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A NOTE ABOUT THE ALMOST CONTINUITY

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ABSTRACT. Some notions of quasicontinuity for Husain's almost continuity are introduced and examined.

Let (X, T_X) and (Y, T_Y) be topological spaces and let (Z, ϱ) be a metric space. A function $f: X \to Z$ is said to be *almost continuous* ([3]) if for each $x \in X$ and each open set $V \subset Z$ containing $f(x)$, the closure $\mathrm{Cl}(f^{-1}(V))$ of the set $f^{-1}(V)$ is a neighbourhood of x. A function $f: X \to Z$ is said to be *cliquish at a point* $x \in X$ ([1, 2]) if for every positive number r and for every open set $U \subset X$ containing x, there exists a nonempty open set $V \subset U$ such that $\operatorname{osc}_V f < r$.

Remark 1. ([2], Corollary 12). If $f: X \to Z$ is a cliquish and almost continuous function, then f is a continuous function.

Remark 2. A function $f: X \times Y \to Z$ having continuous all sections $f_x(t) = f(x,t)$ and $f^y(u) = f(u,y)$ ($x, u \in X$ and $t, y \in Y$) need not be almost continuous. Indeed, if $X = Y = Z = \mathbb{R}$ ($\mathbb{R}$ denotes the set of reals), $T_X = T_Y$ are the Euclidean topologies and ϱ is the Euclidean metric, then there is a function $f: \mathbb{R}^2 \to \mathbb{R}$ having continuous all sections f_x, f^y, which is not continuous. Since f is cliquish, by Remark 1 it is not almost continuous.

The following definitions are some analogies of the quasicontinuity ([1, 2]) for the almost continuity.

A function $f: X \to Z$ has the *property* (P) (resp. (R)) if for each $x \in X$ and each open set $V \subset Z$ containing $f(x)$, $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))))$ ($U \cap f^{-1}(V)$ is of the second category for every open set U containing x). Int denotes the interior operation.

Obviously, every function $f: X \to Z$ having the property (R) has also the property (P).

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THEOREM 1. *Suppose that for every $x \in X$ there is an open neighbourhood $U(x)$ having a countable basis of open sets. Let $f\colon X \times Y \to Z$ be a function. If all sections f_x have the property* (R) *and all sections f^y have the property* (P)*, then the function f has the property* (P).

P r o o f. Fix points $x \in X$, $y \in Y$ and a positive number r. Let $W \subset X \times Y$ be an open set such that $(x,y) \in W$. There are open sets $U \subset X$ and $V \subset Y$ such that $x \in U$, $y \in V$ and $U \times V \subset W$. Since the section f_x has the property (R), there exists a set $A \subset V$ of second category such that

$$\varrho\big(f(x,t),\ f(x,y)\big) < r/2 \qquad \text{for each} \quad t \in A\,.$$

There is an open set $T \subset U$ containing x and having a countable basis of open sets $U_1, U_2, \dots, U_n, \dots$. Since all sections f^y have the property (P), for every $t \in A$ there is an open set $U_{n(t)}$ such that

$$U_{n(t)} \subset \operatorname{Int}\Big(\operatorname{Cl}\Big(\big\{u \in T\colon\ \varrho\big(f(u,t),\ f(x,t)\big) < r/2\big\}\Big)\Big)\,. \tag{1}$$

But the set A is of second category, so there is a positive integer m such that the set

$$B = \big\{t \in A\colon\ n(t) = m\big\}$$

is also of second category. Let

$$C = \big\{(u,t) \in U_m \times B\colon\ \varrho\big(f(u,t),\ f(x,t)\big) < r/2\big\}\,.$$

For each point $(u,t) \in C$ we have

$$\begin{aligned}&\varrho\big(f(u,t),\ f(x,y)\big)\\ &\qquad \le \varrho\big(f(u,t),\ f(x,t)\big) + \varrho\big(f(x,t),\ f(x,y)\big) < r/2 + r/2 = r\,.\end{aligned} \tag{2}$$

From (1) it follows that $\operatorname{Cl}(C) \supset U_m \times B$. Consequently, $\operatorname{Cl}(C) \supset U_m \times \operatorname{Cl}(B)$, and $\operatorname{Int}\big(\operatorname{Cl}(C)\big) \supset U_m \times \big(\operatorname{Int}\big(\operatorname{Cl}(B)\big) \cap V\big)$. But the set B is of second category, so

$$\operatorname{Int}\big(\operatorname{Cl}(C)\big) \supset U_m \times \big(\operatorname{Int}\big(\operatorname{Cl}(B)\big) \cap V\big) \neq \emptyset\,.$$

From this it follows by (2) that

$$(x,t) \in \operatorname{Cl}\Big(\operatorname{Int}\Big(\operatorname{Cl}\Big(\big\{(u,t)\colon\ \varrho\big(f(u,t),\ f(x,y)\big) < r\big\}\Big)\Big)\Big)$$

and the proof is complete.

COROLLARY 1. *Assume that the spaces* (X, T_X), (Y, T_Y) *and* (Z, ϱ) *satisfy the hypothesis of Theorem* 1. *Let* $f\colon X \times Y \to Z$ *be a function such that all sections* f_x *are continuous and all sections* f^y *have the property* (P). *If* (Y, T_Y) *is a Baire space, then* f *has the property* (P).

THEOREM 2. *Let the spaces* (X, T_X), (Y, T_Y), (Z, ϱ) *satisfy the assumptions of Theorem* 1. *Moreover, we suppose that every set* $A \subset X \times Y$ *of first category is such that the set*

$$\{y \in Y\colon\ A^y = \{u \in X\colon\ (u,y) \in A\} \textit{ is of second category}\}$$

is of first category. If all sections f_x *and* f^y *of the function* $f\colon X \times Y \to Z$ *have the property* (R), *then* f *has also the property* (R).

P r o o f . The proof is analogous to the proof of Theorem 1.

COROLLARY 2. *If the spaces* (X, T_X), (Y, T_Y), (Z, ϱ) *satisfy the assumptions of Theorem* 2, *if* (Y, T_Y) *is a Baire space, if all the sections* f_x *of a function* $f\colon X \times Y \to Z$ *are continuous, and all sections* f^y *have the property* (R), *then* f *has also the property* (R).

E x a m p l e . Let $X = Y = Z = \mathbb{R}$, let T_X, T_Y be the euclidean topologies and let ϱ be the euclidean metric. Denote by W the set of all rationals. There are dense sets $W_{nk} \subset W$ $(n, k = 1, 2 \dots)$ such that

$$W_{n_1k_1} \cap W_{n_2k_2} = \emptyset \qquad \text{if} \quad (n_1, k_1) \neq (n_2, k_2), \qquad \text{and} \quad 0 \in W_{11}.$$

Let

$$B_0 = \{b_{01}, b_{02}, \dots, b_{0k}, \dots\} = W_{11} \qquad \text{with} \quad b_{01} = 0$$

and let

$$B_1 = \bigcup_{k=1}^{\infty} \big(((-1/k,\ 1/k) \cap W_{1k}) \times \{b_{0k}\} \big).$$

The set B_1 is nowhere dense in $\mathbb{R}^2$. Let $(P_1, \dots, P_k, \dots)$ be a basis of open sets in $\mathbb{R}^2$. There is a nonempty open set $Q_1 \subset P_1$ such that $B_1 \cap \mathrm{Cl}(Q_1) = \emptyset$. Since the set $B_1 - \{(0,y)\colon\ y \in \mathbb{R}\}$ is countable,

$$B_1 - \{(0,y)\colon\ y \in \mathbb{R}\} = \big((x_{1n},\ y_{1n})\big)_{n=1}^{\infty}.$$

For each point (x_{1n}, y_{1n}), $n = 1, 2, \dots$, there is a positive number $r_{1n} < 1/n$ such that the set

$$B_2 = \bigcup_{n=1}^{\infty} \big(\{x_{1n}\} \times ((y_{1n} - r_{1n},\ y_{1n} + r_{1n}) \cap W_{2n}) \big)$$

does not intersect the set $\mathrm{Cl}(Q_1)$. Since the set B_2 is nowhere dense, there is a nonempty open set $Q_2 \subset P_2$ such that $B_2 \cap \mathrm{Cl}(Q_2) = \emptyset$. Generally, if k is even, then we define the set

$$B_{k+1} = \bigcup_{n=1}^{\infty} \left(\{x_{kn}\} \times \left((y_{kn} - r_{kn},\, y_{kn} + r_{kn}) \cap W_{kn}\right)\right)$$

such that $B_k - B_{k+1} = \left((x_{kn},\, y_{kn})\right)_{n=1}^{\infty}$, $0 < r_{kn} < 1/n$ and

$$B_{k+1} \cap \bigcup_{i=1}^{k} \mathrm{Cl}(Q_i) = \emptyset .$$

Analogously, if k is odd, then we define the set

$$B_{k+1} = \bigcup_{n=1}^{\infty} \left(\left((x_{kn} - r_{kn},\, x_{kn} + r_{kn}) \cap W_{kn}\right) \times \{y_{kn}\}\right)$$

such that $B_k - B_{k+1} = \left((x_{kn},\, y_{kn})\right)_{n=1}$, $0 < r_{kn} < 1/n$ and $B_{k+1} \cap \mathrm{Cl}(Q_i) = \emptyset$ for $i = 1, \dots, k$. Since the set B_{k+1} is nowhere dense, there is a nonempty open set $Q_{k+1} \subset P_{k+1}$ such that $B_{k+1} \cap \mathrm{Cl}(Q_{k+1}) = \emptyset$. Put $B = B_1 \cup B_2 \cup \dots$. Since for every $k = 1, 2, \dots$, $Q_k \cap B = \emptyset$, the set B is nowhere dense. Let us put

$$f(x,y) = \begin{cases} 1 & \text{for } (x,y) \in B \\ 0 & \text{for } (x,y) \in R^2 - B . \end{cases}$$

All sections f_x, f^y are almost continuous, but f has not the property (P).

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