická 1905*
*166 28 Prague 6*
*Czech Republic*
*e-mail: porubsks@vscht.cz*