Math. Slovaca, 42 (1992), No. 4, 437–441

Mathematica Slovaca

ABOUT VARIETIES OF WEAKLY ABELIAN l–GROUPS

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ABSTRACT. For every prime p a variety of weakly abelian l-groups which is not generated by the set of itself nilpotent l-groups is constructed.

A lattice ordered group G is called weakly abelian if the l-group G satisfies the identity $\left(|x|^{-1}|y||x| \wedge |y|^{-2}\right) \vee e = e$. It is well known that every weakly abelian l-group G is representable [1] and that every locally nilpotent l-group G is weakly abelian [2]. The following question is known in the theory of l-varietes:

Let $\mathcal{N}_n$ be the variety of all nilpotent l-groups of class $\leqq n$ and let W_a be the variety of all weakly abelian l-groups. Is this equality $W_a = \bigcup\limits_{n=1}^{\infty} \mathcal{N}_n$ true?

Here for every prime p we construct a variety of weakly abelian l-groups $\mathcal{M}_p$ which is not generated by the set of itself nilpotent l-groups.

Let W be a wreath product $\langle a \rangle \wr \langle b \rangle$ of infinite cyclic groups $\langle a \rangle$, $\langle b \rangle$. It is known that W admits a weakly abelian total order P. Let T denote a subgroup $\prod\limits_{i=-\infty}^{\infty} \langle b^{-i}ab^{i} \rangle$ of group W with total order which is induced on T by the total order P of group W. And let $A = \langle c \rangle \overleftarrow{\times} T$ be a lexicographic product of an infinite cyclic group $\langle c \rangle$ and totally ordered group T. Now we define two automorphisms α, β of group A as follows: $c^{\alpha} = c$, $a_n^{\alpha} = a_{n+1}$, $n \in \mathbb{Z}$, $c^{\beta} = c$,

$$a_n^{\beta} = \begin{cases} a_n c\,, & \text{if} \quad n \equiv 0 \pmod p \\ a_n\,, & \text{if} \quad n \not\equiv 0 \pmod p\,, \end{cases}$$

where a_n denotes an element $b^{-n}ab^{n}$, $n \in \mathbb{Z}$.

AMS Subject Classification (1991): Primary 06F15.
Key words: l-variety, Weakly abelian group, Nilpotent group.

LEMMA 1. *Automorphisms α, β of group A preserve the total order on A.*

Proof. For every element $u = u_1 c^n$, $u \in A$, where $u_1 \in T$, $n \in \mathbb{Z}$, we have $u^\alpha = u_1^\alpha (c^n)^\alpha = u_1^b c^n$. But in the l-group A $u \geq e$ if and only if $u_1 \geq e$ in T, or $u_1 = e$ and $c^n \geq e$ in $\langle c \rangle$. Since conjugation by b in $\langle a \rangle \wr \langle b \rangle$ (and, particularly, in T) is an order automorphism of $\langle a \rangle \wr \langle b \rangle$, then $u \geq e$ follows $u^\alpha \geq e$ in A. Hence, α is an order automorphism of A. Automorphism β acts as the identity in the factor-group $A/\langle c \rangle$ and in the group $\langle c \rangle$, so that

$$u^\beta = u_1^\beta (c^n)^\beta = \begin{cases} u_1 c^m c^n & \text{for some } m \in \mathbb{Z}, \quad \text{if } u_1 \neq e, \\ c^n, & \text{if } u_1 = e. \end{cases}$$

But $u_1 \gg c$ in A and, hence, $u \geq e$ if and only if $u^\beta \geq e$. The proof is completed.

Let now G denote a subgroup $\langle \alpha, \beta \rangle$ of the group order-preserving automorphisms $\operatorname{Aut} A$ of the abelian totally ordered group A.

LEMMA 2. *The group G can be described in terms of generators and relations as:*

$$G = \big\langle \alpha, \beta \,\|\, [\alpha^p, \alpha^{-i}\beta a^i] = e, \ [\alpha^{-i}\beta\alpha^i, \alpha^{-j}\beta\alpha^j] = e, \quad i, j \in \mathbb{Z} \big\rangle .$$

Proof. In the group G we have

$$a_n^{\alpha^p \beta} = a_{n+p}^\beta = \begin{cases} a_{n+p} c, & \text{if } p + n \equiv 0 \pmod{p}, \\ a_{n+p}, & \text{if } p + n \not\equiv 0 \pmod{p}. \end{cases}$$

$$a_n^{\beta \alpha^p} = \begin{cases} a_n^{\alpha^p} c^{\alpha^p} = a_{n+p} c, & \text{if } n \equiv 0 \pmod{p}, \\ a_n^{\alpha^p} = a_{n+p}, & \text{if } n \not\equiv 0 \pmod{p}. \end{cases}$$

But $p + n \equiv 0 \pmod{p}$ if and only if $n \equiv 0 \pmod{p}$, hence $a_n^{\alpha^p \beta} = a_n^{\beta \alpha^p}$ for every $n \in \mathbb{Z}$ and so $\alpha^p \beta = \beta \alpha^p$ in G. For every $i \in \mathbb{Z}$ we now have $\alpha^{-i}\beta\alpha^i = \alpha^{-i}\alpha^{-p}\beta\alpha^p\alpha^i = \alpha^{-p}\alpha^{-i}\beta\alpha^i\alpha^p$, therefore, $[\alpha^p, \alpha^{-i}\beta\alpha^i] = e$.

In the same way we establish that the relations $[\alpha^{-i}\beta\alpha^i, \alpha^{-j}\beta\alpha^j] = e$ for $i, j \in \mathbb{Z}$ are true in G. Now it is not hard to see that every element u in G can be written in the form

$$u = \alpha^m \alpha^{-(p-1)} \beta^{m_1} \alpha^{p-1} \cdot \ldots \cdot \alpha^{-1} \beta^{m_{p-1}} \alpha \beta^{m_p}$$

for some integers $m, m_1, \ldots, m_p$. Let us have in G some relation $u = e$. Then for every $n \in \mathbb{Z}$ we must have in A

$$a_n = a_n^u = a_{n+m}^{\alpha^{-(p-1)}\beta^{m_1}\alpha^{p-1} \cdot \ldots \cdot \alpha^{-1}\beta^{m_{p-1}}\alpha\beta^{m_p}} = a_{n+m} c^{m_i},$$

where $n+m+i \equiv 0 \pmod p$. But in A $a_n = a_{n+m}c^{m_i}$ if and only if $m=0$, $m_i = 0$. Hence, choose $n = 1,2,\ldots,p-1$, we immediately have $m = 0$, $m_1 = 0, \ldots, m_p = 0$. Therefore, every relation in G follows from relations $[\alpha^p, \alpha^{-i}\beta\alpha^i] = e$, $[\alpha^{-i}\beta\alpha^i, \alpha^{-j}\beta\alpha^j] = e$. The proof is completed.

LEMMA 3. *The group G satisfies the identity $[x_1^p, x_2^p] = e$.*

Proof. As it follows from Lemma 2, for any element x_i, $x_i \in G$, we have $x_i = \alpha^{n_i} f_i$ for some integer n_i and some element f_i, $f_i \in G^*$, where

$$G^* = \left\langle \beta, \alpha^{-1}\beta\alpha, \ldots, \alpha^{-(p-1)}\beta\alpha^{p-1} \right\rangle.$$

Hence,

$$x_i^p = \left(\alpha^{n_i} f_i\right)^p = \alpha^{n_i p} u_i,$$

where $u_i = \alpha^{-n_i(p-1)} f_i \alpha^{n_i(p-1)} \cdot \ldots \cdot \alpha^{-n_i} f_i \alpha^{n_i} f_i$. As it follows from Lemma 2,

$$\left[x_1^p, x_2^p\right] = \left[\alpha^{n_1 p} u_1, \alpha^{n_2 p} u_2\right] = \left[\alpha^{n_1 p}, \alpha^{n_2 p}\right]\left[\alpha^{n_1 p}, u_2\right]\left[u_1, \alpha^{n_2 p}\right]\left[u_1, u_2\right] = e.$$

The proof is completed.

The group G is solvable of class 2 (it follows from Lemma 2). Let F be a free solvable of class 2 group with two generators a_α, a_β, and let $\phi\colon F \to G$ be a homomorphism such that $\phi(a_\alpha) = \alpha$, $\phi(a_\beta) = \beta$. Let H denote a semidirect product $A \circ F$ of groups A, F, where for $a \in A$ and $f \in F$ $a^f = a^{\phi(f)}$. It is well known that the free solvable of class 2 group F admits some weakly abelian total order Q. Now we introduce a weakly abelian order on group H as follows: for $fa \in H$, where $a \in A$, $f \in F$ let $fa \geq e$ in H if and only if $f \geq e$ in (F, Q), or $f = e$ and $a \geq e$ in (A, P).

LEMMA 4. *A lattice ordered group H satisfies the identity*

$$\left[[x_1^p, x_2^p], [x_3^p, x_4^p]\right] = e.$$

Proof. Consider a centralizer $C = C_H(A)$ of subgroup A in the group H. It is easy to see that $C \supseteq A$, C is normal in H, and $\ker(\phi) \subseteq C$. Let us shown that C is an abelian subgroup. It is sufficient to show that $\ker(\phi)$ is abelian. A group G admits representation

$$G = \left\langle \alpha, \beta \mid\mid [\alpha^{-i}\beta\alpha^i, \alpha^{-j}\beta\alpha^j] = [\alpha^p, \beta] = e \right\rangle.$$

Therefore, $\ker(\phi)$, as a normal subgroup of F, is generated by the set $X = \left\{[a_\alpha^{-i} a_\beta a_\alpha^i, a_\alpha^{-j} a_\beta a_\alpha^j],\ [a_\alpha^p, a_\beta],\quad i,j \in \mathbb{Z}\right\}$, but $X \subseteq [F, F]$, a subgroup

$[F, F]$ is fully invariant in F and abelian, and, hence, $\ker(\phi) = \mathrm{gr}\langle X\rangle^F$ is abelian. Let now x_1, x_2, x_3, x_4 be any elements in H. As follows from Lemma 3 we have inclusions $[x_1^p, x_2^p] \in C$, $[x_3^p, x_4^p] \in C$. Hence, $\big[[x_1^p, x_2^p], [x_3^p, x_4^p]\big] = e$ because C is abelian. Proof is completed.

Let now $\mathcal{M}_p$ denote a l-variety generated by the l-group H, and let $\mathcal{B}_p$ denote a subvariety of $\mathcal{M}_p$ generated by all nilpotent lattice ordered groups from $\mathcal{M}_p$.

THEOREM. $\mathcal{M}_p \neq \mathcal{B}_p$.

P r o o f. It is not hard to see that the following identities are true in $\mathcal{M}_p$:

$$\big[[y_1^p \cdot \ldots \cdot y_k^p, z_1^p \cdot \ldots \cdot z_s^p], [u_1^p \cdot \ldots \cdot u_n^p, v_1^p \cdot \ldots \cdot v_m^p]\big] = e\,, \qquad (*)$$

where k, s, n, m are integers and y_i, z_j, u_t, v_q are variables. Consider any nilpotent l-group B, $B \in \mathcal{M}_p$. The lattice ordered group B satisfies the identities $(*)$, therefore, the identity $\big[[y, z], [u, v]\big] = e$ is true in subgroup pB of group B, generated by the set $\{x^p, \ \ x \in B\}$. As it follows from theorem of B a u m s l a g [3], every identity of nilpotent torsion free group pB must be true in nilpotent completion $(pB)^*$ of pB. But as it follows from the theorem of M a l'c e v [4], $B \subseteq (pB)^*$, and, hence, the identity $\big[[y, z], [u, v]\big] = e$ is true in the l-group B. So, the identity $\big[[y, z], [u, v]\big] = e$ is true in the l-variety $\mathcal{B}_p$. Let now $y = a_0$, $z = a_\alpha$, $u = a_\beta^{-1}$, $v = a_\alpha^{-1}$. We have $[y, z] = a_0^{-1}a_\alpha^{-1}a_0a_\alpha = a_0^{-1}a_0^\alpha = a_0^{-1}a_1$,

$$\begin{aligned}\big[[y, z], [u, v]\big] &= (a_0^{-1}a_1)^{-1} \cdot (a_0^{-1}a_1)^{[u,v]}\\ &= a_0a_1^{-1}(a_0^{-1}a_1)^{\beta\alpha\beta^{-1}\alpha^{-1}} = a_0a_1^{-1}(a_0^{-1}c^{-1}a_1)^{\alpha\beta^{-1}\alpha^{-1}}\end{aligned}$$

$$= a_0a_1^{-1}\big(a_1^{-1}c^{-1}a_2^{\beta^{-1}}\big)^{\alpha^{-1}} = \begin{cases} a_0a_1^{-1}a_0^{-1}c^{-1}a_1 = c^{-1} & \text{if } \ p \neq 2\,,\\ a_0a_1^{-1}a_0^{-1}c^{-1}(a_2c^{-1})^{\alpha^{-1}} = c^{-2} & \text{if } \ p = 2\,.\end{cases}$$

In both cases we have $\big[[y, z], [u, v]\big] \neq e$ in the l-group H. So $\mathcal{M}_p \neq \mathcal{B}_p$. The proof is completed.

COROLLARY. *The l-variety $\mathcal{M}_p$ has no divisible embedding property.*

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Received November 13, 1991

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