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ON POSITIVE SOLUTIONS OF NONLINEAR RETARDED DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper there are given for nonlinear retarded differential equations $y^{(n)}(t) = f\Big(t, y\big(\tau(t)\big)\Big)$ the conditions of the absence and the existence of solutions that have the property

$$(-1)^i y^{(i)}(t) > 0 \qquad \text{for} \quad t \geq t_1 > 0 \quad (i = 0, \dots, n-1).$$

We will consider the nonlinear differential equation with retarded argument

$$y^{(n)}(t) = f\Big(t, y\big(\tau(t)\big)\Big), \tag{1}$$

where $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$, and $\tau\colon \mathbb{R}_+ \to \mathbb{R}$ are continuous functions, $\tau(t) \leq t$ for $t \geq 0$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

Below we will assume that there exist $\delta > 0$, $\lambda > 1$ and a continuous function $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that one of the next inequalities holds:

$$(-1)^n f(t, y) \geq p(t) y^\lambda \tag{2}$$

for $t \in \mathbb{R}_+$, $y \in [0, \delta]$ or

$$0 < (-1)^n f(t, y) \leq p(t) y^\lambda \tag{3}$$

for $t \in \mathbb{R}_+$, $y \in (0, \delta]$. It is the well-known fact (see, e.g., [2, 4]) that if (2) holds, $p(t) \not\equiv 0$ in any neighbourhood of ∞ and $\tau(t) \equiv t$, then the equation (1) has a solution $y\colon [t_0, \infty) \to \mathbb{R}$ which satisfies the following conditions:

$$\begin{gathered} (-1)^i y^{(i)}(t) > 0 \qquad \text{for} \quad t \geq t_1 \geq t_0 \geq 0, \quad i = 0, \dots, n-1, \\ 0 \leq \lim_{t\to\infty} y(t) < \delta. \end{gathered} \tag{4}$$

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If $\tau(t) < t$, then the solutions that satisfy (4) can be absent (see [3, 5]).

In this paper such conditions will be established under which the equation (1) has not the solutions that satisfy (4) and likewise conditions that guarantee the existence of a solution which has the property (4) (cf. [3]).

Let $r\colon \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and increasing on $[t_0, \infty)$, $t_0 \in \mathbb{R}_+$, $r(t) < t$ for $t \ge t_0$.

For $t > r^{-1}(t_1)$, $t_1 \ge t_0$, we define the function

$$m(t) = m\,, \qquad x_m < t \le x_{m+1}\,, \quad m = 0, 1, \dots\,,$$

where $x_0 = r^{-1}(t_1)$, $x_{m+1} = r^{-1}(x_m)$ and $r^{-1}(t)$ denotes the inverse function of $r(t)$.

We denote the mth iteration of the function $r(t) = r_0(t)$ as $r_m(t)$, $m = 1, 2, \dots$. Thus with regard to the function $m(t)$ for arbitrary $t > r^{-1}(t_1)$ we have

$$t_1 < r_{m(t)}(t) \le r^{-1}(t_1)\,.$$

THEOREM 1. *Suppose that* (2) *holds for some* $\delta > 0$, $\lambda > 1$ *and*

$$\tau(t) < r(t) < t$$

for $t \ge t_0$,

$$\lim_{t\to\infty} \int\limits_{r^{-1}(\tau(t))}^{t} \Big[s - r^{-1}\big(\tau(t)\big)\Big]^{n-1} p(s)\,\mathrm{d}s = \infty\,. \tag{5}$$

Then the equation (1) *has not a solution which satisfies* (4).

P r o o f. Assume the contrary, i.e. the equation (1) has a solution that satisfies (4) and

$$\begin{aligned} y(t) &< \delta && \text{for} \quad t \ge t_1 \ge t_0\,,\\ \tau(t) &\ge t_1 && \text{for} \quad t \ge t_2 \ge t_1\,. \end{aligned} \tag{6}$$

From the identity

$$y(t) = \sum_{j=0}^{n-1} (-1)^j \frac{y^{(j)}(s)}{j!}(s-t)^j + \frac{(-1)^n}{(n-1)!}\int\limits_t^s (\xi - t)^{n-1} y^{(n)}(\xi)\,\mathrm{d}\xi\,,$$

where $s > t$, with regard to (1), (2), (6) we have

$$y(t) \ge \frac{1}{(n-1)!}\int\limits_t^\infty (s-t)^{n-1} p(s)\Big[y\big(\tau(s)\big)\Big]^\lambda\,\mathrm{d}s \tag{7}$$

for $t \ge t_2$. We choose $c > 0$ such that

$$\int\limits_{r^{-1}(\tau(t))}^{t} \left[s - r^{-1}\big(\tau(t)\big)\right]^{n-1} p(s)\,\mathrm{d}s \ge c(n-1)! \tag{8}$$

for $t \ge t_2$ and if for arbitrary $t > r^{-1}(t_2)$ we take ξ such that

$$\tau(\xi) = r(t)\,, \qquad \tau(s) \le r(t)$$

for $s \le \xi$, then according to (7), (8) we get

$$\begin{aligned} y(t) &\ge \frac{1}{(n-1)!} \int\limits_{r^{-1}(\tau(\xi))}^{\xi} \left[s - r^{-1}\big(\tau(\xi)\big)\right]^{n-1} p(s)\left[y\big(\tau(s)\big)\right]^{\lambda} \mathrm{d}s \\ &\ge \frac{1}{(n-1)!} \int\limits_{r^{-1}(\tau(\xi))}^{\xi} \left[s - r^{-1}\big(\tau(\xi)\big)\right]^{n-1} p(s)\,\mathrm{d}s \left[y\big(r(t)\big)\right]^{\lambda} \\ &\ge c\left[y\big(r(t)\big)\right]^{\lambda}, \qquad r(t) > t_2\,. \end{aligned}$$

Using the last inequality we find by iteration

$$y(t) \ge c^{1+\lambda+\dots+\lambda^m}\left[y\big(r_m(t)\big)\right]^{\lambda^{m+1}}, \qquad r_m(t) > t_2\,, \quad m \in \{0\} \cup \mathbb{N}\,.$$

Thus there follows:

$$y(t) \ge \exp\left[\sum_{i=0}^{m} \lambda^i \ln c + \lambda^{m+1} \ln y\big(r_m(t)\big)\right], \qquad r_m(t) > t_2\,, \quad m \in \{0\} \cup \mathbb{N}\,. \tag{9}$$

For arbitrary $t > r^{-1}(t_2)$ we define

$$m(t) = m\,, \qquad x_m < t \le x_{m+1}\,, \qquad m = 0, 1, \dots\,,$$
$$x_0 = r^{-1}(t_2)\,, \qquad x_{m+1} = r^{-1}(x_m)$$

and

$$C_{m(t)} = \sup\left\{C\colon \int\limits_{r^{-1}(\tau(t))}^{t} \left[s - r^{-1}\big(\tau(t)\big)\right]^{n-1} p(s)\,\mathrm{d}s \ge C, \quad x_m < t \le x_{m+1}\right\},$$

where $m \in \{0\} \cup \mathbb{N}$, $x_0 = r^{-1}(t_2)$, $x_{m+1} = r^{-1}(x_m)$.

Hence we have

$$t_2 < r_{m(t)}(t) \leq r^{-1}(t_2) \tag{10}$$

for $t > r^{-1}(t_2)$ and by virtue of (9), (10) we obtain

$$y(t) \geq \exp\left[\sum_{i=0}^{m(t)} \lambda^i \ln C_{m(t)} + \lambda^{m(t)+1} \ln y\big(r^{-1}(t_2)\big)\right]$$

for $t > r^{-1}(t_2)$. Therefore $y(t) \to \infty$ as $t \to \infty$ and this contradicts (6).

COROLLARY 1. *Assume that for some $\delta > 0$, $\lambda > 1$, (2) holds and*

$$\tau^{\gamma}(t) < t \qquad \textit{for} \quad \gamma > 1, \quad t \geq t_0,$$

$$\lim_{t\to\infty} \int\limits_{\tau^{\gamma}(t)}^{t} \left[s - \tau^{\gamma}(t)\right]^{n-1} p(s)\,\mathrm{d}s = \infty.$$

Then the equation (1) has not a solution with the property (4).

P r o o f. The assertion of the Corollary 1 follows from the Theorem 1 if we put $r(t) = t^{\gamma^{-1}}$.

THEOREM 2. *Suppose that for some $\delta > 0$, $\lambda > 1$ (2) holds and*

$$\tau(t) < r(t) < t \qquad \textit{for} \quad t \geq t_0,$$

$$\liminf_{t\to\infty} \frac{1}{\varphi\big(\tau(t)\big)} \ln \int\limits_{r^{-1}(\tau(t))}^{t} \left[s - r^{-1}\big(\tau(t)\big)\right]^{n-1} p(s)\,\mathrm{d}s > 0, \tag{12}$$

$$\liminf_{t\to\infty} \frac{\lambda^{m(t)}}{\sum\limits_{i=0}^{m(t)} \lambda^i \varphi\big(r_i(t)\big)} = 0, \tag{13}$$

where $\varphi(t)$ is a continuous function such that

$$\lim_{t\to\infty} \varphi(t) = \infty. \tag{14}$$

Then the equation (1) has not a solution with the property (4).

P r o o f. We continue as in the proof of Theorem 1. With regard to (12), (14) there is $t_2 \geq t_0$ and $c > 0$ such that

$$\frac{1}{\varphi\big(\tau(t)\big)} \ln \int\limits_{r^{-1}(\tau(t))}^{t} \left[s - r^{-1}\big(\tau(t)\big)\right]^{n-1} p(s)\,\mathrm{d}s \geq \frac{\ln(n-1)!}{\varphi\big(\tau(t)\big)} + c$$

for $t \ge t_2$ and if for arbitrary $t > r^{-1}(t_2)$ we take ξ such that

$$\tau(\xi) = r(t), \quad \tau(s) \le r(t) \qquad \text{for} \quad s \le \xi,$$

we get

$$y(t) \ge \exp\Big[c\,\varphi\big(\tau(\xi)\big)\Big]\Big[y\big(r(t)\big)\Big]^{\lambda} = \exp\Big[c\,\varphi\big(r(t)\big)\Big]\Big[y\big(r(t)\big)\Big]^{\lambda}, \qquad r(t) > t_2 .$$

Using the last inequality we find

$$y(t) \ge \exp\Big[c\sum_{i=0}^{m} \lambda^i \varphi\big(r_i(t)\big)\Big]\Big[y\big(r_m(t)\big)\Big]^{\lambda^{m+1}}, \qquad r_m(t) > t_2, \quad m \in \{0\} \cup \mathbb{N}.$$

With regard to the last inequality and (10) we get

$$y(t) \ge \exp\Big[c\sum_{i=0}^{m(t)} \lambda^i \varphi\big(r_i(t)\big) + \lambda^{m(t)+1} \ln y\big(r^{-1}(t_2)\big)\Big], \qquad t > r^{-1}(t_2).$$

According to (13) and the above inequality we find that

$$\limsup_{t\to\infty} y(t) = \infty,$$

which is a contradiction to (4).

COROLLARY 2. *Assume that for some* $\delta > 0$, $\lambda > 1$, $\gamma > 1$ *and* $\mu \in (0, 1)$ (2) *holds and*

$$\tau^{\gamma}(t) < t \qquad \textit{for} \quad t \ge t_0,$$

$$\liminf_{t\to\infty} \frac{1}{\varphi\big(\tau(t)\big)} \ln \int_{\tau^{\gamma}(t)}^{t} \big[s - \tau^{\gamma}(t)\big]^{n-1} p(s)\, \mathrm{d}s > 0,$$

where $\varphi(t) = (\ln\ln t)^{-\mu} (\ln t)^{\frac{\ln\lambda}{\ln\gamma}}$.
Then the equation (1) *has not a solution which satisfies* (4).

P r o o f. We will show that the condition (13) is satisfied.

If we put $r(t) = t^{\gamma^{-1}}$, we have

$$\sum_{i=0}^{m-1} \lambda^i \varphi\big(r_i(t)\big) = \sum_{i=1}^{m} \lambda^{i-1} \varphi\big(t^{\gamma^{-i}}\big), \qquad t > t_2^{\gamma^m}, \quad m \in \mathbb{N}.$$

According to the fact that

$$\varphi(t) = (\ln\ln t)^{-\mu}(\ln t)^{\frac{\ln\lambda}{\ln\gamma}}\,, \tag{15}$$

we get (cf. [3])

$$\varphi\big(t^{\gamma^{-m}}\big) > \lambda^{-m}\varphi(t)\,, \qquad t > t_2^{\gamma}\,, \quad m \in \mathbb{N}\,.$$

So we obtain

$$\sum_{i=0}^{m-1} \lambda^i \varphi\big(r_i(t)\big) > \frac{m}{\lambda}\varphi(t)\,, \qquad t > t_2^{\gamma^m}, \quad m \in \mathbb{N}\,. \tag{16}$$

We have

$$t_2 < t^{\gamma^{-m(t)}} \le t_2^{\gamma}\,, \qquad \text{for} \quad t > t_2^{\gamma}\,,$$

and

$$\ln\ln t - \ln\ln t_2 - \ln\gamma \le m(t)\ln\gamma < \ln\ln t - \ln\ln t_2\,. \tag{17}$$

With regard to (15), (16), (17) we conclude that

$$\liminf_{t\to\infty} \frac{\lambda^{m(t)-1}}{\sum\limits_{i=0}^{m(t)-1} \lambda^i\varphi\big(r_i(t)\big)} \le \liminf_{t\to\infty} \frac{\lambda^{m(t)}}{m(t)\varphi(t)} = 0\,.$$

THEOREM 3. *Let*

$$\tau(t) < t \qquad \textit{for} \quad t \ge 0\,,$$

and let there be $\delta > 0$, $\lambda > 1$, $\mu \in (0, \lambda)$ *such that* (3) *holds and*

$$\limsup_{t\to\infty} \int_0^t s^{n-1}p(s)\,\mathrm{d}s \left(\int_0^{\tau(t)} s^{n-1}p(s)\,\mathrm{d}s\right)^{-\mu} < \infty\,. \tag{18}$$

Then the equation (1) *has a solution with the property* (4).

Proof. According to (18) there is $c > 0$ and $t_0 > 0$ such that $\tau(t) > 0$ for $t \ge t_0$ and

$$\int_0^t s^{n-1}p(s)\,\mathrm{d}s \le c\left(\int_0^{\tau(t)} s^{n-1}p(s)\,\mathrm{d}s\right)^{\mu}, \qquad t \ge t_0\,.$$

We put

$$v(t) = c_0 \left(\int_0^t s^{n-1} p(s) \, \mathrm{d}s \right)^{\frac{\mu}{\mu-\lambda}} \qquad \text{for} \quad t \ge t_0 \,,$$

where

$$c_0 = \left[(n-1)! \, \mu (\lambda - \mu)^{-1} c^{\frac{\lambda}{\mu-\lambda}} \right]^{\frac{1}{\lambda-1}}$$

and c is so large that $v(t) \le \delta$ for $t \ge t_0$.
By $C_{\mathrm{loc}}\big([t_0, \infty); \mathbb{R}\big)$ we denote the space of continuous functions $x \colon [t_0, \infty) \to \mathbb{R}$ endowed with the topology of local uniform convergence. $S \subset C_{\mathrm{loc}}\big([t_0, \infty); \mathbb{R}\big)$ is the set of functions which satisfy inequalities $0 \le x(t) \le v(t)$ for $t \ge t_0$ and $F \colon S \to C_{\mathrm{loc}}\big([t_0, \infty); \mathbb{R}\big)$ is the operator which is defined by

$$F(x)(t) = \begin{cases} \frac{(-1)^n}{(n-1)!} \int_t^\infty (s-t)^{n-1} f\Big(s, \, x\big(\tau(s)\big)\Big) \, \mathrm{d}s & \text{for} \quad t \ge t_1 \,, \\ v(t) - v(t_1) + F(x)(t_1) & \text{for} \quad t \in [t_0, t_1) \,, \end{cases}$$

where we take $t_1 > t_0$ such that $\tau(t) \ge t_0$ for $t \ge t_1$.

If $x \in S$, we have

$$\begin{aligned} 0 \le F(x)(t) &\le \frac{1}{(n-1)!} \int_t^\infty s^{n-1} p(s) \Big[x\big(\tau(s)\big) \Big]^\lambda \mathrm{d}s \\ &\le \frac{c_0^\lambda}{(n-1)!} \int_t^\infty s^{n-1} p(s) \left[\int_0^{\tau(s)} \xi^{n-1} p(\xi) \, \mathrm{d}\xi \right]^{\frac{\lambda\mu}{\mu-\lambda}} \mathrm{d}s \\ &\le \frac{c_0^\lambda}{(n-1)! \, c^{\frac{\lambda}{\mu-\lambda}}} \int_t^\infty s^{n-1} p(s) \left[\int_0^s \xi^{n-1} p(\xi) \, \mathrm{d}\xi \right]^{\frac{\lambda}{\mu-\lambda}} \mathrm{d}s \le v(t) \,, \\ & \qquad t \ge t_1 \,, \end{aligned}$$

since

$$h'(t) = 0 \,, \qquad t \ge t_1 \,,$$

where

$$h(t) = v(t) - \frac{c_0^\lambda c^{\frac{\lambda}{\lambda-\mu}}}{(n-1)!} \int_t^\infty s^{n-1} p(s) \left[\int_0^s \xi^{n-1} p(\xi) \, \mathrm{d}\xi \right]^{\frac{\lambda}{\mu-\lambda}} \mathrm{d}s \,.$$

Thus $F(S) \subset S$. The operator F is continuous and the functions belonging to the set $F(S)$ are equicontinuous on every compact subinterval of $[t_0, \infty)$. Since the set S is closed and convex, according to the Schauder-Tychonoff fixed point theorem (cf., e.g. [1, p. 231]) F has an element $y \in S$ such that $y = F(y)$. It is easy to see that y satisfies (1) on $[t_1, \infty)$ and has the property (4). The proof is complete.

THEOREM 4. *Let*

$$\tau(t) < t \qquad \text{for} \quad t \geq 0\,,$$

and let there exist $\delta > 0$, $\lambda > 1$, $\mu > \lambda$, $T \in [0, \infty)$ *such that* (3) *holds and*

$$\begin{gathered}\int_T^\infty t^{n-1} p(t) \left(\int_T^t s^{n-1} p(s)\,\mathrm{d}s \right)^{\frac{\lambda}{\lambda-\mu}} \mathrm{d}t \leq A = \left(\frac{\mu-\lambda}{\mu} \right)^{\left(\frac{\mu-\lambda}{\lambda}\right)}, \\ \limsup_{t\to\infty} \int_T^t s^{n-1} p(s)\,\mathrm{d}s \left(\int_T^{\tau(t)} s^{n-1} p(s)\,\mathrm{d}s \right)^{-\mu} < \infty\,. \end{gathered} \tag{19}$$

Then the equation (1) *has a solution with the property* (4).

Proof. With regard to (19) there exist $c \geq 1$ and $t_0 > T$ such that $\tau(t) > T$ for $t \geq t_0$ and

$$\int_T^t s^{n-1} p(s)\,\mathrm{d}s \leq c \left(\int_T^{\tau(t)} s^{n-1} p(s)\,\mathrm{d}s \right)^{\mu}, \qquad t \geq t_0\,.$$

We put

$$v(t) = c_0 \left(A + \int_T^t s^{n-1} p(s)\,\mathrm{d}s \right)^{\frac{\mu}{\lambda-\mu}} \qquad \text{for} \quad t \geq t_0\,,$$

where

$$c_0 = \left[(n-1)!\, c^{\frac{\lambda}{\lambda-\mu}} A^{\frac{2\mu-\lambda}{\lambda-\mu}} \right]^{\frac{1}{\lambda-1}}$$

and we choose c so large that

$$v(t) \leq \delta \qquad \text{for} \quad t \geq t_0\,.$$

Now we proceed as in the proof of the above Theorem and we define the operator $F\colon\ S \to C_{\mathrm{loc}}([t_0, \infty);\ \mathbb{R})$ in the same way.

If $x \in S$, we have

$$\begin{aligned}
0 \le F(x)(t) &\le \frac{c_0^\lambda}{(n-1)!} \int\limits_t^\infty s^{n-1} p(s) \Big(A + \int\limits_T^{\tau(s)} \xi^{n-1} p(\xi)\,\mathrm{d}\xi \Big)^{\frac{\lambda\mu}{\lambda-\mu}} \mathrm{d}s \\
&\le \frac{c_0^\lambda}{(n-1)!} \int\limits_t^\infty s^{n-1} p(s) \Big[A + c^{-\frac{1}{\mu}} \Big(\int\limits_T^{s} \xi^{n-1} p(\xi)\,\mathrm{d}\xi \Big)^{\frac{1}{\mu}} \Big]^{\frac{\lambda\mu}{\lambda-\mu}} \mathrm{d}s \\
&\le \frac{c_0^\lambda}{(n-1)!} \int\limits_t^\infty s^{n-1} p(s) \Big(A + c^{-1} \int\limits_T^{s} \xi^{n-1} p(\xi)\,\mathrm{d}\xi \Big)^{\frac{\lambda}{\lambda-\mu}} \mathrm{d}s \\
&\le \frac{c_0^\lambda c^{\frac{\lambda}{\mu-\lambda}}}{(n-1)!} \int\limits_t^\infty s^{n-1} p(s) \Big(A + \int\limits_T^{s} \xi^{n-1} p(\xi)\,\mathrm{d}\xi \Big)^{\frac{\lambda}{\lambda-\mu}} \mathrm{d}s \le v(t)\,, \\
& \qquad t \ge t_1 > t_0\,,
\end{aligned}$$

since

$$h(T) \ge 0\,, \qquad h'(t) \ge 0\,, \qquad t \ge T\,,$$

where

$$h(t) =$$

$$c_0 \Big(A + \int\limits_T^{t} s^{n-1} p(s)\,\mathrm{d}s \Big)^{\frac{\mu}{\lambda-\mu}} - \frac{c_0^\lambda c^{\frac{\lambda}{\mu-\lambda}}}{(n-1)!} \int\limits_t^\infty s^{n-1} p(s) \Big(A + \int\limits_T^{s} \xi^{n-1} p(\xi)\,\mathrm{d}\xi \Big)^{\frac{\lambda}{\lambda-\mu}} \mathrm{d}s\,,$$
$$t \ge T\,.$$

Thus $F(S) \subset S$.

Now we continue as in the proof of Theorem 3 and we can prove the existence of a solution $y(t)$ of (1) which has the property (4). The proof is complete.

REFERENCES

[1] EDWARDS, R. E: *Functional Analysis.* (Russian translation), Mir, Moscow, 1969.

[2] HARTMAN, P.: *Ordinary Differential Equations.* (Russian translation), Mir, Moscow, 1970.

[3] CHANTURIA, T. A.: *On positive decreasing solutions of nonlinear differential equations with retarded argument.* (Russian), Differentsial'nye Uravneniya 6 (1988), 993–999.

[4] KIGURADZE, I. T: *Some Singular BVPs for Ordinary Differential Equations.* (Russian), Izd. Univ., Tbilisi, 1975.

[5] TRAMOV, M. I.: *Oscillatory conditions of solutions of the first order retarded differential equations.* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. **3** (1975), 92–96.

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