Mathematica Slovaca

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Mathematica Slovaca, Vol. 42 (1992), No. 5, 663--676

Persistent URL: http://dml.cz/dmlcz/136569

Math. Slovaca, **42** (1992), No. 5, 663–676

Mathematica Slovaca
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Slovak Academy of Sciences

WITT EQUIVALENCE OF CYCLOTOMIC FIELDS

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ABSTRACT. It is shown that the Witt equivalence class of the field of m-th roots of unity over the rationals is determined completely by the following three field invariants: the degree over the rationals, the level, and the number of dyadic primes of the field. Using this result, a classification with respect to Witt equivalence is given of all cyclotomic fields of degree not exceeding 400.

Introduction

Two fields are said to be Witt equivalent, if their Witt rings of symmetric bilinear forms are isomorphic. Witt equivalence of global fields has been investigated in several recent papers (see, for instance, [P-S-C-L], [S2], [S3]) leading up to a Hasse Principle for Witt equivalence of global fields and a fairly detailed understanding of the general problem.

In this note we confine ourselves to cyclotomic fields in an attempt to answer a question raised by one of us at the 9-th Czechoslovak Colloquium on Number Theory ([S1]). The question is how to compute the number $w_{\text{cycl}}(2N)$ of Witt equivalence classes of cyclotomic fields of a given degree $2N$. We feel that an exact formula for $w_{\text{cycl}}(2N)$ is out of reach, but we are able to simplify the general criteria and obtain a definitive classification theorem for cyclotomic fields.

We recall that for an algebraic number field F the following field invariants are preserved by Witt equivalence: the degree $[F\colon Q]$ of the field F over the rational field Q, the level $s(F)$, that is, the minimal number of summands in a representation of -1 as the sum of squares of elements of F (if such a representation exists), and the number $g(F)$ of dyadic primes of the field F (see [S3] for details).

It turns out that for cyclotomic fields these are the only invariants needed.

THEOREM. *Two cyclotomic fields E_m and E_n are Witt equivalent if and only if*

(1) $[E_m\colon Q] = [E_n\colon Q]$,

AMS Subject Classification (1991): Primary 11E04. Secondary 11R18.
Key words: Witt classes, Cyclotomic fields.

(2) $s(E_m) = s(E_n)$,
(3) $g(E_m) = g(E_n)$.

Here $E_m = Q(\zeta_m)$ is the m-th cyclotomic field generated by a primitive root of unity ζ_m of degree m, and as is well known, $[E_m : Q] = \phi(m)$, where ϕ is the Euler totient function.

Using a PC computer we have calculated the invariants $\phi(m)$, $s(E_m)$, $g(E_m)$ in the range $\phi(m) \leq 400$ and classified all cyclotomic fields of degrees not exceeding 400 with respect to Witt equivalence. The results are summarized in the Table 1 at the end of the note.

It is remarkable that, while for cyclotomic fields there is no difficulty in pushing up the calculations to the degrees $\leq 15\,000$ (see Table 2), for the general algebraic number fields the classification with respect to Witt equivalence has been obtained so far only for the degrees $n \leq 4$ (see [Cz] and [S2] for $n = 2$, [S3] for $n \leq 3$, and [J-M] for $n = 4$).

We will prove the Theorem in Section 1, and in Section 2 we consider the sum

$$CW(x) := \sum_{2N \leq x} w_{\mathrm{cycl}}(2N),$$

that is, the number of Witt equivalence classes of all cyclotomic fields of degrees $\leq x$. We find explicit estimates for $CW(x)$, but we have been unable to determine the exact order of magnitude. This appears to be an interesting and nontrivial problem related to some open problems on the distribution of values of the ϕ-function. We find, however, an asymptotic formula for $C(x)$, the number of all cyclotomic fields of degrees $\leq x$. Some numerical data on the behaviour of $CW(x)$ are displayed in the Table 2.

Acknowledgement. We want to thank Dr. M. Kula for writing an independent program and checking the numerical data on the number of cyclotomic Witt classes up to degree 5 760. We also thank Professor A. Schinzel for orienting our attention to the elegant formula (2.6).

1. Classification of cyclotomic fields

Here we prove the classification theorem stated in the Introduction. The necessity of (1), (2) and (3) follows from the results in [P-S-C-L] and [S2]. It is proved there that Witt equivalence of two global fields coincides with reciprocity equivalence of the fields, and the latter requires the same numbers of real, complex and dyadic primes in the two global fields. Hence (1) and (3) follow. As to (2), it is well known that for a nonreal field F, the additive group of the Witt ring $W(F)$ has the exponent $2 \cdot s(F)$ (cf. [L], p. 303). It follows that the level is preserved by Witt equivalence of fields. Thus we will be concerned only with the sufficiency part of the proof.

We will use the following notation. Write $m = 2^k \cdot \mu$, where μ is odd. Since $E_m = E_{2m}$ when m is odd, we may assume that $k \neq 1$.

In other words, we assume that m is either odd, or else divisible by 4. We know that the principal ideal (2) has the following decomposition in E_m:

$$(2) = (p_1 \cdot \ldots \cdot p_g)^e ,$$

where the ramification index $e = \phi(2^k)$. The dyadic primes $p_1, \ldots, p_g$ have all the same degree f and $efg = \phi(m)$.

We write $E_m^{(i)}$ for the completion of E_m at the dyadic prime p_i. The fields $E_m^{(i)}$ are pairwise isomorphic (cf. [C-F], p. 163), hence they all have the same level $s^* := s(E_m^{(i)})$, $i = 1, \ldots, g$.

Obviously we have $s^* \leq s := s(E_m)$, since E_m is contained in its completion. Our first goal is to show that actually $s^* = s$. We begin with the following result.

LEMMA 1.1. *$s(E_m) = 1$ if and only if $4 \mid m$.*

P r o o f. The group W_m of roots of unity in E_m has order m if m is even, and order $2m$ if m is odd ([W], p. 267). Now we have $s(E_m) = 1$ if and only if W_m contains an element of order 4 if and only if the order of W_m is divisible by 4 if and only if $4 \mid m$.

LEMMA 1.2. *If $s^* = 1$, then $e > 1$.*

P r o o f. If $s^* = 1$, then we have $2 = \left(1 - \sqrt{-1}\right)^2 \cdot \left(\sqrt{-1}\right)$ in $E_m^{(i)}$, and we conclude that e is even.

PROPOSITION 1.3. *$s = s^*$, that is, the global level $s(E_m)$ is equal to the local dyadic level $s^* = s(E_m^{(i)})$, $i = 1, \ldots, g$.*

P r o o f. It is well known that the level of a nonreal number field can assume only the values 1, 2 or 4 (cf. [L], p. 299). As we have already observed, we always have $s^* \leq s$. Thus $s^* = 4$ implies $s = 4$, and conversely, if $s = 4$, then by the Hasse-Minkowski principle we must have $s^* = 4$ as well. Further, $s = 1$ if and only if $s^* = 1$. Indeed, if $s = 1$, then $s^* = 1$ by the general inequality $s^* \leq s$. On the other hand, if $s^* = 1$, then by Lemma 1.2 we have $\phi(2^k) = e > 1$; that is, $k \geq 2$. Thus $4 \mid m$, and so $s = 1$ by Lemma 1.1. Thus we have proved that $s = s^*$, whenever one of the s, s^* equals 1 or 4. This shows also that $s = 2$ if and only if $s^* = 2$. The proof is now complete.

Proof of the Theorem. Assume that E_m and E_n satisfy (1), (2) and (3). According to the general result in [S3], to prove that E_m and E_n are Witt equivalent, we only have to show that the local degrees and the local levels at dyadic primes coincide. Using (1) and (3), we have

$$\left[E_m^{(i)} : Q\right] = \frac{\phi(m)}{g} = \frac{\phi(n)}{g} = \left[E_n^{(i)} : Q\right],$$

and using (2) and Proposition 1.3, we get

$$s\left(E_m^{(i)}\right) = s(E_m) = s(E_n) = s\left(E_n^{(i)}\right).$$

This proves the Theorem.

Now we make a few remarks about determining the three invariants for a given cyclotomic field E_m. The computation of the degree $\phi(m)$ is routine, and the level $s(E_m)$ and the number $g(E_m)$ of dyadic primes can be found as follows. Write $m = 2^k \cdot \mu$, where μ is odd, and define

$$e(m) := \phi(2^k) = \begin{cases} 1, & \text{if } k = 0, \\ 2^{k-1}, & \text{if } k > 0, \end{cases}$$
$$f(m) := \min\left\{f \in \mathbb{N}\colon\ 2^f \equiv 1 \pmod{\mu}\right\}.$$

Then for $m \not\equiv 2 \pmod 4$, we have

$$s(E_m) = \begin{cases} 1, & \text{if } 4 \mid m, \\ 4, & \text{if } m \text{ is odd and } f(m) \text{ is odd,} \\ 2, & \text{if } m \text{ is odd and } f(m) \text{ is even,} \end{cases} \tag{1.4}$$

$$g(E_m) = \frac{\phi(m)}{e(m) \cdot f(m)}. \tag{1.5}$$

To get the result for the level $s(E_m)$, we combine Lemma (1.1) with the fact that the dyadic local level is equal to 4 if and only if the dyadic local degree $e \cdot f = e(m) \cdot f(m)$ is odd (cf. [L], p 307). The computation of $g(E_m)$ is well known in classical number theory (cf. [W], p. 263).

It follows from the above computation of the Witt equivalence invariants $\phi(m)$, $s(E_m)$, $g(E_m)$ that they are determined completely by the arithmetical invariants $\phi(m)$, $e(m)$, $f(m)$. Thus the classification theorem can be rephrased as follows.

(1.6). *Two cyclotomic fields E_m and E_n are Witt equivalent if and only if $\phi(m) = \phi(n)$, $e(m) = e(n)$, $f(m) = f(n)$.*

2. The number of cyclotomic Witt equivalence classes

According to the classification theorem (1.6) the number $w_{\mathrm{cycl}}(2N)$ of Witt equivalence classes of cyclotomic fields of degree $2N$ can be computed as follows:

$$w_{\mathrm{cycl}}(2N) = \#\{(e(m), f(m)) : m \in \mathbb{N} \quad \text{and} \quad \phi(m) = 2N\}.$$

Since very little is known about the number of solutions of the equation $\phi(m) = 2N$, we cannot expect to find an exact formula for $w_{\mathrm{cycl}}(2N)$. We can, however, compute $w_{\mathrm{cycl}}(2N)$ for moderately large numbers $2N$ (see Tables 1 and 2).

As to the estimates from above, by counting crudely the possible invariants $(2N, s, g)$ we get

$$w_{\mathrm{cycl}}(2N) \le 2 \cdot d(2N),$$

where $d(2N)$ is the number of divisors of $2N$.

On the other hand, the function $w_{\mathrm{cycl}}(2N)$ assumes infinitely often the values 1 and 2. This can be seen as follows.

For each $2N = 2 \cdot 3^{6k+1}$, $k = 1, 2, \dots$, the equation $\phi(m) = 2N$ has precisely two solutions: $m_1 = 3^{6k+2}$ and $m_2 = 2 \cdot 3^{6k+2}$. An elementary proof for that can be found in [Si], p. 255.

It follows that there are at most two cyclotomic fields of degree $2N$. But since m_1 is odd and $m_2 = 2 \cdot m_1$, we have in fact $E_{m_1} = E_{m_2}$ and so there is just one cyclotomic field of the degree $2N$. Hence $w_{\mathrm{cycl}}(2N) = 1$ for $2N = 2 \cdot 3^{6k+1}$, $k = 1, 2, \dots$.

A similar argument shows that there are infinitely many even numbers $2N$ such that $w_{\mathrm{cycl}}(2N) = 2$. For this we will use A. Schinzel's result showing that there are infinitely many even numbers $2N$ such that the equation $\phi(m) = 2N$ has exactly three solutions ([Sc]). For example, if $2N = 12 \cdot 7^{12k+1}$, $k = 0, 1, \dots$, then $m_1 = 3 \cdot 7^{12k+2}$, $m_2 = 4 \cdot 7^{12k+2}$, $m_3 = 2m_1$ are the only solutions to $\phi(m) = 2N$. It follows that E_{m_1} and E_{m_2} are the only cyclotomic fields of degree $2N$. We will show that these fields are not Witt equivalent. It suffices to show that

$$s(E_{m_1}) = 2 \qquad \text{and} \qquad s(E_{m_2}) = 1,$$

the latter being an immediate consequence of the parity of m_2 (see (1.4)). To compute the level of E_{m_1} we observe that $f(7) = 3$, and by elementary number theory,

$$f(7^n) = 3 \cdot 7^{n-1} \qquad \text{for} \quad n \ge 1.$$

Thus $f(m_1) = 6 \cdot 7^{12k+1}$ is even, and so $s(E_{m_1}) = 2$ by (1.4).

R e m a r k . Among the 1260 values of the ϕ-function less than 5 000 there are 512 values $2N$ with $w_{\mathrm{cycl}}(2N) = 1$, and
278 values $2N$ with $w_{\mathrm{cycl}}(2N) = 2$,
which gives the approximate frequency of 40% and 22% for the values 1 and 2 of the function $w_{\mathrm{cycl}}(2N)$. For the value $w_{\mathrm{cycl}}(2N) = 3$ the corresponding frequency in the interval $2 \leq 2N \leq 5000$ is roughly 11%. It is interesting to notice that almost the same frequencies are obtained when $2N$ ranges over any shorter interval. However, we conjecture that the function $w_{\mathrm{cycl}}2N$ is unbounded as $2N \to \infty$.

While the distribution of the values of $w_{\mathrm{cycl}}(2N)$ is quite irregular, the summatory function counting the number of cyclotomic Witt classes of degrees $\leq x$,

$$CW(x) := \sum_{2N \leq x} w_{\mathrm{cycl}}(2N),$$

behaves very much like a linear function of x. This is apparent from the Table 2, where the values of the ratio $CW(x)/x$ are given for some values of $x \leq 15\,000$. To get some estimates for $CW(x)$ we consider the following trivial inequalities:

$$1 \leq w_{\mathrm{cycl}}(2N) \leq c_{2N} \leq a_{2N},$$

for any even number $2N$ for which the equation $\phi(m) = 2N$ is solvable in m, where

c_{2N} is the number of cyclotomic fields of degree $2N$, and
a_{2N} is the number of solutions of the equation $\phi(m) = 2N$.

Adding the inequalities for $2N \leq x$, we get

$$V(x) \leq CW(x) \leq C(x) \leq A(x),$$

where $V(x)$ is the number of the values of the ϕ-function less than or equal to x, $C(x)$ is the number of cyclotomic fields of degrees $\leq x$, and $A(x)$ is the number of integers m with $\phi(m) \leq x$.

The functions V and A attracted the attention of several writers. An asymptotic formula has been found for $A(x)$ with the main term αx, where $\alpha = \zeta(2) \cdot \zeta(3)/\zeta(6)$ (Erdős-Turan-Dressler-Bateman theorem, see [B]). On the other hand, it turns out that $V(x)$ is of smaller order of magnitude. In fact, it is known that there exists a real number $c = 0.8178\ldots$ such that

$$\frac{x}{\log x} \exp\left((c - \varepsilon)(\log\log\log x)^2\right) < V(X) < \frac{x}{\log x} \exp\left((c + \varepsilon)(\log\log\log x)^2\right)$$

for every $\varepsilon > 0$ and sufficiently large x ([M-P]). Thus for $CW(x)$ we get

$$\frac{x}{\log x}\exp\left((c-\varepsilon)(\log\log\log x)^2\right) < CW(x) < (1+\varepsilon)\alpha x$$

for any $\varepsilon > 0$ and sufficiently large x.
A slightly better upper estimate comes from the inequality $CW(x) \leq C(x)$. We have not found any results on the asymptotic behaviour of $C(x)$ in literature and so we prove here the following result.

PROPOSITION 2.1. $\lim_{x\to\infty} C(x)/x = \frac{2}{3}\alpha$, *where* $\alpha = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$.

Apart from a_{2N} and c_{2N} defined above, we will consider

b_{2N} – the number of solutions of the equation $\phi(m) = 2N$ in odd integers m.

We begin with noticing the following fact.

LEMMA 2.2. $a_{2N} = b_{2N} + c_{2N}$.

Proof. For $i \in \{0,1,2,3\}$ we write

$$a_{2N}(i) := \#\{m \in \mathbb{N}\colon\ \phi(m) = 2N \quad\text{and}\quad m \equiv i \pmod 4\}.$$

Observe that $b_{2N} = a_{2N}(2)$. This follows from the fact that for $m \in \mathbb{N}$,

$$m \text{ is odd} \quad\text{and}\quad \phi(m) = 2N \iff 2m \equiv 2 \pmod 4 \quad\text{and}\quad \phi(2m) = 2N.$$

Now since $c_{2N} = a_{2N}(0) + a_{2N}(1) + a_{2N}(3)$, we have

$$a_{2N} = a_{2N}(0) + a_{2N}(1) + a_{2N}(2) + a_{2N}(3) = b_{2N} + c_{2N}.$$

Proof of the Proposition 2.1. We set $B(x) := \sum_{2N\leq x} b_{2N}$. Then, according to the Lemma 2.2, we have

$$A(x) = B(x) + C(x)$$

and since

$$\lim_{x\to\infty} A(x)/x = \alpha \tag{2.3}$$

is a known result (see [B]), it is sufficient to prove that

$$\lim_{x\to\infty} B(x)/x = \frac{1}{3}\alpha. \tag{2.4}$$

We prove (2.4) by modifying slightly the proof of (2.3) in Section 2 of [B]. We consider the Dirichlet series with coefficients $b_1, b_2, \ldots$, (where $b_{2N+1} = 0$ for every N) and factor out $\zeta(s)$ to get

$$\sum_{n=1}^{\infty} b_n n^{-s} = (1 - 2^{-s}) \cdot \zeta(s) \cdot F(s),$$

where

$$F(s) = \prod_{p>2} \left(1 - p^{-s} + (p-1)^{-s}\right).$$

Here $F(s)$ is an analytic function in the right half-plane.

By the Dirichlet-Dedekind theorem, if $\lim_{x \to \infty} B(x)/x$ exists and is equal to γ, then

$$\lim_{s \to 1^+} (s-1) \sum_{n=1}^{\infty} b_n n^{-s} = \gamma.$$

Thus, if the limit exists, it equals

$$\lim_{s \to 1^+} (s-1)\zeta(s)(1 - 2^{-s})F(s) = \frac{1}{2}F(1).$$

Now

$$\begin{aligned} F(1) &= \prod_{p>2} \left(1 - p^{-1} + (p-1)^{-1}\right) \\ &= \prod_{p>2} \left(1 - \frac{1}{p^6}\right)\left(1 - \frac{1}{p^2}\right)^{-1}\left(1 - \frac{1}{p^3}\right)^{-1} \\ &= \frac{2}{3} \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{2}{3}\alpha. \end{aligned}$$

Hence, if γ exists, it is equal to $\frac{1}{2}F(1) = \frac{1}{3}\alpha$.

As in [B], an application of the Wiener-Ikehara theorem shows that the limit of $B(x)/x$ exists, and this proves (2.4).

COROLLARY 2.5. *The number of cyclotomic fields with the prime 2 unramified is asymptotically equal to the number of cyclotomic fields with 2 ramified.*

This follows from Proposition 2.1 and 2.4.

We do not know whether or not a similar statement holds for the number of cyclotomic Witt classes. We show in Table 2 the values of the functions

$CW_{\rm odd}(x)$ and $CW_{\rm even}(x)$ counting the number of Witt equivalence classes of cyclotomic fields of degrees $\leq x$ with 2 unramified or ramified, respectively. Within the limits of computation we have approximately

$$CW_{\rm odd}(x) = \frac{4}{3} CW_{\rm even}(x),$$

in other words, cyclotomic fields with 2 unramified produce more Witt equivalence classes than the remaining cyclotomic fields.

We remark that another proof of the Proposition 2.1 can be obtained by using (2.3) and the following formula communicated to the authors by A. Schinzel:

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} A\left(\frac{x}{2^{k-1}}\right). \tag{2.6}$$

This relation implies also that an asymptotic formula for $A(x)$ produces an asymptotic formula for $C(x)$ with the same error term (the main terms being αx and $\frac{2}{3}\alpha x$, respectively).

Table 1.

Notation. The first column lists even numbers $2N$ with the property that there is at least one cyclotomic field of degree $2N$. The second column gives all $m \not\equiv 2 \pmod 4$ such that $[E_m : \mathbb{Q}] = 2N$. g, s, w, are $g(E_m)$, $s(E_m)$ and $w_{\rm cycl}(2N)$, respectively.

$2N$	m	g	s	w	$2N$	m	g	s	w	$2N$	m	g	s	w
2	3	1	2	2	22	23	2	4	1	42	43	3	2	2
	4	1	1		24	35	2	2	3		49	2	4	
4	5	1	2	2		39	2	2		44	69	2	2	2
	8	1	1			45	2	2			92	2	1	
	12	1	1			52	1	1		46	47	2	4	1
6	7	2	4	2		56	2	1		48	65	4	2	3
	9	1	2			72	1	1			104	1	1	
8	15	2	2	2		84	2	1			105	4	2	
	16	1	1		28	29	1	2	1		112	2	1	
	20	1	1		30	31	6	4	1		140	2	1	
	24	1	1		32	51	4	2	3		144	1	1	
10	11	1	2	1		64	1	1			156	2	1	
12	13	1	2	4		68	2	1			168	2	1	
	21	2	2			80	1	1			180	2	1	
	28	2	1			96	1	1		52	53	1	2	1
	36	1	1			120	2	1		54	81	1	2	1
16	17	2	2	3	36	37	1	2	4	56	87	2	2	2

$2N$	m	g	s	w	$2N$	m	g	s	w	$2N$	m	g	s	w
	32	1	1			57	2	2			116	1	1	
	40	1	1			63	6	2		58	59	1	2	1
	48	1	1			76	1	1		60	61	1	2	4
	60	2	1			108	1	1			77	2	2	
18	19	1	2	1	40	41	2	2	3		93	6	2	
	27	1	2			55	2	2			99	2	2	
20	25	1	2	3		75	2	2			124	6	1	
	33	2	2			88	1	1						
	44	1	1			100	1	1						
						132	2	1						
64	85	8	2	4	102	103	2	4	1		432	1	1	
	128	1	1		104	159	2	2	2		444	2	1	
	136	2	1			212	1	1			456	2	1	
	160	1	1		106	107	1	2	1		468	6	1	
	192	1	1		108	109	3	2	3		504	6	1	
	204	4	1			133	6	2			540	2	1	
	240	2	1			171	6	2		148	149	1	2	1
66	67	1	2	1		189	6	2		150	151	10	4	1
70	71	2	4	1		324	1	1		156	157	3	2	4
72	73	8	4	6	110	121	1	2	1		169	1	2	
	91	6	2		112	113	4	2	3		237	2	2	
	95	2	2			145	4	2			316	2	1	
	111	2	2			232	1	1		160	187	4	2	5
	117	6	2			348	2	1			205	8	2	
	135	2	2		116	177	2	2	2		328	2	1	
	148	1	1			236	1	1			352	1	1	
	152	1	1		120	143	2	2	6		400	1	1	
	216	1	1			155	6	2			440	2	1	
	228	2	1			175	2	2			492	4	1	
	252	6	1			183	2	2			528	2	1	
78	79	2	4	1		225	2	2			600	2	1	
80	123	4	2	3		231	4	2			660	4	1	
	164	2	1			244	1	1		162	163	1	2	1
	165	4	2			248	6	1			243	1	2	
	176	1	1			308	2	1		164	249	2	2	2
	200	1	1			372	6	1			332	1	1	
	220	2	1			396	2	1		166	167	2	4	1
	264	2	1		126	127	18	4	1	168	203	2	2	5
	300	2	1		128	255	16	2	5		215	6	2	
82	83	1	2	1		256	1	1			245	2	2	
84	129	6	2	4		272	2	1			261	2	2	
	147	2	2			320	1	1			344	3	1	
	172	3	1			340	8	1			392	2	1	
	196	2	1			384	1	1			516	6	1	
88	89	8	4	3		408	4	1			588	2	1	
	115	2	2			480	2	1		172	173	1	2	1
	184	2	1		130	131	1	2	1	176	267	8	2	4
	276	2	1		132	161	4	4	3		345	4	2	

$2N$	m	g	s	w
92	141	2	2	2
	188	2	1	
96	97	2	2	6
	119	4	2	
	153	4	2	
	195	8	2	
	208	1	1	
	224	2	1	
	260	4	1	
	280	2	1	
	288	1	1	
	312	2	1	
	336	2	1	
	360	2	1	
	420	4	1	
100	101	1	2	1
	125	1	2	
	357	8	2	
	388	2	1	
	416	1	1	
	448	2	1	
	476	4	1	
	520	4	1	
	560	2	1	
	576	1	1	
	612	4	1	
	624	2	1	
	672	2	1	
	720	2	1	
	780	8	1	
	840	4	1	
196	197	1	2	1
198	199	2	4	1
200	275	10	2	3
	303	2	2	
	375	2	2	
	404	1	1	
	500	1	1	
204	309	2	2	2
	412	2	1	
208	265	4	2	3
	424	1	1	
	636	2	1	
210	211	1	2	1
212	321	2	2	2
	428	1	1	
216	247	6	2	6
	259	6	2	

$2N$	m	g	s	w
	201	2	2	
	207	2	2	
	268	1	1	
136	137	2	2	1
138	139	1	2	1
140	213	2	2	2
	284	2	1	
144	185	4	2	7
	219	8	2	
	273	12	2	
	285	4	2	
	292	8	1	
	296	1	1	
	304	1	1	
	315	12	2	
	364	6	1	
	380	2	1	
238	239	2	4	1
240	241	10	2	7
	287	4	2	
	305	4	2	
	325	4	2	
	369	4	2	
	385	4	2	
	429	4	2	
	465	12	2	
	488	1	1	
	495	4	2	
	496	6	1	
	525	4	2	
	572	2	1	
	616	2	1	
	620	6	1	
	700	2	1	
	732	2	1	
	744	6	1	
	792	2	1	
	900	2	1	
	924	4	1	
250	251	5	2	1
252	301	6	2	3
	381	18	2	
	387	6	2	
	441	6	2	
	508	18	1	
256	257	16	2	6
	512	1	1	
	544	2	1	

$2N$	m	g	s	w
	356	8	1	
	368	2	1	
	460	2	1	
	552	2	1	
178	179	1	2	1
180	181	1	2	4
	209	2	2	
	217	12	4	
	279	6	2	
	297	2	2	
184	235	2	2	2
	376	2	1	
	564	2	1	
190	191	2	4	1
192	193	2	2	7
	221	8	2	
	291	4	2	
280	281	4	2	3
	319	2	2	
	355	2	2	
	568	2	1	
	852	2	1	
282	283	3	2	1
288	323	4	2	9
	365	8	2	
	455	24	2	
	459	4	2	
	555	8	2	
	584	8	1	
	585	24	2	
	592	1	1	
	608	1	1	
	728	6	1	
	740	4	1	
	760	2	1	
	864	1	1	
	876	8	1	
	888	2	1	
	912	2	1	
	936	6	1	
	1008	6	1	
	1080	2	1	
	1092	12	1	
	1140	4	1	
	1260	12	1	
292	293	1	2	1
294	343	2	4	1
296	447	2	2	2

$2N$	m	g	s	w
	327	6	2	
	333	6	2	
	351	6	2	
	399	12	2	
	405	2	2	
	436	3	1	
	532	6	1	
	648	1	1	
	684	6	1	
	756	6	1	
220	253	2	2	2
	363	2	2	
	484	1	1	
222	223	6	4	1
224	339	8	2	4
	435	8	2	
	452	4	1	
	464	1	1	
	580	4	1	
	696	2	1	
226	227	1	2	1
228	229	3	2	1
232	233	8	4	4
	295	2	2	
	472	1	1	
	708	2	1	
	984	4	1	
	1056	2	1	
	1200	2	1	
	1320	4	1	
324	489	2	2	4
	513	18	2	
	567	6	2	
	652	1	1	
	972	1	1	
328	415	2	2	3
	664	1	1	
	996	2	1	
330	331	11	2	1
332	501	2	2	2
	668	2	1	
336	337	16	4	6
	377	4	2	
	609	4	2	
	645	12	2	
	688	3	1	
	735	4	2	
	784	2	1	
	640	1	1	
	680	8	1	
	768	1	1	
	816	4	1	
	960	2	1	
	1020	16	1	
260	393	2	2	2
	524	1	1	
262	263	2	4	1
264	299	2	2	5
	335	2	2	
	483	4	2	
	536	1	1	
	644	4	1	
	804	2	1	
	828	2	1	
268	269	1	2	1
270	271	2	4	1
272	289	2	2	3
	411	4	2	
	548	2	1	
276	277	3	2	4
	329	4	4	
	417	2	2	
	423	2	2	
	556	1	1	
	712	8	1	
	736	2	1	
	920	2	1	
	1068	8	1	
	1104	2	1	
	1380	4	1	
356	537	2	2	2
	716	1	1	
358	359	2	4	1
360	403	6	2	8
	407	2	2	
	427	6	2	
	475	2	2	
	543	2	2	
	549	6	2	
	627	4	2	
	651	12	2	
	675	2	2	
	693	12	2	
	724	1	1	
	836	2	1	
	868	12	1	
	596	1	1	
300	341	30	2	3
	453	10	2	
	604	10	1	
306	307	3	2	1
310	311	2	4	1
312	313	2	2	5
	371	2	2	
	395	2	2	
	471	6	2	
	477	2	2	
	507	2	2	
	628	3	1	
	632	2	1	
	676	1	1	
	948	2	1	
316	317	1	2	1
320	425	8	2	6
	561	8	2	
	615	16	2	
	656	2	1	
	704	1	1	
	748	4	1	
	800	1	1	
	820	8	1	
	880	2	1	
	765	16	2	
	772	2	1	
	776	2	1	
	832	1	1	
	884	8	1	
	896	2	1	
	952	4	1	
	1040	4	1	
	1120	2	1	
	1152	1	1	
	1164	4	1	
	1224	4	1	
	1248	2	1	
	1344	2	1	
	1428	8	1	
	14440	2	1	
	1560	8	1	
	1680	4	1	
388	389	1	2	1
392	591	2	2	2
	788	1	1	
396	397	9	2	4

$2N$	m	g	s	w	$2N$	m	g	s	w	$2N$	m	g	s	w
	812	2	1			1116	6	1			437	2	2	
	860	6	1			1188	2	1			469	6	2	
	980	2	1		366	367	2	4	1		597	2	2	
	1032	6	1		368	705	4	2	2		603	6	2	
	1044	2	1			752	2	1			621	2	2	
	1176	2	1			940	2	1			796	2	1	
342	361	1	2	1		1128	2	1		400	401	2	2	6
344	519	2	2	2	372	373	1	2	1		451	20	2	
	692	1	1		378	379	1	2	1		505	4	2	
346	347	1	2	1	380	573	2	2	2		808	1	1	
348	349	1	2	2		764	2	1			825	20	2	
	413	2	2		382	383	2	4	1		1000	1	1	
	531	2	2		384	485	8	2	7		1100	10	1	
352	353	4	2	5		579	4	2			1212	2	1	
	391	4	2			595	4	2			1500	2	1	
	445	8	2			663	16	2						

Table 2.

N o t a t i o n . $CW(x)$ is the number of Witt equivalence classes of all cyclotomic fields of degrees $\leq x$. $CW_{\text{odd}}(x)$ and $CW_{\text{even}}(x)$ are the numbers of Witt equivalence classes of cyclotomic fields E_m of degrees $\leq x$ with m odd and m divisible by 4, respectively. $C(x)$ is the number of all cyclotomic fields of degrees $\leq x$.

x	$10 \cdot CW(x)/x$	$10 \cdot CW_{\text{odd}}(x)/x$	$10 \cdot CW_{\text{even}}(x)/x$	$10 \cdot CW(x)/C(x)$
100	8.6	5.1	3.5	6.5
500	8.18	4.72	3.46	6.3
1000	7.99	4.59	3.4	6.1
2000	7.7	4.41	3.29	5.9
3000	7.6066	4.3533	3.2533	5.8
4000	7.4500	4.2625	3.1875	5.7
5000	7.3760	4.2120	3.1640	5.690
6000	7.3283	4.1783	3.1500	5.653
7000	7.2285	4.1271	3.1014	5.578
8000	7.1787	4.0862	3.0925	5.532
9000	7.1455	4.0711	3.0744	5.512
10000	7.1130	4.0500	3.0630	5.485
11000	7.0745	4.0254	3.0490	5.469
12000	7.0458	4.0041	3.0416	5.433
13000	7.0115	3.9923	3.0192	5.400
14000	6.9821	3.9764	3.0057	5.388
15000	6.9406	3.9566	2.9840	5.353

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Received September 6, 1991

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