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Mathematica Slovaca, Vol. 41 (1991), No. 4, 393--399

Persistent URL: http://dml.cz/dmlcz/136537

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ON THE EXISTENCE OF HOMOCLINC POINTS

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ABSTRACT. It is shown the existence of transversal homoclinic points for certain perturbed diffeomorphisms if an unperturbed diffeomorphism has a nonhyperbolic fixed point with a homoclinic orbit.

Introduction. Recently the author of this paper has developed a method [1] for the study of *bifurcation of homoclinic points of diffeomorphisms*. An essential assumption was that the *fixed point of an unperturbed diffeomorphism is hyperbolic*. The purpose of this paper is to present a similar method as in [1] for *special mappings when an unperturbed one has a nonhyperbolic fixed point having a homoclinic orbit*. As a model problem we study the existence of a *transversal homoclinic point* of the following n-dimensional mapping

$$G_e \begin{pmatrix} x \\ z \\ y \end{pmatrix} = \begin{pmatrix} 2x - z + e \cdot x + e^3 \cdot g(x, y, e) \\ x \\ f(y) + e \cdot r(x, y, e) \end{pmatrix}, \tag{1}$$

where $x, z \in \mathbb{R}$, $y \in \mathbb{R}^{n-2}$, $e \in \mathbb{R}$ is a parameter, $n \geq 4$.

Let us recall some definitions [3], [4]. Consider a C^1-mapping $F\colon \mathbb{R}^m \to \mathbb{R}^m$. A *fixed point x of F is hyperbolic* if the eigenvalues of $\mathrm{D}\,F(x)$ lie off the unit circle. If F is a diffeomorphism and x is a hyperbolic fixed point of F, then the *stable, unstable manifold of x* $W^s(x)$, $W^u(x)$ is defined to be the set of those y such that $F^j(y) \to x$ as $j \to \infty$, $j \to -\infty$, respectively. A point y is said to be a *transversal homoclinic point* if $y \in W^s(x) \bigcap W^u(x)$ for some fixed point $x \neq y$ of F and $\mathbb{R}^m$ is the direct sum of the tangent spaces to $W^s(x)$ and $W^u(x)$ at y.

Smale [2] shows that *if F has a transversal homoclinic point, then there is a Cantor-like set near it on which some iterate of F is invariant and isomorphic to the Bernoulli shift on a finite number of symbols. This invariant set contains a countable infinity of periodic orbits, an uncountable set of bounded nonperiodic orbits, and a dense orbit.*

Now we return to our problem (1).

AMS Subject Classification (1985): Primary 58F08, 58F14, 58F30
Key words: Dynamical systems, Bifurcations, Homoclinic points

Let us assume for the mapping (1):

(i) $g \in C^1(\mathbf{R} \times \mathbf{R}^{n-2} \times \mathbf{R}, \mathbf{R})$, $r \in C^1(\mathbf{R} \times \mathbf{R}^{n-2} \times \mathbf{R}, \mathbf{R}^{n-2})$
$f \in C^1(\mathbf{R}^{n-2}, \mathbf{R}^{n-2})$, $f^{-1} \in C^1(\mathbf{R}^{n-2}, \mathbf{R}^{n-2})$

(ii) $f(0) = 0$ *and* 0 *is the hyperbolic fixed point of* f

(iii) *there is a homoclinic point* y_0 *of* f *such that*
$f^j(y_0) \to 0$ *as* $j \to \infty$ *or* $j \to -\infty$.

(iv) *The equation*
$v_{j+1} = \mathrm{D} f^j(y_0) v_j$, $j \in \mathbf{Z}$, $v_j \in \mathbf{R}^{n-2}$
has only the trivial bounded solution.

Proposition 1. *Under the above conditions* G_e *has a small fixed point for each small* e.

P r o o f. From the equation

$$x = z, \ x + e^2 \cdot g(x, y, e) = 0$$

$$f(y) + e \cdot r(x, y, e) = y$$

we have

$$G_e(x, z, y) = (x, z, y).$$

Using the implicit function theorem to the first equation we obtain a small fixed point $(x(e), z(e), y(e))$ of G_e for a small e.

By Proposition 1 we can suppose

$$G_e(0,0,0) = (0,0,0)$$

for a small e.

We see that the unperturbed mapping G_0 has a fixed point $(0,0,0)$ which is not hyperbolic and moreover, G_0 has the trivial homoclinic orbit $\Gamma = \{(0, 0, f^j(y_0))\}_{-\infty}^{\infty}$. Hence a general theory from [1], [5] cannot be applied. On the other hand, the mapping

$$Q_e \colon \mathbf{R}^2 \to \mathbf{R}^2 \qquad (x, z) \to (2x - z + e \cdot x, x)$$

has the eigenvalues

$$a_{1,2} = \frac{2 + e \pm \sqrt{e(e+4)}}{2}.$$

We see that for $e > 0$, $a_{1,2}$ do not lie on the unit circle. The purpose of this paper is to present a method which allows us to study the above degenerate case.

Theorem 1. *If the mapping* G_e *satisfies the above conditions* (i)–(iv), *then for each small positive* e, G_e *has a homoclinic point* w_e *such that*

(i) *the orbit* $\{G_e^j(w_e)\}_{-\infty}^{\infty}$ *is near* Γ

(ii) $\lim_{j\to\pm\infty} G_e^j(w_e) = (x(e), z(e), y(e))$.

Now we introduce the following *Banach spaces*

$$X = \left\{ \{x_j\}_{-\infty}^{\infty}, x_j \in \mathbf{R}^2, \sup_j |x_j| < \infty \right\}$$

$$Y = \left\{ \{y_j\}_{-\infty}^{\infty}, y_j \in \mathbf{R}^{n-2}, \sup_j |y_j| < \infty \right\}$$

Lemma 1. *The operator* $A_e : X \to X$

$$\{w_j\}_{-\infty}^{\infty} \to \left\{ w_{j+1} - \begin{pmatrix} 2+e, & -1 \\ 1, & 0 \end{pmatrix} w_j \right\}_{-\infty}^{\infty}$$

is invertible for $e > 0$ *and*

$$|A_e^{-1}| \le \frac{K}{e^2},$$

where $K > 0$ *is a constant.*

P r o o f. In the basis

$$e_1 = \left(\frac{2 + e + \sqrt{(4+e)e}}{2}, 1 \right)$$

$$e_2 = \left(\frac{2 + e - \sqrt{(4+e)e}}{2}, 1 \right)$$

the matrix

$$\mathbf{Q}_e = \begin{pmatrix} 2+e, & -1 \\ 1, & 0 \end{pmatrix}$$

has the form

$$\mathbf{B}_e = \begin{pmatrix} \frac{2+e+\sqrt{(4+e)e}}{2}, & 0 \\ 0, & \frac{2+e-\sqrt{(4+e)e}}{2} \end{pmatrix} = \begin{pmatrix} a_1, & 0 \\ 0, & a_2 \end{pmatrix}.$$

The mapping $C_e : X \to X$

$$\{w_j\}_{-\infty}^{\infty} \to \{w_{j+1} - \mathbf{B}_e w_j\}_{-\infty}^{\infty}$$

is invertible and $|C_e^{-1}| \leq K/e$. Indeed, the equation

$$w_{j+1} = \mathbf{B}_e w_j + h_j, \qquad h = \{h_j\} \in X$$

has a general solution

$$\begin{aligned} w_j^i &= h_{j-1}^i + h_{j-2}^i a_i + \cdots + h_0^i a_i^{j-1} + c_i a_i^j & j \geq 1 \\ w_j^i &= -h_j^i/a_i - h_{j+1}^i/a_i^2 - \cdots - h_{-1}^i/a_i^{-j} + c_i/a_i^{-j} & j \leq -1, \end{aligned} \tag{2}$$

where $i = 1,2$, $w_j = (w_j^1, w_j^2)$, $h_j = (h_j^1, h_j^2)$, $c_i \in \mathbb{R}$.
Since $a_1 > 1$ we have only one c_1 such that $\{w_j^1\}_{-\infty}^{\infty}$ is bounded [5, p. 272]

$$c_1 = -\sum_0^{\infty} h_j^1/a_1^{j+1}.$$

Hence

$$|c_1| \leq |h| \cdot \sum_0^{\infty} 1/a_1^{j+1} \leq \frac{K}{e} \cdot |h|$$

and for $j \leq -1$

$$|w_j^1| \leq |h| \cdot \left(\frac{1}{a_1} + \cdots + \frac{1}{a_1^{-j}} \right) + |c_1| \leq |h| \cdot \frac{K_1}{e}.$$

In the same way we solve other cases.
Finally, we note that $\left|T(e)^{-1}\right| \leq K/e$, where

$$T(e) = \begin{pmatrix} a_1, & 1 \\ a_2, & 1 \end{pmatrix}$$

and also $A_e = D_e^{-1} \cdot C_e \cdot D_e$, where

$$D_e \colon X \to X, \quad \{x_j\}_{-\infty}^{\infty} \to \{T(e) x_j\}_{-\infty}^{\infty}.$$

Proof of Theorem 1. We shall solve the following equation

$$\begin{aligned} x_{j+1} &= 2x_j - z_j + e \cdot x_j + e^3 \cdot g(x_j, y_j, e) \\ z_{j+1} &= x_j \\ y_{j+1} &= f(y_j) + e \cdot r(x_j, y_j, e) \end{aligned}$$

in $X \times Y$ for a small $e > 0$. Using the operators

$$S_e : X \times X \to X,$$
$$S_e(\{(x_j, z_j)\}, \{y_j\}) = \{(g(x_j, y_j + f^j(y_0), e), 0)\},$$
$$R_e : X \times Y \to Y,$$
$$R_e(\{(x_j, z_j)\}, \{y_j\}) = \{y_{j+1} - f(y_j + f^j(y_0)) - e \cdot r(x_j, y_j + f^j(y_0), e)\},$$

we can write the above equation in the form

$$\begin{aligned} A_e x &= e^3 S_e(x, y) \\ 0 &= R_e(x, y), \qquad x \in X, \quad y \in Y. \end{aligned} \tag{3}$$

Lemma 1 implies

$$|A_e^{-1}| \leq K/e^2.$$

Hence the equation (3) has the form

$$\begin{aligned} x &= e^3 A_e^{-1} S_e(x, y) \\ 0 &= R_e(x, y), \qquad e > 0. \end{aligned} \tag{4}$$

Since $e^3 \cdot |A_e^{-1}| \leq K \cdot e$ we apply the Banach fixed point theorem for the first equation of (4) and obtain a solution $x_e(y)$, where $|y| \leq 1$. Note that

$$|x_e(y)| \leq e \cdot M, \quad |\mathrm{D}_y\, x_e(y)| \leq e \cdot M,$$

where M is a constant, e is small positive and $|y| \leq 1$. Thus we can extended $x_e(\cdot)$ on $[0, e_0]$ in the following way

$$x_0(\cdot) = 0.$$

Finally, we solve $0 = R_e(x_e(y), y)$. We see that

$$R_0(x_0(0), 0) = 0$$
$$\mathrm{D}_y\, R_0(x_0(0), 0)\{y_j\}_{-\infty}^{\infty} = \{y_{j+1} - \mathrm{D}\, f(f^j(y_0)) y_j\}_{-\infty}^{\infty}.$$

Since the hypothesis (iv) holds, $f^j(y_0) \to 0$ as $j \to \pm\infty$ and $\mathrm{D}\, f(0)$ is hyperbolic; $\mathrm{D}_y R_0(0)$ is invertible [1], [5]. Hence by the implicit function theorem $R_e(x_e(y), y) = 0$ has a unique small solution $y(e)$ for a small positive e. We have shown that the mapping G_e has a unique orbit near Γ for a small positive e. It is not difficult to see that this orbit is homoclinic also [4, p. 106]. Indeed, let

$$\lim_{j \to \infty} |x_j(e)| + |z_j(e)| + |y_j(e)| = d > 0,$$

where $\{(x_j(e), z_j(e), y_j(e))\}_{-\infty}^{\infty}$ is the above found homoclinic orbit of G_e for small positive e. We have by (2), (3)

$$\begin{aligned}\bar{x}_j(e) &= -\left(h_j^1 \cdot a_1^{-1} + h_{j+1}^1 \cdot a_1^{-2} + \dots\right)\\ \bar{z}_j(e) &= h_{j-1}^2 + h_{j-2}^2 a_2 + \dots + h_0^2 a_2^{j-1} + c_2 a_2^j,\end{aligned}$$

where

$$\begin{aligned}(\bar{x}_j(e), \bar{z}_j(e)) &= T(e)\,(x_j(e), z_j(e))\,,\\ (h_j^1, h_j^2) &= T(e)\left(e^3 g(x_j(e), y_j(e)), 0\right).\end{aligned}$$

Thus for $j \geq j_0$, where j_0 is large fixed, we have $\left|(h_j^1, h_j^2)\right| = O(e^3) \cdot 2d$ and

$$\begin{aligned}&|x_j(e)| + |z_j(e)| = O(e^{-1})(|\bar{x}_j(e)| + |\bar{z}_j(e)|) =\\ &\quad O(e^{-1}) \cdot \left(O(e^3) \cdot 2d \cdot (a_1^{-1} + \dots) + O(e^3) \cdot 2d \cdot (1 + a_2 + \dots) +\right.\\ &\quad \left.O(1) \cdot \left(a_2^{j-j_0} + a_2^{j-j_0+1} + \dots + a_2^j\right)\right) = O(e^{-1}) \cdot \left(O(e^3) \cdot 2d \cdot O(e^{-1}) +\right.\\ &\quad \left.O(e^3) \cdot 2d \cdot O(e^{-1}) + O(e^3) \cdot (a_2^{j-j_0} + \dots + a_2^j)\right) =\\ &\qquad\qquad O(e) \cdot 2d + O(e^2) \cdot \left(a_2^{j-j_0} + \dots + a_2^j\right)\end{aligned}$$

and since $|a_2| < 1$ we have for $j \gg 1$

$$|x_j(e)| + |z_j(e)| = O(e) \cdot (2d + O(e) \cdot o(1)).$$

On the other hand,

$$y_{j+1}(e) = f(y_j(e)) + O(e)$$

and we can apply the same arguments as in [5, p. 295] to show $y_j(e) \to 0$. Hence for e small positive we obtain

$$d = \overline{\lim_{j\to\infty}}\,(|x_j(e)| + |z_j(e)| + |y_j(e)|) < d.$$

Thus $d = 0$. In the same way we study the ca_e $j \to \infty$.
This completes the proof of Theorem 1.

Remark 1. The assumption (iv) for the mapping f is equivalent to the property that y_0 is a transveısal homoclinic point of f (ee [1], [5]). From this it follows that the above found homoclinic point of G_e for a small positive e is a transversal homoclinic point.

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Received August 10, 1989

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