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LINEAR MODEL WITH VARIANCES DEPENDING ON THE MEAN VALUE

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ABSTRACT. The paper shows locally best linear unbiased estimators and uniformly best linear unbiased estimators in a linear model, where the dispersions depend quadratically on the mean value.

Introduction

The process of observing the linear combinations of unknown parameters is characterized by the well-known regression model $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \Sigma)$. The result of the observations is a realization of a random vector $\boldsymbol{Y}_{n,1}$, whose mean value is $E_{\boldsymbol{\beta}}(\boldsymbol{Y}) = \mathbf{X}\boldsymbol{\beta}$ ($\mathbf{X}_{n,k}$ is a known design matrix and $\boldsymbol{\beta}_{k,1} \in \mathbb{R}^k$ the vector of unknown parameters). The covariance matrix of the vector $\boldsymbol{Y}$ in that model does not depend on $\boldsymbol{\beta}$.

The last assumption cannot be satisfied in many situations. In the case when the measuring device has its dispersion characteristic of the form $\sigma^2\big(a + b|E_{\boldsymbol{\beta}}(\mathbf{e}_i'\boldsymbol{Y})|\big)^2$, where σ^2, a and b are known positive constants, $\mathbf{e}_i'$ is the transpose of the ith unity vector, and observations are independent, we get the linear model

$$(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \Sigma),$$

where

$$\Sigma = \sigma^2\Sigma(\boldsymbol{\beta}) = \begin{pmatrix} \sigma^2\big(a + b|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}|\big)^2 & 0 & \dots & 0 \\ 0 & \sigma^2\big(a + b|\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}|\big)^2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \sigma^2\big(a + b|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}|\big)^2 \end{pmatrix}. \tag{1}$$

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The aim of the paper is to find the β_0-locally best linear unbiased estimator (β_0-LBLUE) of a linear function of the parameter β (in Section 2) and also the uniformly best linear unbiased estimator (UBLUE) (in Section 3).

The necessary and sufficient condition for the existence of the β_0-LBLUE is in Lemma 2.4, where also the expression of it can be found.

In Section 3 there are three main results: Necessary and sufficient conditions for the existence of the UBLUE of a linear function of the parameter β in the case when only one additional linearly dependent measurement is made (Corollary 3.9 and Corollary 3.10) and also a solution to that problem in the case of two additional measurements (Corollary 3.11).

These results enable us to find a solution in the case when none or several additional measurements are made.

1. Preliminaries

Let us denote

$$\mathcal{O}_{\text{lin}} = \{\boldsymbol{b}'\boldsymbol{Y}:\ E_{\boldsymbol{\beta}}(\boldsymbol{b}'\boldsymbol{Y}) = 0\ \ \forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}\}$$

the class of all linear unbiased estimators of the function $g(\cdot)\colon \mathbb{R}^k \to \{0\}$.

DEFINITION 1.1. *The linear statistic* $\boldsymbol{p}'\boldsymbol{Y}$ *is said to be*

1. *the* β_0*-locally best linear unbiased estimator* (β_0-LBLUE) *of its mean value* $E_{\boldsymbol{\beta}_0}(\boldsymbol{p}'\boldsymbol{Y})$ *if for any other linear statistic* $\boldsymbol{q}'\boldsymbol{Y}$ *having the property*

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}\quad E_{\boldsymbol{\beta}}(\boldsymbol{p}'\boldsymbol{Y}) = E_{\boldsymbol{\beta}}(\boldsymbol{q}'\boldsymbol{Y}) \tag{$*$}$$

the relation

$$\mathcal{D}_{\boldsymbol{\beta}_0}(\boldsymbol{p}'\boldsymbol{Y}) = E_{\boldsymbol{\beta}_0}\Big(\big(\boldsymbol{p}'\boldsymbol{Y} - E_{\boldsymbol{\beta}_0}(\boldsymbol{p}'\boldsymbol{Y})\big)^2\Big) \leqq \mathcal{D}_{\boldsymbol{\beta}_0}(\boldsymbol{q}'\boldsymbol{Y})$$

holds;

2. *the uniformly best linear unbiased estimator* (UBLUE) *of its mean value* $E_{\boldsymbol{\beta}}(\boldsymbol{p}'\boldsymbol{Y})$ *if for any other linear statistic* $\boldsymbol{q}'\boldsymbol{Y}$ *having the property* $(*)$ *there holds*

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}\quad \mathcal{D}_{\boldsymbol{\beta}}(\boldsymbol{p}'\boldsymbol{Y}) \leqq \mathcal{D}_{\boldsymbol{\beta}}(\boldsymbol{q}'\boldsymbol{Y}).$$

THEOREM 1.2. *In model* (1), $\boldsymbol{p}'\boldsymbol{Y}$ *is the* β_0-LBLUE *of its mean value if and only if*

$$\forall\{\boldsymbol{b}'\boldsymbol{Y} \in \mathcal{O}_{\text{lin}}\}\quad E_{\boldsymbol{\beta}_0}(\boldsymbol{b}'\boldsymbol{Y}\boldsymbol{Y}'\boldsymbol{p}) = 0.$$

The statistic $\boldsymbol{p}'\boldsymbol{Y}$ *is the* UBLUE *of its mean value if and only if*

$$\forall\{\boldsymbol{b}'\boldsymbol{Y} \in \mathcal{O}_{\text{lin}}\}\ \forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}\quad E_{\boldsymbol{\beta}}(\boldsymbol{b}'\boldsymbol{Y}\boldsymbol{Y}'\boldsymbol{p}) = 0.$$

P r o o f. See [1], Theorem 3.1 and the following Corollary.

DEFINITION 1.3. $\mathbf{X}^-$ *is a matrix satisfying the equation* $\mathbf{X}\mathbf{X}^-\mathbf{X} = \mathbf{X}$. *It is a g-inversion of* $\mathbf{X}$.

For any fixed positive definite matrix $\mathbf{W}$ the matrix $\mathbf{G}$ satisfying the equations

$$\mathbf{XGX} = \mathbf{X}, \qquad (\mathbf{GX})'\mathbf{W} = \mathbf{WGX}$$

is said to be the minimum $\mathbf{W}$-norm g-inverse of $\mathbf{X}$. For $\mathbf{G}$ we use the notation $\mathbf{X}^-_{m(\mathbf{W})}$.

2. β_0-LBLUE

LEMMA 2.1. *The statistic* $\boldsymbol{b}'\boldsymbol{Y}$ *belongs to* $\mathcal{O}_{\text{lin}}$ *if and only if* $\boldsymbol{b} \in \operatorname{Ker}\mathbf{X}' = \{\boldsymbol{c} \in \mathbb{R}^n\colon\ \mathbf{X}'\boldsymbol{c} = \boldsymbol{O}\} = \{(\mathbf{I}-(\mathbf{X}')^-\mathbf{X}')\boldsymbol{u}\colon\ \boldsymbol{u} \in \mathbb{R}^n,\ (\mathbf{X}')^-$ *is an arbitrary but fixed g-inverse of* $\mathbf{X}'\}$.

Proof.
$\boldsymbol{b}'\boldsymbol{Y} \in \mathcal{O}_{\text{lin}} \iff E_{\boldsymbol{\beta}}(\boldsymbol{b}'\boldsymbol{Y}) = 0\ \forall\{\beta \in \mathbb{R}^k\} \iff \boldsymbol{b}'\mathbf{X}\beta = 0\ \forall\{\beta \in \mathbb{R}^k\} \iff \boldsymbol{b}'\mathbf{X} = \boldsymbol{O} \iff \boldsymbol{b} \in \operatorname{Ker}\mathbf{X}'$.

The proof of the last equation in Lemma 2.1 is in [2], Theorem 2.3.1.

LEMMA 2.2. $\boldsymbol{p}'\boldsymbol{Y}$ *is the* β_0-LBLUE *of its mean value if and only if* $\boldsymbol{p} \in \{(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}'\boldsymbol{z}\colon\ \boldsymbol{z} \in \mathbb{R}^n,\ (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}$ *is an arbitrary but fixed minimum* $\Sigma(\beta_0)$*-norm g-inverse of* $\mathbf{X}'\}$.

Proof. According to Theorem 1.2 and Lemma 2.1 $\boldsymbol{p}'\boldsymbol{Y}$ is the β_0-LBLUE of its mean value if and only if

$$\forall\{\boldsymbol{b} \in \operatorname{Ker}\mathbf{X}'\}\quad \sigma^2\boldsymbol{b}'\Sigma(\beta_0)\boldsymbol{p} = 0$$

$$\iff \forall\{\boldsymbol{u} \in \mathbb{R}^n\}\quad \boldsymbol{u}'\Big(\mathbf{I} - \mathbf{X}(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\Big)\Sigma(\beta_0)\boldsymbol{p} = 0$$

for an arbitrary but fixed $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}$. The last assertion is valid if and only if $\Big(\mathbf{I} - \mathbf{X}(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\Big)\boldsymbol{p} = \boldsymbol{O}$ ($\Sigma(\beta_0)$ is a p.d. matrix). According to [2], Theorem 2.3.1, $(\mathbf{I} - \mathbf{X}(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))})\boldsymbol{p} = \boldsymbol{O}$ if and only if

$\boldsymbol{p} \in \Big\{\Big[\mathbf{I} - (\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}')^-(\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}')\Big]\boldsymbol{z}\colon\ \boldsymbol{z} \in \mathbb{R}^n,\ (\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}')^-$ is an arbitrary but fixed g-inverse of matrix $\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}'\Big\}$.

One choice of $\big(\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}'\big)^-$ is $\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}'$, that is why

$$\boldsymbol{p} \in \{(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_0))}\mathbf{X}'\boldsymbol{z}\colon\ \boldsymbol{z} \in \mathbb{R}^n\}.$$

The lemma is proved.

COROLLARY 2.3. *One choice of* $(\mathbf{X}')^-_{m(\Sigma(\beta_0))}$ *is* $\Sigma^{-1}(\beta_0)\mathbf{X}\left(\mathbf{X}'\Sigma^{-1}(\beta_0)\mathbf{X}\right)^-$ *and that is why the class of* β_0 *-LBLUE s of its mean value in model* (1) *is*

$$\left\{\boldsymbol{p}'\boldsymbol{Y}:\ \boldsymbol{p}\in\left\{\Sigma^{-1}(\beta_0)\mathbf{X}\left(\mathbf{X}'\Sigma^{-1}(\beta_0)\mathbf{X}\right)^-\mathbf{X}'\boldsymbol{z}:\ \boldsymbol{z}\in\mathbb{R}^n\right\}\right\}.$$

LEMMA 2.4. *For the linear function* $\boldsymbol{f}'\beta$ *of parameter* $\boldsymbol{\beta}=(\beta_1,\dots,\beta_k)'\in\mathbb{R}^k$ *there exists a* $\boldsymbol{\beta}_0$ *-LBLUE if and only if* $\boldsymbol{f}\in\mu(\mathbf{X}')=\{\mathbf{X}'\boldsymbol{u}:\ \boldsymbol{u}\in\mathbb{R}^n\}$.

Proof. If $\boldsymbol{f}\in\mu(\mathbf{X}')$, then there exists a vector $\boldsymbol{u}_0\in\mathbb{R}^n$ so that $\boldsymbol{f}=\mathbf{X}'\boldsymbol{u}_0$ and $\boldsymbol{u}_0'\mathbf{X}\left[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\right]'\boldsymbol{Y}$ is the β_0-LBLUE of $\boldsymbol{f}'\beta$.

Conversely, if $\boldsymbol{p}'\boldsymbol{Y}$ is the β_0-LBLUE of $\boldsymbol{f}'\beta$, then

$$\forall\{\beta\in\mathbb{R}^k\}\quad E_{\beta}(\boldsymbol{p}'\boldsymbol{Y})=\boldsymbol{f}'\beta,\quad \text{i.e.}$$
$$\forall\{\beta\in\mathbb{R}^k\}\quad \boldsymbol{p}'\mathbf{X}\beta=\boldsymbol{f}'\beta.$$

The last assertion yields $\boldsymbol{p}'\mathbf{X}=\boldsymbol{f}'\iff\boldsymbol{f}\in\mu(\mathbf{X}')$. The lemma is proved.

Remark 2.5. One version of the β_0-LBLUE of $\boldsymbol{f}'\beta$, $\beta\in\mathbb{R}^k$ (for $\boldsymbol{f}\in\mu(\mathbf{X}')$) is $\boldsymbol{f}'\left(\mathbf{X}'\Sigma^{-1}(\beta_0)\mathbf{X}\right)^-\mathbf{X}'\Sigma^{-1}(\beta_0)\boldsymbol{Y}$.

3. UBLUE

LEMMA 3.1. *The statistic* $\boldsymbol{p}'\boldsymbol{Y}$ *is the* UBLUE *of its mean value if and only if*

$$\forall\{\beta\in\mathbb{R}^k\}\qquad(\mathbf{I}-\mathbf{X}\mathbf{X}^-)\Sigma(\beta)\boldsymbol{p}=\boldsymbol{O}\tag{2}$$

for an arbitrary but fixed g-inverse $\mathbf{X}^-$.

Proof. The assertion is a consequence of Theorem 1.2 and Lemma 2.1 and is omitted.

LEMMA 3.2. *If the statistic* $\boldsymbol{p}'\boldsymbol{Y}$ *is the* UBLUE *of its mean value, then* $\boldsymbol{p}\in\mu(\mathbf{X})$.

Proof. If in (2) we take $\beta=\boldsymbol{O}$, then the fact that $\boldsymbol{p}'\boldsymbol{Y}$ is the UBLUE of its mean value implies

$$a^2(\mathbf{I}-\mathbf{X}\mathbf{X}^-)\boldsymbol{p}=\boldsymbol{O}\iff(\mathbf{I}-\mathbf{X}\mathbf{X}^-)\boldsymbol{p}=\boldsymbol{O}\iff\boldsymbol{p}\in\mu(\mathbf{X}).$$

The proof is complete.

LEMMA 3.3. *The statistic* $\mathbf{p}'\mathbf{Y}$ *is the* UBLUE *of its mean value if and only if there exists a vector* $\mathbf{w}^0 \in \mathbb{R}^k$ *so that for every* $\boldsymbol{\beta} \in \mathbb{R}^k$ *there exists an* $\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k$ *that the relations*

$$\Sigma(\boldsymbol{\beta})\mathbf{X}\mathbf{w}^0 = \mathbf{X}\boldsymbol{\alpha}(\boldsymbol{\beta}) \qquad \textit{and} \qquad \mathbf{p} = \mathbf{X}\mathbf{w}^0 \tag{3}$$

hold.

Proof. If $\mathbf{p}'\mathbf{Y}$ is the UBLUE of its mean value, then according to Lemma 3.1 and Lemma 3.2 there exists a vector $\mathbf{w}^0 \in \mathbb{R}^k$ that $\mathbf{p} = \mathbf{X}\mathbf{w}^0$ and

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad (\mathbf{I} - \mathbf{X}\mathbf{X}^-)\Sigma(\boldsymbol{\beta})\mathbf{X}\mathbf{w}^0 = \mathbf{O}.$$

The last assertion is equivalent to

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \Sigma(\boldsymbol{\beta})\mathbf{X}\mathbf{w}^0 = \mathbf{X}\mathbf{X}^-\Sigma(\boldsymbol{\beta})\mathbf{X}\mathbf{w}^0,$$

which is satisfied, according to Lemma 2.2.4 in [2], if and only if

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \Sigma(\boldsymbol{\beta})\mathbf{X}\mathbf{w}^0 \in \mu(\mathbf{X}),$$

that is if and only if

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}\ \exists\{\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k\} \quad \text{that} \qquad \Sigma(\boldsymbol{\beta})\mathbf{X}\mathbf{w}^0 = \mathbf{X}\boldsymbol{\alpha}(\boldsymbol{\beta}).$$

Conversely, from the equivalence of the assertions

$$\begin{aligned} &\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad (\mathbf{I} - \mathbf{X}\mathbf{X}^-)\Sigma(\boldsymbol{\beta})\mathbf{p} = \mathbf{O} \\ \iff &\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}\ \exists\{\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k\} \quad \text{that} \qquad \Sigma(\boldsymbol{\beta})\mathbf{p} = \mathbf{X}\boldsymbol{\alpha}(\boldsymbol{\beta}) \end{aligned}$$

and Lemma 3.1 we easy complete the proof.

COROLLARY 3.4. *If* $\mathbf{p}$ *is such a vector from* $\mathbb{R}^n$ *that for each* $i = 1, 2, \dots, n$ $\mathbf{e}_i'\mathbf{p} = 0$ *or* $\mathbf{X}\mathbf{X}^-\mathbf{e}_i = \mathbf{e}_i$ *or simultaneously* $\mathbf{e}_i'\mathbf{p} = 0$ *and* $\mathbf{X}\mathbf{X}^-\mathbf{e}_i = \mathbf{e}_i$ *(i.e. if for* $i = 1, 2, \dots, n$ *the ith component of the vector* $\mathbf{p}$ *is not zero but* $\mathbf{e}_i \in \mu(\mathbf{X})$ *(* $\iff \mathbf{X}\mathbf{X}^-\mathbf{e}_i = \mathbf{e}_i$ *and it does not depend on the choice of* $\mathbf{X}^-$ *)), then* $\mathbf{p}'\mathbf{Y}$ *is the* UBLUE *of its mean value.*

The condition in Corollary 3.4 is only a sufficient one for $\mathbf{w}^{0\prime}\mathbf{X}\mathbf{Y}$ to be the UBLUE of its mean value. The next example shows it.

Example 3.5.

Let $\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$, then $\mathbf{XX}^- = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix}$ i.e. $\mathbf{XX}^-\mathbf{e}_i \neq \mathbf{e}_i$ $i = 2,3$.

The statistic

$$\boldsymbol{p}'\boldsymbol{Y} = (1 \quad 1)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = (2 \quad 3 \quad 3)\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

is the UBLUE of its mean value in spite of $\mathbf{e}_2'\boldsymbol{p}$ and $\mathbf{e}_3'\boldsymbol{p}$ being different from zero. This fact can be seen from the assertion

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^2\} \quad (\mathbf{I} - \mathbf{XX}^-)\boldsymbol{\Sigma}(\boldsymbol{\beta})\boldsymbol{p}$$

$$= \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} (a + b|\beta_1 + \beta_2|)^2 & 0 & 0 \\ 0 & (a + b|\beta_1 + 2\beta_2|)^2 & 0 \\ 0 & 0 & (a + b|\beta_1 + 2\beta_2|)^2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \boldsymbol{O}$$

and Lemma 3.1.

Let us rearrange the rows in the matrix $\mathbf{X}$ to obtain the matrix $\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}$, where $\mathbf{X}_1$ is a matrix of order $R(\mathbf{X}) \times k$ ($R(\mathbf{X})$ is the rank of $\mathbf{X}$) and $\mathbf{X}_2 = \mathbf{EX}_1$, where $\mathbf{E} = \mathbf{X}_2\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}$ is of order $(n - R(\mathbf{X})) \times R(\mathbf{X})$.

In the same way we rearrange the coordinates of $\boldsymbol{Y}$ and the rows of the matrix $\boldsymbol{\Sigma}(\boldsymbol{\beta})$. We obtain the vector $\widetilde{\boldsymbol{Y}}$ and the matrix

$$\begin{pmatrix} \boldsymbol{\Sigma}_1(\boldsymbol{\beta}) & 0 \\ 0 & \boldsymbol{\Sigma}_2(\boldsymbol{\beta}) \end{pmatrix},$$

where

$$\boldsymbol{\Sigma}_1(\boldsymbol{\beta}) = \begin{pmatrix} (a + b|\mathbf{e}_1'\mathbf{X}_1\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a + b|\mathbf{e}_2'\mathbf{X}_1\boldsymbol{\beta}|)^2 & & \\ \vdots & & \ddots & \\ 0 & & & (a + b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{X}_1\boldsymbol{\beta}|)^2 \end{pmatrix}$$

and

$$\boldsymbol{\Sigma}_2(\boldsymbol{\beta}) = \begin{pmatrix} (a + b|\mathbf{e}_1'\mathbf{EX}_1\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a + b|\mathbf{e}_2'\mathbf{EX}_1\boldsymbol{\beta}|)^2 & & \\ \vdots & & \ddots & \\ 0 & & & (a + b|\mathbf{e}_{n-R(\mathbf{X})}'\mathbf{EX}_1\boldsymbol{\beta}|)^2 \end{pmatrix}.$$

From Lemma 3.3 we immediately obtain the next

COROLLARY 3.6. *The statistic* $\boldsymbol{p}'\widetilde{\boldsymbol{Y}}$ *is the* UBLUE *of its mean value if and only if there exists such a vector* $\boldsymbol{w}^0 \in \mathbb{R}^k$ *that for every* $\beta \in \mathbb{R}^k$ *there exists an* $\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k$ *so that the relations*

$$\begin{pmatrix} \Sigma_1(\beta) & 0 \\ 0 & \Sigma_2(\beta) \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{w}^0 = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \alpha(\beta)$$

and

$$\boldsymbol{p} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{w}^0$$

hold.

Finally we have

THEOREM 3.7. *The statistic* $\boldsymbol{p}'\widetilde{\boldsymbol{Y}}$ *is the* UBLUE *of its mean value if and only if* $\boldsymbol{p} = \left(\mathbf{I} \,\vdots\, \mathbf{E}'\right)' \boldsymbol{a}$, *where*

$$\boldsymbol{a} \in \bigcap_{\beta \in \mathbb{R}^k} \operatorname{Ker}\left[\Sigma_2(\beta)\mathbf{E} - \mathbf{E}\Sigma_1(\beta)\right]$$

$$= \bigcap_{j=1}^{n-R(\mathbf{X})} \left(\bigcap_{\beta \in \mathbb{R}^k} \operatorname{Ker}\left\{ \mathbf{e}_j' \left[\Sigma_2(\beta)\mathbf{E} - \mathbf{E}\Sigma_1(\beta)\right] \right\} \right)$$

$$= \bigcap_{j=1}^{n-R(\mathbf{X})} \left(\bigcap_{\mathbf{u} \in \mathbb{R}^{R(\mathbf{X})}} \operatorname{Ker} \left\{ (a + b|\mathbf{e}_j'\mathbf{E}\boldsymbol{u}|)^2 \mathbf{e}_j'\mathbf{E} - \mathbf{e}_j'\mathbf{E} \begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{u}|)^2 & \dots & 0 \\ 0 & & \\ \vdots & \ddots & \\ 0 & & (a+b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{u}|)^2 \end{pmatrix} \right\} \right)$$

$$= \bigcap_{j=1}^{n-R(\mathbf{X})} \mathcal{M}_j = \mathcal{M}\,.$$

P r o o f . From the equality

$$\mathbf{e}_j'\left(\Sigma_2(\beta)\mathbf{E} - \mathbf{E}\Sigma_1(\beta)\right) = \left(a + b|\mathbf{e}_j'\mathbf{E}\mathbf{X}_1\beta|\right)^2 \mathbf{e}_j'\mathbf{E} - \mathbf{e}_j'\mathbf{E}\Sigma_1(\beta)$$

and the fact, that $\mu(\mathbf{X}_1) = \mathbb{R}^{R(\mathbf{X})}$, we obtain

$$\mathcal{M}_j = \bigcap_{\beta \in \mathbb{R}^k} \operatorname{Ker}\left\{ \mathbf{e}_j' \left[\Sigma_2(\beta)\mathbf{E} - \mathbf{E}\Sigma_1(\beta)\right] \right\}$$

$$= \bigcap_{\mathbf{u} \in \mathbb{R}^{R(\mathbf{X})}} \operatorname{Ker} \left\{ (a + b|\mathbf{e}_j'\mathbf{E}\boldsymbol{u}|)^2 \mathbf{e}_j'\mathbf{E} - \mathbf{e}_j'\mathbf{E} \begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{u}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{u}|)^2 & & \\ \vdots & & \ddots & \\ 0 & & & (a+b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{u}|)^2 \end{pmatrix} \right\}. \tag{4}$$

Let now $\boldsymbol{p} = \left(\mathbf{I} \vdots \mathbf{E}'\right)' \boldsymbol{a}$, where $\boldsymbol{a} \in \bigcap_{\boldsymbol{\beta}\in\mathbb{R}^k} \operatorname{Ker}\left[\Sigma_2(\boldsymbol{\beta})\mathbf{E} - \mathbf{E}\Sigma_1(\boldsymbol{\beta})\right]$. The ranks of the matrices $\mathbf{X}_1$ and $\left(\mathbf{X}_1 \vdots \Sigma_1(\boldsymbol{\beta})\mathbf{X}_1\left[\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a}\right]\right)$ are the same for every $\boldsymbol{\beta} \in \mathbb{R}^k$. According to the well-known Cronecker's theorem, for every $\boldsymbol{\beta} \in \mathbb{R}^k$ there exists an $\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k$ that

$$\Sigma_1(\boldsymbol{\beta})\mathbf{X}_1\left[\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a}\right] = \mathbf{X}_1\boldsymbol{\alpha}(\boldsymbol{\beta}). \tag{5}$$

The equality (5) together with the fact that

$$\begin{aligned}\Sigma_2(\boldsymbol{\beta})\mathbf{X}_2\left[\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a}\right] &= \Sigma_2(\boldsymbol{\beta})\mathbf{E}\mathbf{X}_1\left[\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a}\right] = \Sigma_2(\boldsymbol{\beta})\mathbf{E}\boldsymbol{a}\\ &= \mathbf{E}\Sigma_1(\boldsymbol{\beta})\boldsymbol{a} = \mathbf{E}\Sigma_1(\boldsymbol{\beta})\mathbf{X}_1\left[\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a}\right] = \mathbf{E}\mathbf{X}_1\boldsymbol{\alpha}(\boldsymbol{\beta}) = \mathbf{X}_2\boldsymbol{\alpha}(\boldsymbol{\beta})\end{aligned}$$

imply that for every $\boldsymbol{\beta} \in \mathbb{R}^k$ there exists an $\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k$ so that

$$\begin{pmatrix}\Sigma_1(\boldsymbol{\beta}) & 0\\ 0 & \Sigma_2(\boldsymbol{\beta})\end{pmatrix}\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a} = \begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\alpha}(\boldsymbol{\beta})$$

and $\boldsymbol{p} = \left(\mathbf{I} \vdots \mathbf{E}'\right)'\boldsymbol{a} = \left(\mathbf{I} \vdots \mathbf{E}'\right)'\mathbf{X}_1\left[\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a}\right] = \left(\mathbf{X}_1' \vdots \mathbf{X}_2'\right)'\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\boldsymbol{a}$. According to Lemma 3.6, $\boldsymbol{p}'\widetilde{\boldsymbol{Y}}$ is the UBLUE of its mean value.

Conversely, if $\boldsymbol{p}'\widetilde{\boldsymbol{Y}}$ is the UBLUE of its mean value, then, according to Corollary 3.6, there exists a vector $\boldsymbol{w}^0 \in \mathbb{R}^k$ that for every $\boldsymbol{\beta} \in \mathbb{R}^k$ there exists an $\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k$ so that the relations

$$\begin{pmatrix}\Sigma_1(\boldsymbol{\beta}) & 0\\ 0 & \Sigma_2(\boldsymbol{\beta})\end{pmatrix}\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{w}^0 = \begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\alpha}(\boldsymbol{\beta})$$

and

$$\boldsymbol{p} = \begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{w}^0$$

hold. That is why for every $\boldsymbol{\beta} \in \mathbb{R}^k$ there exists an $\boldsymbol{\alpha}(\boldsymbol{\beta}) \in \mathbb{R}^k$ that the equations

$$\begin{aligned}\Sigma_1(\boldsymbol{\beta})\boldsymbol{a} &= \mathbf{X}_1\boldsymbol{\alpha}(\boldsymbol{\beta})\\ \Sigma_2(\boldsymbol{\beta})\mathbf{E}\boldsymbol{a} &= \mathbf{E}\mathbf{X}_1\boldsymbol{\alpha}(\boldsymbol{\beta})\end{aligned}$$

and $\boldsymbol{p} = \left(\mathbf{I} \vdots \mathbf{E}'\right)'\boldsymbol{a}$ are valid, where $\boldsymbol{a} = \mathbf{X}_1\boldsymbol{w}^0$. That is why $\boldsymbol{p} = \left(\mathbf{I} \vdots \mathbf{E}'\right)'\boldsymbol{a}$, where $\boldsymbol{a} \in \bigcap_{\boldsymbol{\beta}\in\mathbb{R}^k} \operatorname{Ker}\left[\Sigma_2(\boldsymbol{\beta})\mathbf{E} - \mathbf{E}\Sigma_1(\boldsymbol{\beta})\right]$. The theorem is proved.

LEMMA 3.8. *Let* $\mathbf{e}_j'\mathbf{E} = t\mathbf{e}_i'$ *(i.e. the* $(R(\mathbf{X})+j)$*th row of the matrix* $\mathbf{X}$ *is its* i*th row multiplied by* t*), where* $t \neq 0$, $j \in \{1,2,\dots,n-R(\mathbf{X})\}$, $i \in \{1,2,\dots,R(\mathbf{X})\}$.

1. $\mathcal{M}_j = \mathbb{R}^{R(\mathbf{X})}$ *if and only if* $|t| = 1$ *(c.f. the notation from Theorem 3.7).*
2. *If* $|t| \neq 1$, *then* $\mathcal{M}_j = \{\boldsymbol{a} \in \mathbb{R}^{R(\mathbf{X})}:\ \mathbf{e}_i'\boldsymbol{a} = 0\}$.

Proof. If we denote $\mathbf{X}_1\beta = \boldsymbol{u}$, then

$$\begin{aligned}
&(a+b|\mathbf{e}_j'\mathbf{E}\boldsymbol{u}|)^2\mathbf{e}_j'\mathbf{E} - \mathbf{e}_j'\mathbf{E}\begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{u}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{u}|)^2 & & \\ \vdots & & \ddots & \\ 0 & & & (a+b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{u}|)^2 \end{pmatrix} \\
&= t\big(a+b|tu_i|\big)^2\mathbf{e}_i' - t\big(a+b|u_i|\big)^2\mathbf{e}_i' \\
&= \mathbf{e}_i'\Big[t\big(a^2+2ab|tu_i|+b^2(tu_i)^2\big) - t\big(a^2+2ab|u_i|+b^2u_i^2\big)\Big] \\
&= \mathbf{e}_i'\Big[2ab|u_i|t\big(|t|-1\big) + b^2u_i^2t(t^2-1)\Big] \\
&= \mathbf{e}_i'\big(|t|-1\big)tb\Big[2a|u_i| + b^2u_i^2\big(|t|+1\big)\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
&(a+b|\mathbf{e}_j'\mathbf{E}\boldsymbol{u}|)^2\mathbf{e}_j'\mathbf{E} - \mathbf{e}_j'\mathbf{E}\begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{u}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{u}|)^2 & & \\ \vdots & & \ddots & \\ 0 & & & (a+b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{u}|)^2 \end{pmatrix} \\
&\qquad = \mathbf{e}_i'\big(|t|-1\big)tb\Big[2a|u_i| + b^2u_i^2\big(|t|+1\big)\Big].
\end{aligned} \tag{6}$$

From (4) and (6) we have

$$\boldsymbol{a} \in \mathcal{M}_j \iff \forall\{u_i \in \mathbb{R}\}\Big[\big(|t|-1\big)tb\big(2a|u_i| + b^2u_i^2\big(|t|+1\big)\big)\Big]\mathbf{e}_i'\boldsymbol{a} = 0. \tag{7}$$

Both assertions of the lemma are a simple consequence of (7).

COROLLARY 3.9. *Let* $n = R(\mathbf{X})+1$, $\mathbf{E} = t\mathbf{e}_i'$ ($t \neq 0$ *and* $i \in \{1,2,\dots,R(\mathbf{X})\}$).

1. *If* $|t| = 1$, $\boldsymbol{p}'\widetilde{\boldsymbol{Y}}$ *is the* UBLUE *of its mean value if and only if*

$$\boldsymbol{p} = \big(\mathbf{I}_{R(\mathbf{X}),R(\mathbf{X})} \ \vdots\ \pm \mathbf{e}_i\big)'\boldsymbol{a},$$

where $\mathbf{a} \in \mathbb{R}^{R(\mathbf{X})}$. *It means in this case that* $\mathbf{p}'\widetilde{\mathbf{Y}} = \mathbf{a}'(\mathbf{I} \vdots t\mathbf{e}_i)\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if and only if it belongs to the class*

$$\{a_1\widetilde{Y}_1 + \cdots + a_i(\widetilde{Y}_i \pm \widetilde{Y}_n) + \cdots + a_{R(\mathbf{X})}\widetilde{Y}_{R(\mathbf{X})}:\ a_i \in \mathbb{R},\ i = 1,2,\ldots,R(\mathbf{X})\}.$$

2. *If* $|t| \neq 1$, *the* UBLUE*s of its mean value are all the linear functions of* $\widetilde{\mathbf{Y}}$ *which do not contain* $\widetilde{Y}_i$ *and* $\widetilde{Y}_n$.

We only remark that in this case the mean value of the n*th measurement is equal to* t *times the mean value of the* i*th measurement.*

The proof is a consequence of Theorem 3.7 and Lemma 3.8.

THEOREM 3.9. *Let* $\mathbf{e}_l'\mathbf{E} = \gamma'$, *where* $\gamma = \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}$, $t \geqq 2$, $\gamma_i \neq 0$ $i = 1,2,\ldots,t \leqq R(\mathbf{X})$, $s_i \in \{1,2,\ldots,R(\mathbf{X})\}$, $l \in \{1,2,\ldots,n-R(\mathbf{X})\}$. *Then*

$$\mathcal{M}_l = \{\mathbf{a} \in \mathbb{R}^{R(\mathbf{X})}:\ \mathbf{e}_{s_i}'\mathbf{a} = 0,\ i = 1,2,\ldots,t\}.$$

Proof. From (4) we have

$$\mathcal{M}_l =$$

$$\bigcap_{\mathbf{u}\in\mathbb{R}^{R(\mathbf{X})}} \operatorname{Ker}\left\{(a + b|\mathbf{e}_l'\mathbf{E}\mathbf{u}|)^2\mathbf{e}_l'\mathbf{E} - \mathbf{e}_l'\mathbf{E}\begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{u}|)^2 & 0 & \cdots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{u}|)^2 & & \\ \vdots & & \ddots & \\ 0 & & & (a+b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{u}|)^2 \end{pmatrix}\right\} =$$

$$\bigcap_{\mathbf{u}\in\mathbb{R}^{R(\mathbf{X})}} \operatorname{Ker}\left\{\left(a + b\left|\sum_{i=1}^{t}\gamma_i\mathbf{e}_{s_i}'\mathbf{u}\right|\right)^2 \sum_{i=1}^{t}\gamma_i\mathbf{e}_{s_i}' - \sum_{i=1}^{t}\gamma_i\mathbf{e}_{s_i}'\begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{u}|)^2 & 0 & \cdots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{u}|)^2 & & \\ \vdots & & \ddots & \\ 0 & & & (a+b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{u}|)^2 \end{pmatrix}\right\} =$$

$$= \bigcap_{u_q\in\mathbb{R}:\ q\in\{1,2,\ldots,R(\mathbf{X})\}} \operatorname{Ker}\left\{\sum_{i=1}^{t}\mathbf{e}_{s_i}'\gamma_i\left[\left(a + b\left|\sum_{i=1}^{t}\gamma_i u_{s_i}\right|\right)^2 - (a + b|u_{s_i}|)^2\right]\right\}.$$

That is why

$$\boldsymbol{a} \in \mathcal{M}_l \iff \sum_{i=1}^{t} \boldsymbol{e}'_{s_i} \boldsymbol{a} \gamma_i \left[\left(a + b \left| \sum_{i=1}^{t} \gamma_i u_{s_i} \right| \right)^2 - \left(a + b |u_{s_i}| \right)^2 \right] = 0 \tag{8}$$
$$\text{for all } u_{s_i} \in \mathbb{R} \quad i = 1, 2, \ldots, t\,.$$

1. Let $t = 2$.
From (8) it is easy to see that in this case if $\boldsymbol{a} \in \mathcal{M}_l$, then for an arbitrary choice of $u_{s_1}^{(1)}$, $u_{s_2}^{(1)}$, $u_{s_1}^{(2)}$ and $u_{s_2}^{(2)}$

$$\mathbf{R} \begin{pmatrix} a_{s_1} \\ a_{s_2} \end{pmatrix} =$$
$$\begin{pmatrix} \gamma_1 \left[\left(a + b|\gamma_1 u_{s_1}^{(1)} + \gamma_2 u_{s_2}^{(1)}| \right)^2 - \left(a + b|u_{s_1}^{(1)}| \right)^2 \right], & \gamma_2 \left[\left(a + b|\gamma_1 u_{s_1}^{(1)} + \gamma_2 u_{s_2}^{(1)}| \right)^2 - \left(a + b|u_{s_2}^{(1)}| \right)^2 \right] \\ \gamma_1 \left[\left(a + b|\gamma_1 u_{s_1}^{(2)} + \gamma_2 u_{s_2}^{(2)}| \right)^2 - \left(a + b|u_{s_1}^{(2)}| \right)^2 \right], & \gamma_2 \left[\left(a + b|\gamma_1 u_{s_1}^{(2)} + \gamma_2 u_{s_2}^{(2)}| \right)^2 - \left(a + b|u_{s_2}^{(2)}| \right)^2 \right] \end{pmatrix} \begin{pmatrix} a_{s_1} \\ a_{s_2} \end{pmatrix}$$
$$= \mathbf{O}_{2,1}\,. \tag{9}$$

1a. Case $|\gamma_1| \neq |\gamma_2|$.
For $u_{s_1}^{(1)} \neq 0$, $u_{s_2}^{(1)} = -\dfrac{\gamma_1}{\gamma_2} u_{s_1}^{(1)}$, $u_{s_2}^{(2)} \neq 0$ and $u_{s_1}^{(2)} = -\dfrac{\gamma_2}{\gamma_1} u_{s_2}^{(2)}$ the determinant of the matrix $\mathbf{R}$ in (9) is

$$\det \mathbf{R} =$$
$$\gamma_1 \gamma_2 b^2 |u_{s_1}^{(1)}||u_{s_2}^{(2)}| \left\{ (2a + b|u_{s_1}^{(1)}|)(2a + b|u_{s_2}^{(2)}|) - \left(2a + b\left|\frac{\gamma_1}{\gamma_2}\right| |u_{s_1}^{(1)}| \right) \left(2a + b\left|\frac{\gamma_2}{\gamma_1}\right| |u_{s_2}^{(2)}| \right) \right\}$$
$$= \gamma_1 \gamma_2 b^2 |u_{s_1}^{(1)}||u_{s_2}^{(2)}| 2ab \left[|u_{s_1}^{(1)}| \left(1 - \left|\frac{\gamma_1}{\gamma_2}\right| \right) + |u_{s_2}^{(2)}| \left(1 - \left|\frac{\gamma_2}{\gamma_1}\right| \right) \right] \neq 0$$
$$\text{if} \quad |u_{s_1}^{(1)}||\gamma_1| \neq |u_{s_2}^{(2)}||\gamma_2|$$

and that is why, according to (9),

$$\text{if} \quad \boldsymbol{a} \in \mathcal{M}_l \implies a_{s_1} = \boldsymbol{e}'_{s_1} \boldsymbol{a} = a_{s_2} = \boldsymbol{e}'_{s_2} \boldsymbol{a} = 0\,.$$

The converse implication is trivial and the theorem is in the case $|\gamma_1| \neq |\gamma_2|$ proved.

1b. Case $|\gamma_1| = |\gamma_2|$.
For $u_{s_1}^{(1)} \neq 0$ and $u_{s_2}^{(1)} \neq 0$ satisfying the inequality $|u_{s_1}^{(1)}| > |u_{s_2}^{(1)}|$ and for

$u^{(2)}_{s_1} \neq 0$ and $u^{(2)}_{s_2} \neq 0$ satisfying the equation $\gamma_1 u^{(2)}_{s_1} + \gamma_2 u^{(2)}_{s_2} = 0$ the determinant of the matrix $\mathbf{R}$ in (9) is

$$\det \mathbf{R} = \gamma_1\gamma_2 \Big\{ \big[2ab|\gamma_1 u^{(1)}_{s_1} + \gamma_2 u^{(1)}_{s_2}| + b^2(\gamma_1 u^{(1)}_{s_1} + \gamma_2 u^{(1)}_{s_2})^2 - 2ab|u^{(1)}_{s_1}| - b^2(u^{(1)}_{s_1})^2\big]\big[-2ab|u^{(2)}_{s_2}| - b^2(u^{(2)}_{s_2})^2\big] - \big[2ab|\gamma_1 u^{(1)}_{s_1} + \gamma_2 u^{(1)}_{s_2}| + b^2(\gamma_1 u^{(1)}_{s_1} + \gamma_2 u^{(1)}_{s_2})^2 - 2ab|u^{(1)}_{s_2}| - b^2(u^{(1)}_{s_2})^2\big]\big[-2ab|u^{(2)}_{s_1}| - b^2(u^{(2)}_{s_1})^2\big] \Big\}.$$

Because of $|u^{(2)}_{s_1}| = |u^{(2)}_{s_2}|$ we obtain that

$$\det \mathbf{R} = -\gamma_1\gamma_2 b^2 |u^{(2)}_{s_1}| \Big\{ (2a + b|u^{(2)}_{s_1}|)\big[-2a|u^{(1)}_{s_1}| - b(u^{(1)}_{s_1})^2 + 2a|u^{(1)}_{s_2}| + b(u^{(1)}_{s_2})^2\big] \Big\}$$
$$= -\gamma_1\gamma_2 b^2 |u^{(2)}_{s_1}| (2a + b|u^{(2)}_{s_1}|)\big(|u^{(1)}_{s_2}| - |u^{(1)}_{s_1}|\big)\Big[2a + b\big(|u^{(1)}_{s_1}| + |u^{(1)}_{s_2}|\big)\Big] \neq 0 .$$

That is why, according to (9),

$$\text{if} \quad \boldsymbol{a} \in \mathcal{M}_l \implies a_{s_1} = \mathbf{e}'_{s_1}\boldsymbol{a} = a_{s_2} = \mathbf{e}'_{s_2}\boldsymbol{a} = 0 .$$

The converse implication is trivial again and the theorem for $t = 2$ is proved.

2. Let $t \geqq 3$.

From (8) it is easy to see that if $\boldsymbol{a} \in \mathcal{M}_l$, then for an arbitrary choice of

$$\begin{matrix} u^{(1)}_{s_1}, & u^{(1)}_{s_2}, & \dots, & u^{(1)}_{s_t} \\ u^{(2)}_{s_1}, & u^{(2)}_{s_2}, & \dots, & u^{(2)}_{s_t} \\ u^{(3)}_{s_1}, & u^{(3)}_{s_2}, & \dots, & u^{(3)}_{s_t} \end{matrix}$$

$$\sum_{i=1}^{t} \mathbf{e}'_{s_i}\boldsymbol{a}\gamma_i \bigg[\Big(a + b\Big|\sum_{i=1}^{t} \gamma_i u^{(c)}_{s_i}\Big|\Big)^2 - \big(a + b|u^{(c)}_{s_i}|\big)^2 \bigg] = 0 , \qquad c = 1,2,3. \qquad (10)$$

2a. Case $t = 3c$ $(c \geqq 1)$.

For

$$u^{(1)}_{s_{(3v+2)}} \neq 0, \quad u^{(1)}_{s_{(3v+3)}} = -\frac{\gamma_{(3v+2)}}{\gamma_{(3v+3)}} u^{(1)}_{s_{(3v+2)}}, \quad u^{(1)}_{s_q} = 0 \quad q \notin \{3v+2,\, 3v+3\}$$

$$u^{(2)}_{s_{(3v+3)}} \neq 0, \quad u^{(2)}_{s_{(3v+1)}} = -\frac{\gamma_{(3v+3)}}{\gamma_{(3v+1)}} u^{(2)}_{s_{(3v+3)}}, \quad u^{(2)}_{s_q} = 0 \quad q \notin \{3v+1,\, 3v+3\}$$

$$u^{(3)}_{s_{(3v+1)}} \neq 0, \quad u^{(3)}_{s_{(3v+2)}} = -\frac{\gamma_{(3v+1)}}{\gamma_{(3v+2)}} u^{(3)}_{s_{(3v+1)}}, \quad u^{(3)}_{s_q} = 0 \quad q \notin \{3v+1,\, 3v+2\}$$

we obtain from (10) that

$\boldsymbol{a} \in \mathcal{M}_l \implies$ for an arbitrary choice of $u^{(3)}_{s_{(3v+1)}} \neq 0$, $u^{(1)}_{s_{(3v+2)}} \neq 0$ and $u^{(2)}_{s_{(3v+3)}} \neq 0$

$$\mathbf{S}\begin{pmatrix} a_{s_{(3v+1)}} \\ a_{s_{(3v+2)}} \\ a_{s_{(3v+3)}} \end{pmatrix} = \left(\mathbf{S}_1 \vdots \mathbf{S}_2 \vdots \mathbf{S}_3\right)\begin{pmatrix} a_{s_{(3v+1)}} \\ a_{s_{(3v+2)}} \\ a_{s_{(3v+3)}} \end{pmatrix} = \mathbf{O}_{3,1}\,,$$

where

$$\mathbf{S}_1 = \begin{pmatrix} 0 \\ -\gamma_{(3v+1)}b\left|\frac{\gamma_{(3v+3)}}{\gamma_{(3v+1)}}\right||u^{(2)}_{s_{(3v+3)}}|\left(2a+b\left|\frac{\gamma_{(3v+3)}}{\gamma_{(3v+1)}}\right||u^{(2)}_{s_{(3v+3)}}|\right) \\ -\gamma_{(3v+1)}b|u^{(3)}_{s_{(3v+1)}}|(2a+b|u^{(3)}_{s_{(3v+1)}}|) \end{pmatrix},$$

$$\mathbf{S}_2 = \begin{pmatrix} -\gamma_{(3v+2)}b|u^{(1)}_{s_{(3v+2)}}|(2a+b|u^{(1)}_{s_{(3v+2)}}|) \\ 0 \\ -\gamma_{(3v+2)}b\left|\frac{\gamma_{(3v+1)}}{\gamma_{(3v+2)}}\right||u^{(3)}_{s_{(3v+1)}}|\left(2a+b\left|\frac{\gamma_{(3v+1)}}{\gamma_{(3v+2)}}\right||u^{(3)}_{s_{(3v+1)}}|\right) \end{pmatrix}$$

and

$$\mathbf{S}_3 = \begin{pmatrix} -\gamma_{(3v+3)}b\left|\frac{\gamma_{(3v+2)}}{\gamma_{(3v+3)}}\right||u^{(1)}_{s_{(3v+2)}}|\left(2a+b\left|\frac{\gamma_{(3v+2)}}{\gamma_{(3v+3)}}\right||u^{(1)}_{s_{(3v+2)}}|\right) \\ -\gamma_{(3v+3)}b|u^{(2)}_{s_{(3v+3)}}|(2a+b|u^{(2)}_{s_{(3v+3)}}|) \\ 0 \end{pmatrix}.$$

Because of

$$\begin{aligned}\det\mathbf{S} = -\gamma_{(3v+1)}\gamma_{(3v+2)}\gamma_{(3v+3)}b^3|u^{(3)}_{s_{(3v+1)}}||u^{(1)}_{s_{(3v+2)}}||u^{(2)}_{s_{(3v+3)}}|\cdot \\ \cdot\Big\{(2a+b|u^{(1)}_{s_{(3v+2)}}|)(2a+b|u^{(2)}_{s_{(3v+3)}}|)(2a+b|u^{(3)}_{s_{(3v+1)}}|) \\ +\left(2a+b\left|\frac{\gamma_{(3v+2)}}{\gamma_{(3v+3)}}\right||u^{(1)}_{s_{(3v+2)}}|\right)\left(2a+b\left|\frac{\gamma_{(3v+3)}}{\gamma_{(3v+1)}}\right||u^{(2)}_{s_{(3v+3)}}|\right)\cdot \\ \cdot\left(2a+b\left|\frac{\gamma_{(3v+1)}}{\gamma_{(3v+2)}}\right||u^{(3)}_{s_{(3v+1)}}|\right)\Big\} \neq 0\end{aligned}$$

we obtain for $v = 1,2,\dots,c-1$ the implication

$$\boldsymbol{a}\in\mathcal{M}_l \implies \boldsymbol{e}'_{s_{(3v+1)}}\boldsymbol{a} = \boldsymbol{e}'_{s_{(3v+2)}}\boldsymbol{a} = \boldsymbol{e}'_{s_{(3v+3)}}\boldsymbol{a} = 0\,. \quad i.e.$$

$$\boldsymbol{a}\in\mathcal{M}_l \implies \boldsymbol{e}'_{s_i}\boldsymbol{a} = 0 \qquad i = 1,2,\dots t\,.$$

The converse implication is trivial and we have proved the theorem in the case $t = 3c$ $(c \geqq 1)$.

2b. Case $t = 3c + 1$ $(c \geqq 1)$.
If we proceed as in 2a., we get the implication

$$\boldsymbol{a} \in \mathcal{M}_l \implies \mathbf{e}'_{s_i}\boldsymbol{a} = 0 \qquad i = 1, 2, \dots, 3c = t - 1 .$$

If we choose $u_{s_1}, u_{s_2}, \dots, u_{s_t}$ in such a way that $u_{s_t} \neq 0$ and $\left|\sum_{i=1}^{t} \gamma_i u_{s_i}\right| \neq |u_{s_t}|$, we have from (8) that if $\boldsymbol{a} \in \mathcal{M}_l$ $\mathbf{e}'_{s_t}\boldsymbol{a} = 0$ is also valid.

The converse implication (i.e. $\mathbf{e}'_{s_i}\boldsymbol{a} = 0$, $i = 1, 2, \dots, t \implies \boldsymbol{a} \in \mathcal{M}_l$) is trivial and we have proved the theorem in the case $t = 3c + 1$ $(c \geqq 1)$.

2c. Case $t = 3c + 2$ $(c = 1)$.
If we proceed as in 2a., we get the implication

$$\boldsymbol{a} \in \mathcal{M}_l \implies \mathbf{e}'_{s_i}\boldsymbol{a} = 0 \qquad i = 1, 2, \dots, 3c = t - 2 .$$

Let us choose $u^{(1)}_{s_1} = \dots = u^{(1)}_{s_{t-2}} = 0$, $u^{(2)}_{s_1} = \dots = u^{(2)}_{s_{t-2}} = 0$, and we get from (8) that if $\boldsymbol{a} \in \mathcal{M}_l$, then for an arbitrary choice of $u^{(1)}_{s_{(t-1)}}$, $u^{(2)}_{s_{(t-1)}}$, $u^{(1)}_{s_t}$ and $u^{(2)}_{s_t}$

$$\mathbf{T}\begin{pmatrix} a_{s_{t-1}} \\ a_{s_t} \end{pmatrix} = \left(\mathbf{T}_1 \vdots \mathbf{T}_2\right)\begin{pmatrix} a_{s_{t-1}} \\ a_{s_t} \end{pmatrix} = \mathbf{O}_{2,1} ,$$

where

$$\mathbf{T}_1 = \begin{pmatrix} \gamma_{(t-1)}\left[\left(a + b|\gamma_{(t-1)}u^{(1)}_{s_{(t-1)}} + \gamma_t u^{(1)}_{s_t}|\right)^2 - \left(a + b|u^{(1)}_{s_{(t-1)}}|\right)^2\right] \\ \gamma_{(t-1)}\left[\left(a + b|\gamma_{(t-1)}u^{(2)}_{s_{(t-1)}} + \gamma_t u^{(2)}_{s_t}|\right)^2 - \left(a + b|u^{(2)}_{s_{(t-1)}}|\right)^2\right] \end{pmatrix}$$

and

$$\mathbf{T}_2 = \begin{pmatrix} \gamma_t\left[\left(a + b|\gamma_{(t-1)}u^{(1)}_{s_{(t-1)}} + \gamma_t u^{(1)}_{s_t}|\right)^2 - \left(a + b|u^{(1)}_{s_t}|\right)^2\right] \\ \gamma_t\left[\left(a + b|\gamma_{(t-1)}u^{(2)}_{s_{(t-1)}} + \gamma_t u^{(2)}_{s_t}|\right)^2 - \left(a + b|u^{(2)}_{s_t}|\right)^2\right] \end{pmatrix} .$$

If we proceed as in 1., we obtain that if $\boldsymbol{a} \in \mathcal{M}_l$, also $\mathbf{e}'_{s_{(t-1)}}\boldsymbol{a} = \mathbf{e}'_{s_t}\boldsymbol{a} = 0$ is valid.

The converse implication (i.e. $\mathbf{e}'_{s_i}\boldsymbol{a} = 0$ $i = 1, 2, \dots, t \implies \boldsymbol{a} \in \mathcal{M}_l$) is trivial again and we have proved the theorem for this last case as well.

COROLLARY 3.10. *Let* $n = R(\mathbf{X}) + 1$, $\mathbf{E} = \gamma' = \sum_{i=1}^{t} \gamma_i \mathbf{e}'_{s_i}$, $t \geqq 2$, *where* $\gamma_i \neq 0$, $s_i \in \{1, 2, \ldots, R(\mathbf{X})\}$ $(i = 1, \ldots, t \leqq R(\mathbf{X}))$. *The random variable* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if and only if* $\mathbf{p} = \left(\mathbf{I}_{R(\mathbf{X}),R(\mathbf{X})} \vdots \gamma\right)' \mathbf{a}$, *where* $\mathbf{a}'\mathbf{e}_{s_i} = 0$, $i = 1, 2, \ldots, t$. *It means in this case that* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if and only if* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *does not contain* $\widetilde{Y}_{s_1}, \ldots, \widetilde{Y}_{s_t}, \widetilde{Y}_n$.

The proof is an easy consequence of Theorem 3.7 and Theorem 3.9.

COROLLARY 3.11. *Let*

$$\mathbf{E} = \begin{pmatrix} \gamma'_1 \\ \gamma'_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{k_1} \gamma_{1_i} \mathbf{e}'_{s_i} \\ \sum_{i=1}^{k_2} \gamma_{2_i} \mathbf{e}'_{l_i} \end{pmatrix},$$

where $1 \leqq k_1 \leqq R(\mathbf{X})$, $1 \leqq k_2 \leqq R(\mathbf{X})$, $\gamma_{ij} \neq 0$ *for all* i, j; s_i *and* l_i *belong to* $\{1, 2, \ldots, R(\mathbf{X})\}$ *for all* i.

1. *If* $k_1 = k_2 = 1$, $\mathbf{e}_{s_1} = \mathbf{e}_{l_1} = \mathbf{e}_s$, $|\gamma_{11}| = |\gamma_{21}| = 1$, *then* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if and only if*

$$\mathbf{p}'\widetilde{\mathbf{Y}} = a_1\widetilde{Y}_1 + \cdots + a_{s-1}\widetilde{Y}_{s-1} + a_s(\widetilde{Y}_s \pm \widetilde{Y}_{n-1} \pm \widetilde{Y}_n) + a_{s+1}\widetilde{Y}_{s+1} + \cdots + a_{R(\mathbf{X})}\widetilde{Y}_{R(\mathbf{X})},$$

where $a_i \in \mathbb{R}$ $i = 1, 2, \ldots, R(\mathbf{X})$ (*the sign* $+$ *or* $-$ *corresponds to one of* γ_{11} *and* γ_{21}).

2. *If* $k_1 = k_2 = 1$, $s_1 < l_1$, $|\gamma_{11}| = |\gamma_{21}| = 1$, *then* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if and only if*

$$\mathbf{p}'\widetilde{\mathbf{Y}} = a_1\widetilde{Y}_1 + \cdots + a_{s_1-1}\widetilde{Y}_{s_1-1} + a_{s_1}(\widetilde{Y}_{s_1} \pm \widetilde{Y}_{n-1}) + a_{s_1+1}\widetilde{Y}_{s_1+1} + \ldots$$
$$+ a_{l_1-1}\widetilde{Y}_{l_1-1} + a_{l_1}(\widetilde{Y}_{l_1} \pm \widetilde{Y}_n) + a_{l_1+1}\widetilde{Y}_{l_1+1} + \cdots + a_{R(\mathbf{X})}\widetilde{Y}_{R(\mathbf{X})},$$

where $a_i \in \mathbb{R}$ $i = 1, 2, \ldots, R(\mathbf{X})$.

(*The case* $l_1 < s_1$ *is a trivial modification.*)

3. *If* $k_1 = 1$, $k_2 > 1$, $|\gamma_{11}| = 1$, $\mathbf{e}_{s_1} \neq \mathbf{e}_{l_i}$ $i = 1, 2, \ldots, k_2$, *then* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if and only if* $\mathbf{p}'\widetilde{\mathbf{Y}} = a_1\widetilde{Y}_1 + \cdots + a_{s_1}(\widetilde{Y}_{s_1} \pm \widetilde{Y}_{n-1}) + \cdots + a_{R(\mathbf{X})}\widetilde{Y}_{R(\mathbf{X})}$ *does not contain* $\widetilde{Y}_{l_1}, \ldots, \widetilde{Y}_{l_{k_2}}, \widetilde{Y}_n$ *and the coefficients are arbitrary real numbers.* (*The case* $k_1 > 1$, $k_2 = 1$, $|\gamma_{21}| = 1$, $\mathbf{e}_{l_1} \neq \mathbf{e}_{s_i}$, $i = 1, 2, \ldots, k_1$ *is a trivial modification.*)

4. *If* $k_1 = 1$, *but* $|\gamma_{11}| \neq 1$, $k_2 > 1$, *then* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if an only if* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *does not contain* $\widetilde{Y}_{s_1}$, $\widetilde{Y}_{l_1}, \dots$, $\widetilde{Y}_{l_{k_2}}$, $\widetilde{Y}_{n-1}$ *and* $\widetilde{Y}_n$.

(*The case* $k_1 > 1$, $k_2 = 1$ *and* $|\gamma_{21}| \neq 1$ *is similar.*)

5. *If* $k_1 > 1$ *and* $k_2 > 1$, *then* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *is the* UBLUE *of its mean value if and only if* $\mathbf{p}'\widetilde{\mathbf{Y}}$ *does not contain* $\widetilde{Y}_{s_1}, \dots$, $\widetilde{Y}_{s_{k_1}}$, $\widetilde{Y}_{l_1}, \dots, \widetilde{Y}_{l_{k_2}}$, $\widetilde{Y}_{n-1}$ *and* $\widetilde{Y}_n$.

The proof is an easy consequence of Theorem 3.7, Lemma 3.8 and Theorem 3.9.

An easy generalization is for the case with $\mathbf{E}$ containing $n - R(\mathbf{X}) \geqq 2$ rows.

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