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BUCK'S MEASURE DENSITY AND SETS OF POSITIVE INTEGERS CONTAINING ARITHMETIC PROGRESSION

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ABSTRACT. The concept of measure density μ was introduced by R. C. Buck in 1946. In this paper some further properties of μ are established.

In [1] the concept of measure density of sets $A \subseteq \mathbf{N} = \{1, 2, \dots, n, \dots\}$ is introduced. Denote by $\mathcal{D}_0$ the class of all sets $A \subseteq \mathbf{N}$ which are finite unions of arithmetic progressions, or which differ from these by finite sets (the empty set $\emptyset$ belongs to $\mathcal{D}_0$, too).

If $A = \{an + b : \ n \geq 0, \ a, b \in \mathbf{N}\}$, then we put $\triangle(A) = \frac{1}{a}$ and if $A = A_1 \cup A_2 \cup \dots \cup A_m$ where the sets A_j $(j = 1, 2, \dots, m)$ are mutually disjoint and of the previous form, then we put $\triangle(A) = \sum_{j=1}^{m} \triangle(A_j)$. For $\emptyset$ we put $\triangle(\emptyset) = 0$.

The symbol $A \dot{\subseteq} B$ denotes that $A \subseteq B$ holds if we omit a finite number of elements from A (i.e. $A \dot{\subseteq} B$ means that the set $A \setminus B$ is finite). Then $A \doteq B$ means that the set $(A \setminus B) \cup (B \setminus A)$ is finite. If $A \in \mathcal{D}_0$ and $B \doteq A$, then B belongs to $\mathcal{D}_0$, too and we put $\triangle(B) = \triangle(A)$.

For $S \subseteq \mathbf{N}$ we define

$$\mu^*(S) = \inf_{A \in \mathcal{D}_0, \ S \dot{\subseteq} A} \triangle(A).$$

The number $\mu^*(S)$ is said to be the outer measure density of the set S. The function $\mu^* : 2^{\mathbf{N}} \to [0, 1]$ has the following properties:

a) $\mu^*(\emptyset) = 0$

b) If $S \subseteq \bigcup_{j=1}^{m} S_j$, then $\mu^*(S) \leq \sum_{j=1}^{m} \mu^*(S_j)$.

Denote by $\mathcal{D}_\mu$ the class of all $S \subseteq \mathbf{N}$ which satisfy the following condition:

$$\mu^*(Z) = \mu^*(Z \cap S) + \mu^*(Z \cap S') \qquad \text{for all } Z \subseteq \mathbf{N} \tag{1}$$

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where $S' = \mathbf{N} \setminus S$. Then the class $\mathcal{D}_\mu$ is an algebra of sets and the set function $\mu = \mu^*/\mathcal{D}_\mu$ is a finitely additive measure on $\mathcal{D}_\mu$ (c.f. [8], pp. 226–228).

The number $\mu(S) \in [0,1]$ is called the measure density of the set $S \in \mathcal{D}_\mu$.

It can be shown that the condition (1) is equivalent to the following condition:

$$\mu^*(S) + \mu^*(S') = 1. \tag{1'}$$

This fact is recalled (without proof) in [1] (p. 562 (i)). We shall prove it using the following simple observation.

Proposition A. *A set* $S \subseteq \mathbf{N}$ *satisfies the condition* (1') *if and only if*

$$\inf_{A \dot{\supseteq} S,\ A \in \mathcal{D}_0} \Delta(A) = \sup_{B \dot{\subseteq} S,\ B \in \mathcal{D}_0} \Delta(B). \tag{A}$$

P r o o f. The set S satisfies the condition (1') if and only if

$$\inf_{A \dot{\supseteq} S,\ A \in \mathcal{D}_0} \Delta(A) = 1 - \inf_{C \dot{\supseteq} S',\ C \in \mathcal{D}_0} \Delta(C).$$

Consider that $C \dot{\supseteq} S'$ holds if and only if $\mathbf{N} \setminus C \dot{\subseteq} S$. Put $B = \mathbf{N} \setminus C$. Then $B \in \mathcal{D}_0$ (c.f. [1], (A1), p. 561), $B \dot{\subseteq} S$ and $\Delta(B) = 1 - \Delta(C)$. It is obvious from this that the set S satisfies (1') if and only if the equality (A) holds. ∎

Corollary. (a) *The conditions* (1), (1') *are equivalent.*

P r o o f. Evidently (1) implies (1') (it suffices to put $Z = \mathbf{N}$ in (1)). Assume that (1') holds. Then according to Proposition A we get for an arbitrary $Z \subseteq \mathbf{N}$

$$\mu^*(Z \cap S) = \sup_{F \dot{\subseteq} Z \cap S,\ F \in \mathcal{D}_0} \Delta(F); \tag{2}$$

$$\mu^*(Z \cap S') = \sup_{E \dot{\subseteq} Z \cap S',\ E \in \mathcal{D}_0} \Delta(E); \tag{2'}$$

$$\mu^*(Z) = \sup_{G \dot{\subseteq} Z,\ G \in \mathcal{D}_0} \Delta(G). \tag{2''}$$

Let $\varepsilon > 0$. According to (2),(2') there exist $F_1, E_1 \in \mathcal{D}_0$ such that

$$\mu^*(Z \cap S) - \frac{\varepsilon}{2} < \Delta(F_1); \tag{3}$$

$$\mu^*(Z \cap S') - \frac{\varepsilon}{2} < \Delta(E_1); \tag{3'}$$

$F_1 \dot{\subseteq} Z \cap S$, $E_1 \dot{\subseteq} Z \cap S'$. But then $F_1 \cup E_1 \dot{\subseteq} Z$ and $F_1 \cap E_1 \doteq \emptyset$. Therefore

$$\Delta(F_1 \cup E_1) = \Delta(F_1) + \Delta(E_1). \qquad (3")$$

Adding (3),(3') we get on account of (3")

$$\mu^*(Z \cap S) + \mu^*(Z \cap S') - \varepsilon < \Delta(F_1 \cup E_1) \leq \mu^*(Z).$$

From this by $\varepsilon \to 0^+$ we get

$$\mu^*(Z) \geq \mu^*(Z \cap S) + \mu^*(Z \cap S').$$

The opposite inequality holds too because μ^* is an outer measure. Thus (1) follows. ∎

Corollary. (b) *A set $S \subseteq \mathbf{N}$ belongs to $\mathcal{D}_\mu$ if and only if for each $\varepsilon > 0$ there exist two sets $A, B \in \mathcal{D}_0$ such that $B \dot{\subseteq} S \dot{\subseteq} A$ and $\Delta(A) - \Delta(B) < \varepsilon$.*

For $A \subseteq \mathbf{N}$ we define the asymptotic densities $\underline{\mathrm{d}}(A)$ (the lower density of A), $\overline{\mathrm{d}}(A)$ (the upper density of A) as follows: Denote by $A(n)$ the number of elements of A not exceeding n. Then we put

$$\underline{\mathrm{d}}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}, \qquad \overline{\mathrm{d}}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}.$$

If there exists $\lim_{n\to\infty} \frac{A(n)}{n}$, then we denote this limit by $\mathrm{d}(A)$. The number $\mathrm{d}(A)$ is called the asymptotic density of the set A.

Denote by $\mathcal{D}$ the class of all sets $S \subseteq \mathbf{N}$ for which $\mathrm{d}(S)$ exists. In [1] (p. 571) the inclusion $\mathcal{D}_\mu \subseteq \mathcal{D}$ is proved and if $S \in \mathcal{D}_\mu$, then $\mu(S) = \mathrm{d}(S)$.

In this paper we introduce some considerations about a possible extension of the class $\mathcal{D}$, further we prove that the measure density μ has the Darboux property and introduce some simple results concerning the relation between the positivity of $\mu(S)$ and the fact that S contains an infinite arithmetic progression.

1. On an extension of the class $\mathcal{D}$

Put for $S \subseteq \mathbf{N}$

$$\omega(S) = \inf_{A \supseteq S,\ A \in \mathcal{D}} \mathrm{d}(A)$$

Denote by $\mathcal{D}_\omega$ the class of all sets $S \subseteq \mathbf{N}$ for which

$$\omega(S) + \omega(S') = 1$$

holds.

It is proved in [1] (Theorem 8 in [1], p. 572) that $\mathcal{D}_\omega = \mathcal{D}$. We now give a new proof of the quoted result from [1] which shows that the class $\mathcal{D}$ is not extendable in the described way.

Theorem 1.1. *We have* $\mathcal{D}_\omega = \mathcal{D}$.

P r o o f. Since evidently $\mathcal{D} \subseteq \mathcal{D}_\omega$, it suffices to prove that

$$\mathcal{D}_\omega \subseteq \mathcal{D} \tag{4}$$

Let $S \in \mathcal{D}_\omega$. Then $\omega(S) + \omega(S') = 1$ ($S' = \mathbf{N} \setminus S$). Hence

$$\inf_{B \supseteq S,\ B \in \mathcal{D}} \mathrm{d}(B) = 1 - \inf_{C \supseteq S',\ C \in \mathcal{D}} \mathrm{d}(C). \tag{5}$$

From $C \supseteq S'$ we get $A = \mathbf{N} \setminus C \subseteq S$ and $1 - \mathrm{d}(C) = \mathrm{d}(\mathbf{N} \setminus C) = \mathrm{d}(A)$. From this we get

$$\sup_{A \subseteq S,\ A \in \mathcal{D}} \mathrm{d}(A) = 1 - \inf_{C \supseteq S',\ C \in \mathcal{D}} \mathrm{d}(C). \tag{5'}$$

The equalities (5),(5') yield

$$\inf_{B \supseteq S,\ B \in \mathcal{D}} \mathrm{d}(B) = \sup_{A \subseteq S,\ A \in \mathcal{D}} \mathrm{d}(A) \qquad (= v). \tag{6}$$

Let $\epsilon > 0$. According to (6) there exist two sets $A_0, B_0 \in \mathcal{D}$ such that

$$A_0 \subseteq S \subseteq B_0 \tag{7}$$

$$\mathrm{d}(B_0) < v + \frac{\epsilon}{2}, \qquad \mathrm{d}(A_0) > v - \frac{\epsilon}{2}. \tag{7'}$$

From (7),(7') we get

$$v - \frac{\epsilon}{2} < \mathrm{d}(A_0) \le \liminf_{n \to \infty} \frac{S(n)}{n} \le \limsup_{n \to \infty} \frac{S(n)}{n} \le \mathrm{d}(B_0) < v + \frac{\epsilon}{2}.$$

This is true for each $\epsilon > 0$. Therefore there exists $\mathrm{d}(S)$ and $\mathrm{d}(S) = v$. Hence (4) holds. ∎

Finally we mention the cardinalities of the investigated classes. Denote by $|M|$ the cardinal number of the set M. Already in [1], p. 580, the equalities

$$|\mathcal{D}_0| = \aleph_0, \qquad |\mathcal{D}_\mu| = |\mathcal{D}| = c$$

are proved (c is the cardinal number of the continuum).

Further, we have seen that $\mathcal{D}_0 \subseteq \mathcal{D}_\mu \subseteq \mathcal{D}$. Therefore the question arises how large the cardinalities of the classes $\mathcal{D}_\mu \setminus \mathcal{D}_0$, $\mathcal{D} \setminus \mathcal{D}_0$, $\mathcal{D} \setminus \mathcal{D}_\mu$ are. We have

$$|\mathcal{D}_\mu \setminus \mathcal{D}_0| = |\mathcal{D} \setminus \mathcal{D}_0| = c$$

on the basis of the well-known result of the set theory according to which, if P is an uncountable set and M is a countable set, then the set $P \setminus M$ and P have the same cardinality.

The cardinality of $\mathcal{D} \setminus \mathcal{D}_\mu$ is also equal to c. This follows from the fact that each set of the form $A = \{[\alpha n + \beta] : n \in \mathbf{N}\}$ ($[t]$ denotes the integer part of t), where $\alpha > 1$, $\beta \ge 0$, β is real and α irrational, belongs to $\mathcal{D}$ (the density of A being $\frac{1}{\alpha}$), but does not belong to $\mathcal{D}_\mu$ (cf. [1], Theorem 7, p. 570).

2. Darboux property of the measure density

In this part of the paper we shall give a proof of the fact that μ has the Darboux property. This proof is quite different from that given in [6].

We use the concept of the Darboux property in agreement with the terminology contained in [2] pp. 25–32. Let $\mathcal{S}$ be a class of sets and $\nu : \mathcal{S} \to [0, +\infty]$ a set function on $\mathcal{S}$. The set $E \in \mathcal{S}$ is said to have the Darboux property with respect to ν provided that for each $a \in [0, \nu(E)]$ there exists a set $A \subseteq E$, $A \in \mathcal{S}$ such that $\nu(A) = a$. The set function ν is said to have the Darboux property provided that each set $E \in \mathcal{S}$ has the Darboux property with respect to ν.

Instead of "the Darboux property of ν" also the terminology "ν is full-valued" can be used (cf. [5]).

The proof of the following theorem is based on a modification of a procedure used in [5]. This method enables us to prove a more general result (see Theorem 2.2).

Theorem 2.1. *The measure density μ has the Darboux property.*

The proof is based on the following auxiliary result.

Lemma 2.1. *Let $M \subseteq E$, $M, E \in \mathcal{D}_\mu$ and $\epsilon > 0$. Then there exist mutually disjoint sets $D_j \in \mathcal{D}_\mu$ $(j = 1, 2, \ldots, s)$ such that*

$$M = \bigcup_{j=1}^{s} D_j, \qquad \mu(D_j) < \epsilon \qquad (j = 1, 2, \ldots, s).$$

P r o o f. Choose an $s \in \mathbf{N}$ such that $\frac{1}{s} < \epsilon$. Put

$$D_j = R_j \cap M \qquad (j = 1, 2, \ldots, s),$$

where R_j $(j = 1, 2, \ldots, s)$ denotes the set of positive elements of the residue class $\overline{j}$ $(\bmod s)$. Then evidently $D_j \in \mathcal{D}_\mu$ since $\mathcal{D}_\mu$ is an algebra of sets. It is easy to check that the sets D_j have the desired properties. ∎

P r o o f o f T h e o r e m 2.1. Let $E \in \mathcal{D}_\mu$ and $0 < a < \mu(E)$. Suppose that there is no $M \in \mathcal{D}_\mu$, $M \subseteq E$ such that $\mu(M) = a$.

We shall construct two sequences $\{B_n\}_{n=1}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$ of sets from $\mathcal{D}_\mu$ such that

$$B_1 \subseteq B_2 \subseteq \ldots; \qquad C_1 \supseteq C_2 \supseteq \ldots \tag{8}$$

$$B_n \subseteq C_n \qquad (n = 1, 2, \ldots) \tag{9}$$

$$a - \frac{1}{n} < \mu(B_n) < a < \mu(C_n) < a + \frac{1}{n} \qquad (n = 1, 2, \ldots) \tag{10}$$

$$B_n \subseteq E, \qquad C_n \subseteq E \qquad (n = 1, 2, \ldots). \tag{11}$$

In the first step we put $B_1 = \emptyset$, $C_1 = E$. Let us suppose that the construction of the sets B_k, C_k is already finished in such a way that the conditions (8)–(11) (for $n = k$) are satisfied. We shall construct the sets B_{k+1}, C_{k+1}.

According to the assumption of induction we have

$$B_k \subseteq C_k, \qquad B_k, C_k \in \mathcal{D}_\mu, \qquad B_k, C_k \subseteq E$$

Put $M = C_k \setminus B_k$ and

$$\epsilon = \min\{a - \mu(B_k), \ \frac{1}{k+1}\}$$

in Lemma 2.1. On account of Lemma 2.1 there exist mutually disjoint sets $D_j \in \mathcal{D}_\mu$ $(j = 1, 2, \ldots, s)$ such that $D_j \subseteq E$ $(j = 1, 2, \ldots, s)$ and

$$C_k \setminus B_k = \bigcup_{j=1}^{s} D_j \tag{12}$$

and for each $j = 1, 2, \ldots, s$ we have $\mu(D_j) < \epsilon$.

Consider that

$$\mu(B_k \cup D_1) \le \mu(B_k) + \mu(D_1) < \mu(B_k) + (a - \mu(B_k)) = a$$

and simultaneously according to (12)

$$\mu(B_k \cup \bigcup_{j=1}^{s} D_j) = \mu(C_k) > a\,.$$

Therefore there exists a positive integer t such that $1 \le t < s$ and

$$\mu(B_k \cup \bigcup_{j=1}^{t} D_j) < a\,; \tag{13}$$

$$\mu(B_k \cup \bigcup_{j=1}^{t+1} D_j) \ge a\,. \tag{13'}$$

Since the set $M = B_k \cup \bigcup_{j=1}^{t+1} D_j$ belongs to $\mathcal{D}_\mu$ and $M \subseteq E$ we cannot have $\mu(M) = a$. Therefore in (13') the strict inequality $>$ holds.

Put

$$B_{k+1} = B_k \cup \bigcup_{j=1}^{t} D_j \tag{14}$$

$$C_{k+1} = B_{k+1} \cup D_{t+1} = B_k \cup \bigcup_{j=1}^{t+1} D_j \,. \tag{14'}$$

It follows from (14),(14') that $B_{k+1}, C_{k+1} \in \mathcal{D}_\mu$, $B_{k+1}, C_{k+1} \subseteq E$. Further, from (13),(13') we get $\mu(B_{k+1}) < \mu(C_{k+1})$.

Consider that

$$\mu(C_{k+1}) \leq \mu(B_{k+1}) + \mu(D_{t+1}) < a + \frac{1}{k+1}$$

and according to (13),(13') we have

$$a < \mu(C_{k+1}) \leq \mu(B_{k+1}) + \mu(D_{t+1}) < \mu(B_{k+1}) + \frac{1}{k+1} \,.$$

From this we get

$$\mu(B_{k+1}) > a - \frac{1}{k+1} \,.$$

Hence we have

$$a - \frac{1}{k+1} < \mu(B_{k+1}) < a < \mu(C_{k+1}) < a + \frac{1}{k+1} \,.$$

Further, from (14),(14') we get $B_{k+1} \subseteq C_{k+1}$ and evidently $B_k \subseteq B_{k+1}$. It follows from the definition of C_{k+1} that

$$C_{k+1} = B_k \cup \bigcup_{j=1}^{t} D_j \cup D_{t+1} \subseteq B_k \cup \bigcup_{j=1}^{s} D_j = C_k \,,$$

hence $C_{k+1} \subseteq C_k$.

This ends construction (by induction) of the sequences $\{B_n\}_{n=1}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$.

Put $A = \bigcup_{j=1}^{\infty} B_j$. Then according to (11) we have $A \subseteq E$.

For each $n \in \mathbf{N}$ we have $A = \bigcup_{j=1}^{n} B_j \cup \bigcup_{j=n+1}^{\infty} B_j$. Since $B_j \subseteq B_n \subseteq C_n$ for $j \leq n$ and $B_j \subseteq C_j \subseteq C_n$ for $j > n$, we see that $A \subseteq C_n$.

Obviously we have $B_n \subseteq A$ and therefore

$$B_n \subseteq A \subseteq C_n \qquad (n = 1, 2, \ldots) \,. \tag{15}$$

We prove that the set A belongs to $\mathcal{D}_\mu$. Let $\epsilon > 0$. Choose an $n \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\epsilon}{4}$. Since the sets B_n, C_n belong to $\mathcal{D}_\mu$, we can choose by Proposition A the sets $B^*, C^* \in \mathcal{D}_0$ such that $B^* \dot{\subseteq} B_n$, $C_n \dot{\subseteq} C^*$ and

$$\Delta(B^*) > \mu(B_n) - \frac{\epsilon}{4}, \qquad \Delta(C^*) < \mu(C_n) + \frac{\epsilon}{4}$$

According to (10) and (15) we get $B^* \dot{\subseteq} A \dot{\subseteq} C^*$ and

$$\Delta(C^*) - \Delta(B^*) < \mu(C_n) - \mu(B_n) + \frac{\epsilon}{2} < \frac{2}{n} + \frac{\epsilon}{2} < \epsilon .$$

On the basis of Proposition A the set A belongs to $\mathcal{D}_\mu$.

We obtain a contradiction showing that $\mu(A) = a$.

Let n be an arbitrary positive integer. According to (10) and (15) we have $|\mu(A) - a| < \frac{2}{n}$. From this by $n \to \infty$ we get $\mu(A) = a$. This ends the proof. ∎

The detailed analysis of the foregoing proof shows that by an analogous procedure the following more general result can be proved.

Theorem 2.2. *Let $S \subseteq 2^{\mathbf{N}}$ be an algebra of sets and let ν be a finitely additive measure on S. Let ν satisfy the following two conditions:*

(i) *If $A \subseteq \mathbf{N}$ and*

$$\inf_{C \supseteq A,\ C \in S} \nu(C) = \sup_{B \subseteq A,\ B \in S} \nu(B) \qquad (= v),$$

then A belongs to S and $\nu(A) = v$.

(ii) *For each $M \in S$ and $\epsilon > 0$ there exist mutually disjoint sets $D_j \in S$ such that $M = \bigcup_{j=1}^{s} D_j$ and $\nu(D_j) < \epsilon$ $(j = 1, 2, \dots, s)$.*

Then the measure ν has the Darboux property.

3. The measure density μ and the sets $A \subseteq \mathbf{N}$ containing arithmetic progressions

The set $A \subseteq \mathbf{N}$ is said to contain an arithmetic progression of the length $k \geq 3$ $(k \in \mathbf{N})$ if there is an arithmetic progression $a_1 < a_2 < \cdots < a_k$ with k terms such that $\{a_1, a_2, \dots, a_k\} \subseteq A$. Analogously we say that

$$B = \{b_1 < b_2 < \cdots < b_n < \dots\} \subseteq \mathbf{N}$$

contains an infinite arithmetic progression if there exists a sequence $k_1 < k_2 < \dots < k_n < \dots$ of indices such that

$$b_{k_1} < b_{k_2} < \dots < b_{k_n} < \dots$$

forms an arithmetic progression.

It is well known (cf. [9]) that a set $A \subseteq \mathbf{N}$ contains arithmetic progressions of the length k for each $k \geq 3$ provided that $\overline{\mathrm{d}}(A) > 0$. The following simple theorem gives a sufficient condition for a set $A \subseteq \mathbf{N}$ contains an infinite arithmetic progression.

Theorem 3.1. *If $S \in \mathcal{D}_\mu$ and $\mu(S) > 0$, then S contains an infinite arithmetic progression.*

P r o o f. According to Proposition A and Corollary (a) after it we have

$$0 < \mu(S) = \sup_{A \dot{\subseteq} S,\ A \in \mathcal{D}_0} \Delta(A)$$

Put $\epsilon = \frac{\mu(S)}{2} > 0$. Then on the basis of the definition of the least upper bound there exists a set $A_0 \in \mathcal{D}_0$ such that $A_0 \dot{\subseteq} S$ and

$$\Delta(A_0) > \mu(S) - \frac{\epsilon}{2} > 0.$$

It is clear from this that $A_0 \neq \emptyset$ and therefore A_0 contains an infinite arithmetic progression. But then by $A_0 \dot{\subseteq} S$ the set S contains such a progression, too. ∎

We shall show that in Theorem 3.1 the measure density cannot be replaced by the outer measure μ^*.

Theorem 3.2. *There exists a set $S_0 \subseteq \mathbf{N}$ such that $\mu^*(S_0) = 1$ and S_0 does not contain any arithmetic progression of the length 3.*

P r o o f. Put

$$S_0 = \{1 + 1!, 2 + 2!, \dots, n + n!, \dots\}.$$

Let $\{aj + b\}_{j=1}^{\infty}$, $a, b \in \mathbf{N}$, be an arbitrary arithmetic progression. Denote by A the set of all its terms. Put $n_k = ak + b$ $(k = 1, 2, \dots)$. Then it is easy to see that the elements $n_k + n_k!$ $(k = 1, 2, \dots)$ of S_0 belong to A. Thus the set $A \cap S_0$ is infinite and so S_0 cuts each arithmetic progression in infinitely many terms. From this we get obviously that $\mu^*(S_0) = 1$.

We shall show that S_0 does not contain any arithmetic progression of length 3.

We shall proceed indirectly. Suppose that

$$1 \le a_1 < a_2 < a_3 \tag{16}$$

is an arithmetic progression such that $\{a_1, a_2, a_3\} \subseteq S_0$. Then by definition of the set S_0 there exist positive integers $1 \le n_1 < n_2 < n_3$ such that $a_k = n_k + n_k!$ $(k = 1, 2, 3)$. The difference of the sequence (16) is equal to $d = n_2 + n_2! - (n_1 + n_1!)$. The following simple estimation yields

$$\begin{aligned} a_3 = a_1 + 2d = n_1 + n_1! + 2\,[(n_2 + n_2!) - (n_1 + n_1!)] = \\ = 2n_2 + 2n_2! - n_1 - n_1! < 2n_2 + 2n_2! < \\ < (n_2 + 1) + (n_2 + 1)! \le n_3 + n_3! = a_3\,. \end{aligned}$$

Hence we have a contradiction. ∎

Finally let us remark that even the positivity of the asymptotic density of a set $A \subseteq \mathbf{N}$ does not guarantee that A contains an infinite arithmetic progression. According to Theorem 3.1 such a sufficient condition is the following: $\mathrm{d}(A) > 0$ and simultaneously $A \in \mathcal{D}_\mu$. An example of a set $A \subseteq \mathbf{N}$ with a positive $\mathrm{d}(A)$ which does not contain any infinite arithmetic progression is given in [3], pp. 159–160. Here we give another example of this kind.

Example 3.1. Denote by Q the set of all $a \in \mathbf{N}$ such that there is no prime number p with p^2 dividing the number a (quadratfreie Zahlen). It is well known that $\mathrm{d}(Q) = \frac{6}{\pi^2} > 0$ (cf. [4], p. 269). Suppose that Q contains an infinite arithmetic progression $\{a_k\}_{k=1}^{\infty}$. Then according to Exercise 1, pp. 243–244 from [7] there exists a geometric progression $\{aq^n\}_{n=1}^{\infty}$ $(q \ge 2)$ as a subsequence of $\{a_k\}_{k=1}^{\infty}$. But then Q contains the numbers aq^n $(n \ge 2)$, which contradicts the definition of Q.

Remark 3.1. It follows from Example 3.1 and Theorem 3.1 that the set Q does not belong to $\mathcal{D}_\mu$. More generally, if $A \subseteq \mathbf{N}$, $\mathrm{d}(A) > 0$ and A does not contain any infinite arithmetic progression, then A does not belong to the class $\mathcal{D}_\mu$.

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REFERENCES

[1] BUCK, R. C.: The measure theoretic approach to density. Amer. J. Math. LXVIII (1946), 560–580.

[2] DINCULEANU, N.: Vector Measures. VEB Deutscher Verlag der Wissen, Berlin, 1966

[3] ERDÖS, P.—NATHANSON, M. B.—SÁRKÖZY, A.: Sumsets containing infinite arithmetic progressions. J. Number Theory 28 (1988), 159–166.

[4] HARDY, G. H.—WRIGHT, E. M.: An Introduction to the Theory of Numbers. 3rd ed., Oxford, 1954.

[5] MAHARAM, D.: Finitely additive measures on the integers. Sankhya: The Indian J. Stat. Ser. A 38 (1976), 44–59.

[6] PAŠTÉKA, M.: Some properties of Buck's measure density. (To appear.).

[7] SIERPIŃSKI, W.: Elementary Theory of Numbers. PWN, Warszawa, 1964.

[8] SIKORSKI, R.: Funkcje rzeczywiste I. PWN, Warszawa, 1958.

[9] SZEMERÉDI, E.: On sets of integers containing no k elements in arithmetic progression. Acta Arithm. 27 (1975), 199–245.

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